APPLICATIONS OF THE TOPOLOGICAL METHOD OF WAZEWSKI TO CERTAIN PROBLEMS OF ASYMPTOTIC BEHAVIOR IN ORDINARY DIFFERENTIAL EQUATIONS

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Introduction. The main objective of this paper is to present some results concerning the asymptotic behavior of the integrals of some systems of ordinary differential equations.

As Wazewski's theorem, used in our work, is not very well known, we state it here, giving first some definitions and notations.

HYPOTHESIS H. (a) *The real-valued functions* $f_i(t, x_1, \cdots, x_n)$, $i = 1, \cdots, n$, *of the real variables* $t, x_1, \cdots, x_n$, *are continuous in an open set* $\Omega \subset R^{n+1}$.

(b) *Through every point of* Ω *passes only one integral of the system*

$$\dot{x} = f(t, x) \quad \left(\cdot = \frac{d}{dt} \right) \quad \textit{where}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} f_1(t, x_1, \cdots, x_n) \\ \vdots \cdots\cdots\cdots\cdots \\ f_n(t, x_1, \cdots, x_n) \end{pmatrix} \qquad \textit{and } (t, x) \in \Omega .$$

Let ω be an open set of R^{n+1}, $\omega \subset \Omega$ and let us denote by $B(\omega, \Omega)$ the boundary of ω in Ω.

Let P_0: $(t_0, x_0) \in \Omega$. We write $I(t, P_0) = (t, x(t, P_0))$, where $x(t, P_0)$ is the integral of the system $\dot{x} = f(t, x)$ passing through the point P_0.

Let $(\alpha(P_0), \beta(P_0))$ be the maximal open interval in which the integral passing through P_0 exists. We write

$$I(\Delta, P_0) = \{(\mathrm{t}, x(t, P_0)) \mid t \in \Delta\}$$

for every set Δ contained in $(\alpha(P_0), \beta(P_0))$.

We say that the point P_0: $(t_0, x_0) \in B(\omega, \Omega)$ is a *point of egress* from ω (with respect to the system $\dot{x} = f(t, x)$ and the set Ω) if there exists a positive number δ such that $I([t_0 - \delta, t_0), P_0) \subset \omega$; P_0 is a *point of strict egress* from ω if P_0 is a point of egress and if there exists a positive number δ such that $I((t_0, t_0 + \delta], P_0) \subset \Omega - \overline{\omega}$. The set of all points of egress (strict egress) is denoted by $S(S^*)$.

If $A \subset B$ are any two sets of a topological space and $K: B \to A$ is

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a continuous mapping from B onto A such that $K(P) = P$ for every $P \in A$, then K is said to be a *retraction* from B into A and A a *retract* of B.

THEOREM OF WAŻEWSKI. *Suppose that the system* $\dot{x} = f(t, x)$ *and the open sets* $\omega \subset \Omega \subset R^{n+1}$ *satisfy the following hypotheses:*

(1) *Hypothesis H.*

(2) $S = S^*$.

(3) *There exists a set* $Z \subset \omega \cup S$ *such that* $Z \cap S$ *is a retract of* S *but is not a retract of* Z.

Then there is at least one point P_0: $(t_0, x_0) \in Z - S$ *such that* $I(t, P_0) \subset \omega$ *for every* $t_0 \leqq t < \beta(P_0)$.

The theorem of Ważewski [6, Théorème 1, p. 299] is actually more general than the one stated above.

If $f_i(t, x_1, \cdots, x_n)$, $i = 1, \cdots, n$, are complex-valued functions of the real variable t and of the complex variables $x_1, \cdots, x_n$, the n-dimensional complex system $\dot{x} = f(t, x)$ can be considered as a $2n$-dimensional real system, so that the theorem of Ważewski is also extensible, in a natural way, to complex systems [5, p. 19. § 1 and p. 21, § 2].

The most difficult part in the applications of the method of Ważewski is, in general, to verify that $S = S^*$. To accomplish this Ważewski introduced the concept of a regular polyfacial set [6, § 14 p. 307 and § 15, p. 309]. However the distinction established by Ważewski between positive and negative faces has certain inconveniences. In some applications of the method of Ważewski there appear sets ω such that $S = S^*$ but whose faces are only "almost positive" and "almost negative". We thus have to work sometimes with sets ω that are similar, in some sense, to the regular polyfacial sets and that satisfy the condition $S = S^*$.

In the first part of our work we give a generalization of polyfacial regular sets eliminating the distinction between positive and negative faces and such that the main theorem concerning the polyfacial regular sets [6, Théorème 5, p. 310] remains valid. We observe that the sets ω considered in Z. Szmydtówna's paper [5, § 4, Théorème 1, p. 24][1], in our Theorem II–1 and in Barbălat's paper [1, Théorème 1, p. 303; Théorème 2, p. 305] are generalized regular polyfacial sets, in our sense, but are not regular polyfacial sets.

Szmydtówna [5, Corollaire 1-Remarque 2, p. 30] proves a theorem

[1] Szmydtówna's Theorem 1 is false. We observed that the proof is wrong because the statement: "La frontière de ω touchant celle de Ω exclusivement sur le plan $t = \infty$ $\cdots$" [5, p. 28] is false.

J. Lewowics [3], developing a counter-example suggested by J. L. Massera, has shown that the theorem is actually false. Nevertheless, Theorems 2 and 3 deduced from Theorem 1 are correct because, in the particular case of linear systems $\dot{x} = A(t)x$, with $A(t)$ defined for $T \leqq t < \infty$, the solutions are defined for all $T \leqq t < \infty$.

which generalizes a theorem of Perron. In part II of our work (Theorem II-1) we obtain the same conclusion but starting from hypotheses different from those of Szmydtówna.

Note[2]. Our Theorem II-1 improves a result of N. I. Gavrilov. I. M. Rapoport in his book "On some asymptotic methods in the theory of differential equations", Kiev (1954) has also studied problems of this type. For some reference to their work to see "Forty years of Soviet Mathematics", Moscow (1959), Vol. i., pp. 520–521.

Our Theorem II-2 follows the same line of ideas.

Theorem II-3, due to Professor J. L. Massera, shows that in the case $n = 2$ the asymptotic behavior can be described more completely.

