PRIMAL CLUSTERS OF TWO-ELEMENT ALGEBRAS

EDWARD S. O'KEEFE

1. Introduction. The development of a structure theory for universal algebras which subsumes the familiar structure theory of Boolean and Post algebras and p-rings (Foster, [1]-[4]) has focused attention on certain classes of functionally complete universal algebras, called primal clusters. A primal cluster is a set of primal algebras in which every finite subset is strictly independent (see definitions § 2, below). Each such cluster determines a unique subdirect factorization for each algebra satisfying all the identities common to some finite subset of the cluster. In other words, every function over a direct product of strictly independent primal algebras, expressible in terms of the algebras' operations, has a decomposition and reconstruction analogous both to the Boolean theory and the Fourier transform theory. In order to broaden the domain of application of the generalized theory, we must find strictly independent sets of primal algebras.

The purpose of this paper is to present the theory of independence of primal algebras in a new dimension. Simple necessary and sufficient conditions for strict independence of primal algebras of one primitive operation, regardless of the number of elements, have been obtained [5]. We now give necessary and sufficient conditions for strict independence of certain two-element primal algebras of the same species, regardless of the number of primitive operations.

2. Basic notions, the ϕ-condititition. The following definitions are stated for easy reference.

Let $\mathfrak{A} = (A, o_1, o_2, \cdots)$ be a universal algebra.

2.1. The *species* $Sp = \{n_1, n_2, \cdots\}$ of $\mathfrak{A}$ is the sequence of ranks of the primitive operations o_i of $\mathfrak{A}$, where n_i is the rank of o_i.

2.2. An *expression* $\phi(\xi_1, \cdots, \xi_n)$ of species Sp is a finite set of one or more indeterminate symbols ξ_i, composed by operation-symbols of Sp.

2.3. *A strict $\mathfrak{A}$-function* is an expression interpreted in algebra $\mathfrak{A}$. The notation $\phi = \chi(\mathfrak{A})$ means that the strict function represented by ϕ *in algebra $\mathfrak{A}$ is the same as that for χ.*

2.4. $\mathfrak{A}$ is a *primal* algebra if every transformation of $A \times A \times \cdots \times A$ into A can be represented by a strict $\mathfrak{A}$-function, and Sp is denumerable.

Received April 25, 1960

2.5. A finite set of algebras $\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_p\}$, all of the same species Sp, is *strictly independent* if each given set of strict functions ϕ_i has a single expression ψ which reduces to the given function ϕ_i in the algebra $\mathfrak{A}_i$; i.e., $\psi = \phi_i(\mathfrak{A}_i)$.

2.6. $\widetilde{\mathfrak{A}}$ is a primal cluster if $\widetilde{\mathfrak{A}}$ is a set of primal algebras and every finite subset of $\widetilde{\mathfrak{A}}$ is strictly independent. The totality of pairwise non-isomorphic primal algebras of species $[s]$ constitutes a primal cluster **[5]**. Various other categories of primal clusters are known, largely of species **[2, 1]**.

The ϕ-condition is analogous to the factorization of functions of real numbers. It is simply that any strict function may be represented by any expression operating on some set of strict functions.

2.7. *The ϕ-condition.* For every strict $\mathfrak{A}$-function, $\theta(\xi, \eta, \cdots, \zeta)$, and every strict $\mathfrak{A}$-function, $\kappa(\xi_1, \cdots, \xi_m)$, provided that no variable ξ_i occurs twice in κ, there exist strict $\mathfrak{A}$-functions, $\psi_1(\xi, \eta, \cdots, \zeta)$, $\cdots$, $\psi_m(\xi, \eta, \cdots, \zeta)$, such that

$$\kappa(\psi_1(\xi, \eta, \cdots, \zeta), \cdots, \psi_m(\xi, \eta, \cdots, \zeta)) = \theta(\xi, \eta, \cdots, \zeta) . \tag{2.1}$$

Formerly primal algebras were defined to be finite. However, this property is now derived from the denumerability of Sp.

THEOREM 2.8. *Every primal algebra is finite.*

Proof. Let $\mathfrak{A} = \langle A, o_1, \cdots, o_n, \cdots \rangle$ be a primal algebra. The two-valued functions on any infinite set have a larger cardinal number than the set of expressions made of a denumerable set of operations. Therefore, the fact that the functions on $A \times A$ to A are represented by expressions in the operations of A means that A is not infinite.

From [5], we require the following basic results.