Consider two systems

$$\dot{y} = A(t)y \tag{1}$$

$$\dot{x} = A(t)x + g(t, x) \tag{2}$$

where $A(t)$ is a continuous matrix for $t \geqq T$ and $g(t, x)$ a continuous vector-function in $\Omega = [T, \infty) \times R^{2n}$.

Suppose that $g(t, x)$ satisfies some condition ensuring the uniqueness of the solution through each point $P_0 \in \Omega$ and that all solutions are defined for $T \leqq t < \infty$. We say that (1) and (2) are asymptotically equivalent if there exists a homeomorphism ϕ from the plane $t = T$ onto itself such that if $Q_0 = \phi(P_0)$ then $\lim_{t\to\infty} [x(t, P_0) - y(t, Q_0)] = 0$ [4, Cap. IX, § 4, p. 634].

In part III of our work the main result is the establishment of a condition that implies the asymptotic equivalence between two linear systems (Theorem III-3).

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PART I

Let the real-valued functions

$$f_i(t, x_1, \cdots, x_n) \, , \qquad i = 1, \cdots, n \, ,$$

of real variables $t, x_1 \cdots, x_n$ belong to C^p, $p \geqq 1$, on an open set $\Omega \subset R^{n+1}$, i.e., all partial derivatives

$$\frac{\partial^k_{f_i}}{\partial t^{p_0} \partial x_1^{p_1} \cdots \partial x_n^{p_n}} \qquad (p_0 + p_1 + \cdots + p_n = k \leqq p)$$

[2] The information given in this Note is due to the referee. We have not had access to the above works. We are indebted to him for this.

exist and are continuous on Ω.

Consider the differential system

$$\text{(I)} \qquad \dot{x} = f(t, x)$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } f(t, x) = \begin{pmatrix} f_1(t, x_1, \cdots, x_n) \\ \vdots \cdots\cdots\cdots\cdots \\ f_n(t, x_1, \cdots, x_n) \end{pmatrix}$$

with $(t, x) \in \Omega$.

Let $g(t, x)$ be a real-valued function belonging to C^{p+1} on Ω, let $P_0: (t_0, x_0) \in \Omega$ and let $x(t)$ be the integral of system (I) passing through the point P_0. We set $\varphi(t) = g(t, x(t))$; since $f(t, x) \in C^p$ and $g(t, x) \in C^{p+1}$ it follows $\varphi(t) \in C^{p+1}$ on $(\alpha(P_0), \beta(P_0))$.

The qth derivative, $q \leqq p+1$, of $g(t, x)$ at the point $P_0: (t_0, x_0)$ with respect to the system (I), is by definition

$$\left[\frac{d^q}{dt^q}\varphi(t)\right]_{t_0} \text{ and is denoted by } [D^q_{(I)}g(P)]_{P_0}\,.$$

Let $H_i(P) = H_i(t, x)$, $i = 1, \cdots, m$, be functions $\in C^{p+1}$ on the open set $\Omega \subset R^{n+1}$.

Let

$$\omega = \{P \in \Omega \mid H_i(P) < 0, i = 1, \cdots, m\}$$
$$\Gamma_i = \{P \in \Omega \mid H_i(P) = 0, H_j(P) \leqq 0, j = 1, \cdots, m\}$$

The Γ_i are called *faces* of ω.

Such a set ω will be called a *generalized regular polyfacial set* relative to (I) if, for each $i = 1, \cdots, m$ and each $P_0: (t_0, x_0) \in \Gamma_i$, the following alternative holds:

(1) The smallest index $q \leqq p+1$ such that $[D^q_{(I)}H_i(P)]_{P_0} \neq 0$ is odd and the corresponding derivative is positive;

(2) P_0 is not a point of egress.

Let L_i, M_i be the corresponding sets of points. Useful criteria to verify $P_0 \in M_i$ are:

(*a*) the smallest index $q \leqq p+1$ such that $[D^q_{(I)}H_i(p)]_{P_0} \neq 0$ is either odd with a negative value of the derivative or even with a positive value of the derivative;

(b) There exists $[a, b] \subset (\alpha(P_0), \beta(P_o))$ such that $a < t_0 \leqq b$ and $I([a, b], P_0) \subset \Gamma_i$.

LEMMA 1. *If ω is a generalized regular polyfacial set relative to* (I),

$$S = S^* = \bigcup_{i=1}^{m} L_i - \bigcup_{i=1}^{m} M_i .$$

Proof. Since $\Gamma_i = L_i \cup M_i$, $B(\omega, \Omega) \subset \bigcup_{i=1}^{m} \Gamma_i$,

$$S^* \subset S \subset \bigcup_{i=1}^{m} L_i - \bigcup_{i=1}^{m} M_i ,$$

it is enough to show that any point P_0 belonging to this last set is a point of strict egress. For such a P_0, $J = \{j \mid P_0 \in L_j\} \neq \phi$. If $j \in J$, $H_j(P_0) = 0$ and there exists a $\delta > 0$ such that $H_j(t, I(t, P_0)) < 0$ in $[t_0 - \delta, t_0)$ and $H_j(t, I(t, P_0)) > 0$ in $(t_0, t_0 + \delta]$. If $j \notin J$, $P_0 \notin \Gamma_j$ whence $H_j(P_0) < 0$ and there exists a $\delta > 0$ such that $H_j(t, I(t, P_0)) < 0$ in $[t_0 - \delta, t_0)$. There exists therefore a $\delta > 0$ such that $H_j(t, I(t, P_0)) < 0, j = 1, \cdots, m$, $t \in [t_0 - \delta, t_0)$, and, for at least one $j (\varepsilon J)$, $H_j(t, P_0)) > 0, t \in (t_0, t_0 + \delta]$, so that $P_0 \in S^*$.

PART II

Consider the linear differential system

$$\dot{y}_i = f_i(t) y_i + \sum_{j=1}^{n} g_{ij}(t) y_j , \qquad i = 1, \cdots, n$$

where the coefficients $f_i, g_{ij}, T \leqq t < \infty$, are continuous functions (in general complex-valued) of the real variable t.

By using Ważewski's method Z. Szmydtówna proved that if

$$R(f_k - f_{k+1}) > 0 , \quad \int_T^\infty R(f_k - f_{k+1}) dt = \infty , \qquad k = 1, \cdots, n-1 ,$$

and

$$\lim_{t \to \infty} \frac{g_{ij}}{R(f_k - f_{k+1})} = 0 , \qquad i, j = 1, \cdots, n , \quad k = 1, \cdots, n-1 ,$$

then there is a system of n linearly independent solutions $(y_{1k}, \cdots, y_{nk})$, $k = 1, \cdots, n$, with $\lim_{t \to \infty} y_{ik}/y_{kk} = 0$ for $i \neq k$ [5, Corollaire 1, Remarque 2, p. 30]. This theorem generalizes a theorem of Perron who obtains the same result requiring the existence of a constant $c > 0$ such that $R(f_k) > R(f_{k+1}) + c$, $k = 1, \cdots, n-1$, and $\lim_{t \to \infty} g_{ij} = 0$.

We notice that Szmydtówna allows the f_i, $i = 1, \cdots, n$, to be large and the g_{ij} to be small in some sense. In the following theorem we obtain the same result allowing also the f_i to be large and the g_{ij} to be small but in a sense completely different from Szmydtówna's.

THEOREM II-1. *Suppose that the system*

$$\text{(II)}\quad \dot{x}_i = f_i(t)x_i + \sum_{j=1}^{n} g_{ij}(t)x_j\ , \qquad i = 1, \cdots, n\ ,$$

satisfies the following hypotheses:

(1) *The coefficients f_i, g_{ij}, $T \leqq t < \infty$, are continuous functions (in general complex-valued) of the real variable t.*

(2) *There exists a real-valued continuous function $h(t)$, $T \leqq t < \infty$, such that for all $i \neq j$ we have*

$$| R(f_i - f_j) | \leqq h(t)\ ,$$
$$\int_T^\infty | g_{ij}(t) | e^{H(t)} dt < \infty$$

and

$$\int_T^\infty | R(g_{ii} - g_{jj}) | e^{H(t)} dt < \infty\ ,$$

where $H(t) = \int_T^t h(s)ds$

Then there is a system of n linearly independent solutions

$$(x_1(t), \cdots, x_n(t)) = \begin{pmatrix} x_{11}(t), \cdots, x_{1n}(t) \\ \cdots\cdots\cdots\cdots\cdots \\ x_{n1}(t), \cdots, x_{nn}(t) \end{pmatrix}$$

with $\lim_{t\to\infty} x_{ik}/x_{kk} = 0$ for all $i \neq k$.

Proof.

For every fixed integer p, $0 < p \leqq n$, we set

$$\omega_p = \{P: (t, x) \mid |x_i|^2 - |x_p|^2 \varphi^2(t) < 0, i \neq p, t > t_0 \geqq T\}$$

where $\varphi(t)$ and t_0 will be conveniently chosen so that, for every $t \geqq t_0$, $\varphi(t) > 0$, φ is differentiable, $\lim_{t\to\infty}\varphi(t) = 0$ and ω_p is a generalized regular polyfacial set.

Let

$$H_i(P) = |x_i|^2 - |x_p| \varphi^2(t)\ , \qquad i \neq p\ ,$$
$$H_p(P) = t_0 - t\ ,$$

it follows that $\omega_p = \{P \mid H_i(P) < 0, i = 1, \cdots, n\}$.

Set, for $q \neq p$,

$$\tilde{\Gamma}_p = \Gamma_p - \{Q: (t, x) \mid x = 0\}$$
$$= \{P \mid |x_q| = |x_p| \varphi(t), |x_i| \leqq |x_p| \varphi(t) \text{ for } i \neq p, t \geqq t_0, x_p \neq 0\}\ .$$

An easy computation shows that

$$\frac{1}{2}[D_{(\mathrm{II})}H_q(P)]P \in \tilde{\Gamma}_q \geqq |x_p|^2\varphi^2(t)[R(f_q - f_p + g_{qq} - g_{pp})]$$
$$- |x_p|^2\varphi(t)\dot{\varphi}(t) - |x_p|^2\varphi^2(t)\sum_{j\neq p}|g_{pj}|\frac{|x_j|}{|x_p|}$$
$$- |x_p|^2\sum_{j\neq q}|g_{qj}|\frac{|x_j|}{|x_q|}\cdot\frac{|x_q|}{|x_p|} .$$

Since $|x_q| = |x_p|\varphi(t) \geqq |x_j|$ for $j \neq p$ it follows that $|x_j|/|x_p| \leqq \varphi(t)$. As we want $\varphi(t) > 0$ and $\mathrm{lin}_{t\to\infty}\varphi(t) = 0$ we can take t_0 such that $\varphi(t) < 1$ for $t \geqq t_0$. Then

$$\frac{1}{2}[D_{(\mathrm{II})}H_q(P)]P \in \tilde{\Gamma} \geqq |x_p|^2\varphi^2(t)R(f_q - f_p + g_{qq} - g_{pp})$$
$$- |x_p|^2\varphi(t)\dot{\varphi}(t) - |x_p|^2\varphi^2(t)\sum_{j\neq p}|g_{pj}| - |x_p|^2\varphi(t)\sum_{j\neq q}|g_{qj}| .$$

since

$$\varphi(t)R(f_q - f_p + g_{qq} - g_{pp}) - \dot{\varphi}(t) - \varphi(t)\sum_{j\neq p}|g_{pj}| - \sum_{j\neq q}|g_{qj}| >$$
$$- \dot{\varphi}(t) - \varphi(t)h(t) - g(t) ,$$

where

$$g(t) = \{\sum_{i\neq j}|R(g_{ii} - g_{jj})| + |g_{ij}|\} + e^{-H(t)-t} ,$$

in order to have, for $q \neq p$, $[D_{(\mathrm{II})}H_q(P) \in \tilde{\Gamma}_q > 0$, it is sufficient to choose $\varphi(t)$ such that

(A) $\dot{\varphi}(t) + \varphi(t)h(t) + g(t) = 0.$

$\varphi(t) = e^{-H(t)}\int_t^\infty g(s)e^{H(s)}\,ds$ is indeed a solution of (A) satisfying the conditions $\varphi(t) > 0$, φ differentiable and $\lim_{t\to\infty}\varphi(t) = 0$.

If ω_p is defined in this way, taking into account that $[D_{(\mathrm{II})}H_p(P)]_{P\in\Gamma_p} = -1$ and that the set $\{P \in \Gamma_q \mid x_p = 0\} \subset M_q$, for $q \neq p$, it follows that ω_p is a generalized regular polyfacial set.

For $i \neq p$ we have

$$L_i = \tilde{\Gamma}_i \text{ and } L_p = \phi ,$$
$$M_i = M = \{P\colon (t, x) \mid t \geqq t_0, x = 0\} \text{ and } M_p = \Gamma_p .$$

By Lemma 1

$$S = S = \bigcup_{i\neq p}\tilde{\Gamma}_i - \Gamma_p - M .$$

We choose

$$Z_p = \{P\colon (t, x) \mid t = \tau > t_0, x_p = x_p^0 \neq 0, |x_i| \leqq |x_p^0|\varphi(\tau), i \neq p\} = \prod_{j\neq p} B_j^2 ,$$

where B_j^2 is a solid sphere in R^2. We have

$$Z_p \cap S = Z_p \cap [\bigcup_{i \neq p} \tilde{\Gamma}_i - \Gamma_p - M] = \bigcup_{i \neq p} z_p \cap [\tilde{\Gamma}_i - \Gamma_p - M] .$$

For $i \neq p$

$$Z_p \cap [\tilde{\Gamma}_i - \Gamma_p - M] = \{P: (t, x) \mid t = \tau, x_p = x_p^0, \mid x_i \mid = \mid x_p^0 \mid \varphi(\tau),$$
$$\mid x_j \mid \leqq \mid x_p^0 \mid \varphi(\tau), j \neq p\} = B_1^2 \times \cdots \times B_{i-1}^2 \times S_i^1 \times B_{i+1}^2 \times \cdots \times B_n^2$$

(in the cartesian product above B_p^2 is exclued) where S_i^1 is the boundary of B_i^1 in R^2.

Modulo homeomorphisms we have therefore $Z_p = B^{2n-2}$ (solid sphere in R^{2n-2}) and $Z_p \cap S = S^{2n-3} =$ Boundary of B^{2n-2} in R^{2n-2}, so that $Z_p \cap S$ is not a retract of Z_p.

There is however a retraction $\phi: S \rightarrow Z_p \cap S$ given by $\phi(P) = P^*$, with $t^* = \tau, x_p^* = x_p^0, x_i^* = \varphi(\tau)/\varphi(t) \mid x_p^0 \mid / \mid x_p \mid \cdot x_i, i \neq p$. The verification is trivial.

By using the theorem of Ważewski we can conclude the existence of at least one point P_0: $(\tau, x_0) \in Z_p - S$ with $I(t, P_0) \subset \omega_p$ for every $t \geqq \tau$. This means that the solution $x_p(t) = (x_{1p}(t), \cdots, x_{np}(t))$ of (II) passing through P_0 satisfies

$$\frac{\mid x_{ip}(t) \mid}{\mid x_{pp}(t) \mid} < \varphi(t) \text{ for } t \geqq \tau \text{ and } i \neq p .$$

Letting $p = 1, \cdots, n$ we find n solutions $(x_1(t), \cdots, x_n(t))$ with the required property. Let us show that these solutions can be taken linearly independent.

By choosing Z_p with sufficiently large τ and $x_{pp}^0 = 1$ the absolute values of the coordinates x_{ip}, $i \neq p$, of the points of Z_p can be made arbitrarily small. We then have

$$\begin{pmatrix} x_1(\tau) \\ x_2(\tau) \\ \vdots \\ x_n(\tau) \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} \cdots \varepsilon_{1n} \\ \varepsilon_{21} \cdots \varepsilon_{2n} \\ \cdots\cdots\cdots \\ \varepsilon_{n1} \cdots \varepsilon_{nn} \end{pmatrix}$$

where $\varepsilon_{ii} = 1$ and the $\mid \varepsilon_{ij} \mid$ are smaller than any given positive number for all $i \neq j$. This completes the proof

In the following theorem we will look for linearly independent solutions of (II) with similar properties to those of Theorem II-I but not necessarily requiring that they form a fundamental set of solutions of (II).

THEOREM II-2. *Suppose that the system (II) satisfies the following hypotheses*:

(1) *The coefficients* f_i, g_{ij}, $T \leqq t < \infty$, *are continuous functions (in general complex-valued) for the real variable* t.

(2) *There exists a natural number* $r \leqq n$ *such that* $R(f_r) = \cdots = R(f_n)$, $R(f_i) \geqq R(f_r)$ *for all* $i < r$, $\int_T^\infty | g_{ij}(t) | dt < \infty$ *for all* $i \neq j$ *and* $\int_T^\infty | R(f_{ii} - g_{jj}) | dt < \infty$.

Then there exists $s + 1$ $(r + s = n)$ *linearly independent solutions*

$$(x_r(t), \cdots, x_n(t)) = \begin{pmatrix} x_{1r}(t) \cdots x_{1n}(t) \\ \cdots\cdots\cdots\cdots \\ x_{nr}(t) \cdots x_{nn}(t) \end{pmatrix}$$

such that $\lim_{t \to \infty} x_{ik}/x_{kk} = 0$ for all $i \neq k$, $k = r, \cdots, n$.

Proof. Given an integer p, $r \leqq p \leqq n$, we prooceed exactly as in Theorem II-1 up to the point where we got the expression:

$$\varphi(t)R(f_q - f_p + g_{qq} - g_{pp}) - \dot{\varphi}(t) - \varphi(t) \sum_{j \neq p} | g_{pj} | - \sum_{j \neq q} | g_{qj} |$$

which we denote by B_q.

As we have $R(f_q - f_p) \geqq 0$ for all q, $0 < q \leqq n$, it follows $(\varphi(t) < 1)$

$$B_q \geqq - \dot{\varphi}(t) - g(t) \text{ where } \int_{t0}^\infty g(t) dt < \infty .$$

Making $\varphi(t) = \int_t^\infty [g(s) + e^{-s}] ds$ it follows that $\varphi(t) > 0$, φ is differentiable, $\lim_{t \to \infty} \varphi(t) = 0$ and $B_q > 0$.

Proceeding as in Theorem II-1 we find a set of $(s + 1)$ solutions $(x_r(t), \cdots, x_n(t))$. Still by a similar reasoning we may show that these solutions can be so chosen that for a sufficiently large τ we have

$$(x_r(\tau), \cdots, x(\tau)) = \begin{pmatrix} \varepsilon_{1r}, \cdots, \varepsilon_{1n} \\ \cdots\cdots\cdots \\ \varepsilon_{rr}, \cdots, \varepsilon_{rn} \end{pmatrix}$$

with $\varepsilon_{ii} = 1$ and the $| \varepsilon_{ij} |$, $i \neq j$, smaller than any given positive number, so that, they are linearly independent.

If $n = 2$ Theorem II-2, with some supplementary hypotheses, leads us to a deeper result. As already mentioned in the Introduction the following theorem is due to Professor J. L. Massera with whose permission it is reproduced here.

THEOREM II-3. *Suppose that the system*

$$\dot{x} = f_1(t)x + g_{11}(t)x + g_{12}(t)y$$

$$\dot{y} = f_2(t)y + g_{21}(t)x + g_{22}(t)y$$

satisfies the following hypotheses:

(1) *The coefficients* $f_i, g_{ij}, T \leqq t < \infty$, *are continuous real-valued functions of the real variable* t.

(2) $f_1(t) \geqq f_2(t)$, $\int_T^\infty (f_1(t) - f_2(t))dt = \infty$, $\int_T^\infty | g_{ij}(t) | dt < \infty$ *for* $i \neq j$ *and* $\int_T^\infty | g_{11}(t) - g_{22}(t) | dt < \infty$.

Then there exists a solution $(x_1(t), y_1(t))$ *satisfying* $\lim_{t\to\infty} x_1(t)/y_1(t) = 0$ *and, for any other solution* $(x(t), y(t))$ *which is not proportional to* $(x_1(t), y_1(t))$, *we have* $\lim_{t\to\infty} y(t)/x(t) = 0$.

Proof. The existence of a solution $(x_1(t), y_1(t))$ with the required property follows from Theorem II-2.

Without loss of generality we may assume $g_{11} = g_{22} = 0$. Choose $t_0 \geqq T$ so large that $\int_{t0}^\infty (| g_{12} | + | g_{21} |)dt < \pi/4$. Let $(x_2(t), y_2(t))$ be the solution which satisfies $x_2(t_0) = 1$, $y_2(t_0) = 0$. Setting $\theta(t) = \arg (x_2(t), y_2(t))$, we claim, in the first place, that $| \theta(t) | < \pi/4$ for $t \geqq t_0$. Assume that this were not the case. It then follows that there exists an interval (t_1, t_2), $t_1 \geqq t_0$, such that $\theta(t_1) = 0$, $| \theta(t_2) | = \pi/4$, $0 < | \theta(t) | < \pi/4$ for $t_1 < t < t_2$, say, $\theta(t_2) = \pi/4$, $0 < \theta(t) < \pi/4$ for $t_1 < t < t_2$, whence $x_2(t) \cdot y_2(t) > 0$ in (t_1, t_2). Since

$$\dot{\theta} = \frac{\dot{y}_2 x_2 - \dot{x}_2 y_2}{x_2^2 + y_2^2} = \frac{g_{21}x_2^2 - g_{12}y_2^2 + (f_2 - f_1)x_2 y_2}{x_2^2 + y_2^2},$$

we arrive to the contradiction $\pi/4 = \theta(t_2) - \theta(t_1) = \int_{t_1}^{t_2} \dot{\theta} dt < \pi/4$.

We next prove that $\lim_{t\to\infty} y_2(t)/x_2(t) = 0$, or equivalently $\lim_{t\to\infty}\theta(t) = 0$. There exsists a sequence $t_n \to \infty$ with $\theta(t_n) \to 0$, otherwise $\theta(t) > \theta_0 > 0$, say, which leads to the contradiction

$$\theta(t) - \theta(t_0) \leqq - \int_{t_0}^t (f_1(t) - f_2(t)) \sin \theta(t) \cos \theta(t) dt$$
$$+ \int_{t_0}^t (| g_{12} | + | g_{21} |)dt \to - \infty .$$

Now, given $\varepsilon > 0$, choose t_n such that $| \theta(t_n) | < \varepsilon/2$, $\int_{tn}^\infty (| g_{12} | + | g_{21} |)dt < \varepsilon/2$. An argument similar to the one used to prove $| \theta(t) | < \pi/4$ then shows that $| \theta(t) | < \varepsilon$ for $t \geqq t_n$.

Assume t_0 large enough so that $| x_1(t) |/| y_1(t) | < 1$, $| y_2(t) |/| x_2(t) | < 1$ for $t \geqq t_0$ and, say, $y_1(t) > 0$, $x_2(t) > 0$; then

$$\dot{y}_1(t) \leqq (f_2(t) + | g_{21}(t) |). y_1(t) ,$$

$$\dot{x}_2(t) \geqq (f_1(t) - | g_{12}(t) |) . \; x_2(t) \; ,$$

whence

$$y_1(t) \leqq y_1(t_0) . \exp\left(\int_{t_0}^{t} (f_2(t) + | g_{21}(t) |) dt\right) \; ,$$

$$x_2(t) \geqq x_2(t_0) . \exp\left(\int_{t_0}^{t} (f_1(t) - | g_{12}(t) |) dt\right)$$

and

$$\frac{y_1(t)}{x_2(t)} \leqq \frac{y_1(t_0)}{x_2(t_0)} . \exp\left(\int_{t_0}^{t} (f_2(t) - f_1(t) + | g_{12}(t) | + | g_{21}(t) |) dt\right) \to 0 \; .$$

Finally, any solution $(x(t), y(t))$ which is not proportional to $(x_1(t), y_1(t))$ satisfies, for a certain constant value k,

$$\frac{y(t)}{x(t)} = \frac{y_2(t) + ky_1(t)}{x_2(t) + ky_2(t)} = \frac{(y_2(t)/x_2(t)) + k(y_1(t)/x_2(t))}{1 + k(x_1(t)/y_1(t))(y_1(t)/x_2(t))} \to 0 \; .$$

PART III

Consider the linear differential systems

(III) $$\dot{x} = A(t)x + B(t)x$$

(III′) $$\dot{y} = A(t)y$$

where $A(t)$, $B(t)$, $T \leqq t < \infty$, are continuous complex matrix functions.

Conti [2, Theorem I, p. 589] proved that: *if* $\int^{\infty} | B(t) | \, dt < \infty$ *where* $B(t) = (b_j^i(t))$ *and* $| B(t) | = \sum_{i,j} | b_j^i(t) |$ *and if (III′) is uniformly stable, then the system (III) and (III′) are asymptotically equivalent*[3].

The theorem of Wintner [7, 7-i, p. 423] stating that:

If $B(t) = (b_j^i(t))$, $T \leqq t < \infty$, $i, j = 1, \cdots, n$, *is a matrix of* n^2 *continuous functions satisfying* $\int^{\infty} | B(t) | \, dt < \infty$, *then every solution of* $\dot{x} = B(t)x$ *tends to a finite limit as* $t \to \infty$, is a particular case of Conti's result $(A(t) = 0)$.

Our Theorem III-3, is also a generalization of Wintner's theorem but different from that of Conti.

Theorems III-1 and III-2, which are preliminary to Theorem III-3, give us some information, though less than asymptotic equivalence, concerning the behavior of two systems, one of which not necessarily linear.

THEOREM III-1. *Suppose that the systems*

[3] The theorem of Conti is actually more general. We have considered the theorem applied to linear systems only.

$$\text{(III-1)} \qquad \dot{x}_i = f_i(t)x_i + g_i(t, x) ,$$
$$\text{(III-2)} \qquad \dot{y}_i = f_i(t)y_i ,$$
$$i = 1, \cdots, n , \qquad g(t, x) = (g_i(t, x))$$

satisfy the following hypotheses:

(1) *$f_i(t)$, $T \leqq t < \infty$, are continuous functions (in general complex-valued) of the real variable t; $g_i(t, x)$ are functions (in general complex-valued) continuous in*

$$\Omega = \{(t, x) \mid t > T, \mid x \mid = \sum_{i=1}^{n} \mid x_i \mid < \infty\}$$

and satisfy some condition which implies the existence of only one integral passing through each point of Ω.

(2) $\mid g(t, x) \mid \leqq \mid x \mid F(t)$ *on* Ω.

(3) *There exists a negative constant K such that*

$$K \leqq \int_t^v R(f_i(\tau))d\tau$$

for all $v \geqq t > T$ and

$$\int^{\infty} F(t) \exp\left[\int_T^t R(f_i(s))ds\right]dt < \infty$$

for all $i = 1, \cdots, n$.

Then for every solution $y(t)$ of III-2 there is a solution $x(t)$ of III-1 such that $\lim_{t\to\infty}[x(t) - y(t)] = 0$.

Proof. We define $\omega = \{P \in \Omega \mid \mid x_i - y_i(t) \mid < \varphi_i(t), t > t_0 \geqq T\}$ where the $\varphi_i(t)$ and t_0 will be adequately chosen so that for all $t \geqq t_0$, $i = 1, \cdots, n$, we have: $\varphi_i(t) > 0$, φ_i differentiable, $\lim_{t\to\infty}\varphi_i(t) = 0$ and ω a generalized regular polyfacial set.

If we put

$$H_i(P) = \mid x_i - y_i(t) \mid^2 - \varphi_i^2(t) , \qquad i = 1, \cdots, n$$
$$H_{n+1}(P) = t_0 - t$$

it follows that $\omega = \{P \mid H_i(P) < 0 , i = 1, \cdots, n + 1\}$.

For all $i, 1 \leqq i \leqq n$,

$$\Gamma_i = \{P \in \Omega \mid\mid x_i - y_i(t) \mid = \varphi_i(t), \mid x_j - y_j(t) \mid \leqq \varphi_j(t), j = 1, \cdots, n, t \geqq t_0\} .$$

An easy computation shows that

$$\frac{1}{2}[D_{(\mathrm{III}-1)}H_i(P)]P\in\Gamma_i \geqq \varphi_i^2(t)R[f_i(t)] - \varphi_i(t)\dot{\varphi}_i(t)$$
$$- \varphi_i(t)F(t)\left[\sum_{k=1}^{n} | x_i - y_k | + | y_k |\right] .$$

As we want $\varphi_i(t) > 0$ and $\lim_{t\to\infty}\varphi_i(t) = 0$, we can take t_0 such that $\sum_{i=1}^{n}\varphi_i(t) < 1$ for all $t \geqq t_0$. Then,

$$\frac{1}{2}\left[D_{(\mathrm{III}-1)}H_i(P)\right]P\in\Gamma_i \geqq \varphi_i^2(t)R[f_i(t)] - \varphi_i(t)\dot{\varphi}_i(t) - \varphi_i(t)F(t)$$
$$- \varphi_i(t)F(t)\sum_{k=1}^{n} | c_k | \exp\int_{t_0}^{t} R[f_k(s)]ds \geqq \varphi_i^2(t)R[f_i(t)] - \varphi_i(t)\dot{\varphi}_i(t)$$
$$- \varphi_i(t)F(t)\sum_{k=1}^{n} | d_k | \exp\int_{t_0}^{t} R[f_k(s)]ds \geqq \varphi_i^2(t)R[f_i(t)] - \varphi_i(t)\dot{\varphi}_i(t)$$
$$- \varphi_i(t)h(t)$$

where we can assume $\int^{\infty} h(t)dt < \infty$ and, without loss of generality, $h(t) > 0$ for all $t \geqq t_0$.

In order to have, for all $i = 1, \cdots, n$, $[D_{(\mathrm{III}-1)}H_i(P)]P\in\Gamma_i > 0$ it is sufficient to choose $\varphi(t)$ such that

$$- \dot{\varphi}_i(t) + R[f_i(t)]\varphi_i(t) - h(t) > 0 .$$

The problem is then to look for a solution $z(t)$ of $\dot{z} < \sigma(t)z - \gamma(t)$ satisfying $z(t) > 0$ for all $t \geqq t_0$, $\lim_{t\to\infty} z(t) = 0$, knowing that $\gamma(t) > 0$ for all $t \geqq t_0$, $\int^{\infty}\gamma(t)dt < \infty$ and $\int_t^v \sigma(s)ds \geqq K$ for some constant K and all $v \geqq t \geqq t_0$. If $W(t)$ is a solution of $\dot{W} = \sigma(t)W - \gamma(t)$ it follows that $z(t) = 2W(t)$ is a solution of $\dot{z} < \sigma(t)z - \gamma(t)$. It is then sufficient to find a solution $W(t)$ satisfying $W(t) > 0$ for all $t \geqq t_0$ and $\lim_{t\to\infty} W(t) = 0$. The solution $W(t) = \exp\left(\int_{t_0}^{t}\sigma(s)ds\right)$. $\int_t^{\infty}\gamma(v)\exp\left(-\int_{t_0}^{v}\sigma(s)ds\right)dv$ exists and indeed $W(t) \to 0$ as $t \to \infty$ because

$$W(t) = \int_t^{\infty}\gamma(v)\exp\left(-\int_t^{v}\sigma(s)ds\right)dv \leqq e^{-K}\int_t^{\infty}\gamma(v)dv .$$

Since $[D_{(\mathrm{III}-1)}H_{n+1}(P)] = -1$ it follows that ω is a generalized regular polyfacial set and $S = S^* = \bigcup_{i=1}^{n}\Gamma_i - \Gamma_{n+1}$.

If we choose

$$Z = \{(t, x) \mid t = \tau > t_0, | x_j - y_j(\tau) | \leqq \varphi_j(\tau), j = 1, \cdots, n\}$$

it follows that $S \cap Z = \bigcup_{i=1}^{n}\Gamma_i \cap Z - \Gamma_{n+1}$

$$\Gamma_i \cap Z = \{t, x) \mid t = \tau, | x_i - y_i(\tau) |$$
$$= \varphi_i(\tau), | x_j - y_j(\tau) | \leqq \varphi_j(\tau), j = 1, \cdots, n\} .$$

Then $Z = \prod_{j=1}^{n}B_j^2$

$$Z \cap S = \bigcup_{j=1}^{n} B_1^2 \times \cdots \times B_{j-1}^2 \times S_j^1 \times B_{j+1}^2 \times \cdots \times B_n^2$$

and, modulo homomorphisms, $Z = B^{2n}$, $Z \cap S = S^{2n-1}$ so that $Z \cap S$ is not a retract of Z. However, it is easily seen that $\phi : S \to S \cap Z$ given by $\phi(P) = P^*$, with $t^* = \tau$, $x_i^* = y_i(\tau) + [x_i - y_i(t)]\varphi_i(\tau)/\varphi_i(t)$, is a retraction.

Using the theorem of Ważewski we can conclude the existence of at least one point $P_0 : (\tau, x_0) \in Z - S$ such that $(t, x(t, P_0)) = I(t, P_0) \subset \omega$ for all $t \geqq t_0$.

Since $x(t, P_0)$ is defined in the future, i.e., $\beta(P_0) = \infty$ (because $\beta(P_0) < \infty$ implies $\{I(t, P_0) \mid t_0 \leqq t < \beta(P_0)\}$ bounded, which is not possible), it follows that $\lim_{t\to\infty}[x(t, P_0) - y(t)] = 0$.

COROLLARY 1. *Suppose that the systems*

(III–1′) $$\dot{x}_i = f_i(t)x_i + \sum_{j=1}^{n} g_{ij}(t)x_j$$

(III–2′) $$\dot{y}_i = f_i(t)y_i$$

$$i = 1, \cdots, n\,, \quad g(t) = (g_{ij}(t))$$

satisfy the following hypotheses:

(1) *The coefficients f_i, g_{ij}, $T \leqq t < \infty$, are continuous functions (in general complex-valued) of the real variable t.*

(2) *There exists a constant K such that*

$$K \leqq \int_t^v R[f_i(s)]ds \text{ for all } v \geqq t \geqq T \text{ and}$$

$$\int^\infty | g(t) | \exp\left\{\int_T^t R[f_i(s)]ds\right\}dt < \infty\,, \qquad i = 1, \cdots, n\,.$$

Then for every solution $y(t)$ of (III–2′) there exists a solution $x(t)$ of (III–1′) such that $\lim_{t\to\infty}[x(t) - y(t)] = 0$.

The theorem of Wintner mentioned before follows a once from Corollary 1.

THEOREM III–2. *Suppose that the systems*

(III-A) $$\dot{x}_i = \sum_{j=1}^{n} f_{ij}(t)x_j + g_i(t, x)$$

(III-B) $$\dot{y}_i = \sum_{j=1}^{n} f_{ij}(t)y_j$$

$$i, j = 1, \cdots, n\,, \quad g(t, x) = (g_j(t, x))$$

satisfy the following hypotheses:

(1) *$f_{ij}(t)$, $T \leqq t < \infty$, are continuous functions (in general complex-valued) of real variable t; $g_i(t, x)$ are functions (in general complex-valued) continuous in*

$$\Omega = \{(t, x) \mid t < T, \mid x \mid < \infty\}$$

and satisfy some condition which implies the existence of only one integral passing through each point of Ω.

(2) $\mid g(t, x) \mid \leqq \mid x \mid F(t)$ *in Ω.*

(3) *There exists a constant K such that*

$$K \leqq \int_t^v R[f_{ii}(s)]ds \text{ for all } v \geqq t \geqq T \text{ and}$$

$$\int^\infty F(t) \exp\left\{\int_T^t R[f_{ii}(s)]ds\right\}dt < \infty\ , \qquad i = 1, \cdots, n$$

$$\int^\infty \mid f_{ij}(t) \mid \exp\left\{\int_T^t R[f_{kk}(s)]ds\right\}dt < \infty\ , \qquad k = 1, \cdots, n, i \neq j\ .$$

Then for every solution $y(t)$ of (III-B) there is a solution $x(t)$ of (III-A) such that $\lim_{t\to\infty}[x(t) - y(t)] = 0$

Proof. Consider the systems

(III-A) $\quad \dot{x}_i = f_{ii}(t)x_i + \tilde{g}_i(t, x)$ where $\tilde{g}_i(t, x) = g_i(t, x) + \sum_{j\neq i} f_{ij}(t)x_j$

(III-C) $\quad \dot{z}_i = f_{ii}(t)z_i\ .$

These systems satisfy the condition of Theorem III-1. Hence for every solution $z(t)$ of (III-C) there is a solution $x(t)$ of

(III-A) $\qquad$ such that $\lim_{t\to\infty}[z(t) - x(t)] = 0$

Consider now the systems

(III-B) $$\dot{y}_i = \sum_{j=1}^n f_{ij}(t)y_j$$

(III-C) $$\dot{z}_i = f_{ii}(t)z_i\ .$$

It is easy to see that they also satisfy the hypotheses of Theorem III-1. Hence for every solution $z(t)$ of (III-C) there is a solution $y(t)$ of (III-B) such that $\lim_{t\to\infty}[y(t) - z(t)] = 0$. But we can also prove that for every solution $y(t)$ of (III-B) there is a solution $z(t)$ of (III-C) such that $y(t) - z(t) \to 0$ as $t \to \infty$. For that purpose it is enough to show that there is a fundamental set $z^1(t), \cdots, z^n(t)$ of solutions of (III-C) such that the solutions $y^1(t), \cdots, y^n(t)$ satisfying $y^i(t) - z^i(t) \to 0$ as $t \to \infty$, for all $i = 1, \cdots, n$, are a fundamental set of solutions of (III B).

Let us take $z^i(t) = \begin{pmatrix} z_1^i(t) \\ \vdots \\ z_n^i(t) \end{pmatrix}$ such that $z_j^i(t) = 0$ for all $j \neq i$ and $z_j^i(t)$ $= \exp\left(\int_T^t f_{ii}(s)ds\right)$ for all $i = 1, \cdots, n$.

The corresponding $y^i(t)$, $i = 1, \cdots, n$, satisfy $\lim_{t\to\infty} y_j^i(t) = 0$ if $j \neq i$ and $\lim_{t\to\infty} | y_1^i(t) - \exp\left[\int_T^t f_{ii}(s)ds\right| = 0$. Hence, there exists t_0 such that $t \geqq t_0$ implies

$$| y_i^i(t) - \exp\left[\int_T^t f_{ii}(s)ds\right] | < \frac{1}{2}e^K .$$

Whence

$$| y_i^i(t) | > \exp\left\{\int_T^t R[f_{ii}(s)]ds\right\} - \frac{1}{2}e^K \geqq \frac{1}{2}e^K .$$

Therefore, for any $\varepsilon > 0$ there is a $t(\varepsilon)$ such that $t \geqq t(\varepsilon)$ implies $| y_i^i(t) | > 1/2\, e^K$, $i = 1, \cdots, n$, and $| y_j^i(t) | < \varepsilon$ for all $i \neq j$. This implies the existence of a $\tau \geqq T$ with $\det(y^1(\tau), \cdots, y^n(\tau)) \neq 0$ and $(y^1(t), \cdots, y^n(t))$ is a fundamental set of solutions of (III-B).

From the results concerning the systems (III-A), (III-C) and (III-B), (III-C) we conclude that for every solution $y(t)$ of (III-B) there is a solution $x(t)$ of (III-A) such that $\lim_{t\to\infty}[x(t) - y(t)] = 0$.

THEOREM III-3. *Suppose that the systems*

$$\text{(III-}\alpha\text{)} \qquad \dot{x}_i = \sum_{j=1}^n f_{ij}(t)x_j(t)x_j + \sum_{j=1}^n g_{ij}(t)x_j$$

$$\text{(III-}\beta\text{)} \qquad \dot{y}_i = \sum_{j=1}^n f_{ij}(t)y_j \qquad i, j = 1, \cdots, n$$

satisfy the following hypotheses:

(1) *The coefficients f_{ij}, g_{ij}, $T \leqq t < \infty$, are continuous functions (in general complex-valued) of the real variable t.*

(2) *There exists constant K such that $K \leqq \int_t^v R[f_{ii}(s)]ds$ for all $v \geqq t \geqq T$, $i = 1, \cdots, n$, and*

$$\int^\infty | g_{ij}(t) | \exp\left\{\int_T^t R[f_{kk}(s)]ds\right\}dt < \infty , \qquad i, j, k = 1, \cdots, n$$

$$\int^\infty | f_{ij}(t) | \exp\left\{\int_T^t R[f_{kk}(s)]ds\right\}dt < \infty , \qquad i, j, k = 1, \cdots, n, i \neq j .$$

Then the systems (III-α) and (III-β) are asymptotically equivalent.

Proof. By Theorem III-2 for every solution $y(t)$ of (III-β) there is a solution $x(t)$ of (III-α) such that $\lim_{t\to\infty}[x(t) - y(t)] = 0$.

Let us show that given a fundamental set $(y^1(t), \cdots, y^n(t))$ of solutions of (III-β) the corresponding solutions $(x^1(t), \cdots, x^n(t))$ of (III-α) satisfying $\lim_{t\to\infty}[x^i(t) - y^i(t)] = 0, i = 1, \cdots, n$, also form a fundamental set of solutions.

Consider the auxiliary system

$$\text{(III-}\gamma\text{)} \qquad \dot{z}_i = f_{ii}(t)z_i \, , \qquad i = 1, \cdots, n \, .$$

Applying the argument used in Theorem III-2 to the systems (III-β), (III-γ) we conclude that there exists a fundamental set $(y^1(t), \cdots, y^n(t))$ of solutions of (III-β) and a t_0 such that $t \geqq t_0$ implies

$$| y^i_i(t) | \geqq \frac{1}{2}e^K \text{ and } y^i_j(t) \to 0 \text{ as } t \to \infty \text{ for all } i \neq j \, .$$

Let $(x^1(t), \cdots, x^n(t))$ be the solutions of (III-α) such that $\lim_{t\to\infty} [x^i(t) - y^i(t)] = 0$ (the existence of which follows from Theorem III-2). Then $\lim_{t\to\infty} x^i_j(t) = 0$ for all $i \neq j$ and there exists $\tau \geqq t_0$ such that $t \geqq \tau$ implies $| x^i_i(t) | > 1/4e^K$.

For sufficiently large t we have therefore

$$\det (x^1(t), \cdots, x^n(t)) \neq 0$$

and this means that $(x^1(t), \cdots, x^n(t))$ is a fundamental set of solutions of (III-α).

The systems (III-α) and (III-β) being linear this implies that they are asymptotically equivalent.

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