THEOREM 2.9. *In any primal algebra in which the primitive operations are onto transformations, the ϕ-condition holds.*

THEOREM 2.10. *Let $\mathfrak{A} = (A, o, \cdots)$ and $\mathfrak{A} = (B, o, \cdots)$ be two non-isomorphic primal algebras of the same species, Sp. Then there exists a set of unary expressions $\{\phi_i\} = \{\phi_1, \cdots, \phi_p\}$ of species Sp such that*

$$\phi_1 = \phi_2 = \cdots = \phi_p(\mathfrak{A}) \tag{2.2}$$

and such that every unary $\mathfrak{B}$-function is equivalent modulo $\mathfrak{B}$ to one of the $\phi_1, \cdots \phi_p$.

THEOREM 2.11. *Let $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_n\}$ be a set of universal algebras of species Sp, in which every pair of algebras is strictly independent. If the ϕ-condition holds in each algebra, the set is strictly independent.*

3. The two-element independence theorem.

Our main result is

THEOREM 3.1. *Every set of primal algebras is a primal cluster if:*

(i) *every algebra in the set has exactly two elements,*

(ii) *no two algebras are isomorphic,*

(iii) *no primitive operation is constant,*

(iv) *all algebras in the set are in the same species, $Sp = [n_1, \cdots, n_m]$.*

The proof of Theorem 3.1 is preceded by three lemmas.

LEMMA 3.2. *Let $\mathfrak{B} = (\{\beta_1, \beta_2\}, o, \cdots)$ be a two-element primal algebra with no constant primitive operations. Every expression $\phi(\xi_1, \cdots, \xi_n)$, in which no variable occurs twice may be changed, modulo algebra $\mathfrak{B}$, to any given function $\chi(\xi)$ by replacing some variable by a properly chosen strict $\mathfrak{B}$-function $\psi(\xi)$, and all others by constant strict $\mathfrak{B}$-functions.*

Proof. If the expression $\phi(\xi_1, \cdots, \xi_n)$ has but one operation-symbol o_i, then, since no operation-symbol represents a constant, there are constants δ_i and γ_i such that

$$o_i(\delta_1, \cdots, \delta_{n_i}) = \chi(\beta_1)$$
$$o_i(\gamma_1, \cdots, \gamma_{n_i}) = \chi(\beta_2) \,. \tag{3.1}$$

We alter δ_1 to γ_1, δ_2 to γ_2, etc. until the function changes value. Some jth argument must give the change from $\chi(\beta_1)$ to $\chi(\beta_2)$. We choose the expression $\psi(\xi)$ so that

$$\psi(\beta_1) = \delta_j$$
$$\psi(\beta_2) = \gamma_j \tag{3.2}$$
$$o_i(\gamma_1, \cdots, \gamma_{j-1}, \psi(\beta_k), \delta_{j+1}, \cdots, \delta_{ni}) = \chi(\beta_k) \,, \qquad (k = 1, 2) \,.$$

Since there are but two elements in the algebra $\mathfrak{B}$, χ is now completely represented

$$o_i(\gamma_1, \cdots, \gamma_{j-1}, \psi(\xi), \delta_{j+1}, \cdots, \delta_{n_i}) = \chi(\xi) \,. \tag{3.3}$$

On the other hand, let $\phi(\xi_1, \cdots, \xi_n)$ be composed of m operation-symbols. Assume that the theorem holds for all expressions with fewer than m operation-symbols. ϕ is a set of expressions $\phi_1, \cdots, \phi_{n_i}$ composed by primitive operations o_j: $\phi = o_j(\phi_1, \cdots, \phi_{n_i})$. $\phi_1, \cdots, \phi_{n_i}$ have fewer than m operation-symbols, so by assumption,

$$\begin{cases} \phi_k = \gamma_k, \text{ for } k = 1, \cdots, j-1 \\ \phi_j = \psi(\xi) \\ \phi_{j+1} = \delta_{j+1}, \cdots, \phi_{n_i} = \delta_{n_i}, \end{cases} \tag{3.4}$$

where all variables but one have been replaced by constants. But, obviously, in ϕ_k, $k \neq j$, the last variable may also be replaced by a constant, since a constant result is desired and is given by either value of the variable. This leaves only one variable in ϕ_j; but with these replacements

$$\phi(\phi_1, \cdots, \phi_{n_i}) = \chi(\xi) \tag{3.5}$$

and the proof is complete.

LEMMA 3.3. *If in two primal algebras $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}$ satisfies the ϕ-condition, then, for every $\beta \in B$ and every $a(\xi)$, there is an expression $\Pi(\xi)$ such that*

$$\Pi(\xi) = \begin{cases} a(\xi)(\mathfrak{A}) \\ \beta\ (\mathfrak{B})\ . \end{cases} \tag{3.6}$$

Proof. Modulo $\mathfrak{B}$, there must exist expressions for constants in B. Therefore, letting $\kappa_\beta(\xi) = \beta(\mathfrak{B})$, replace each occurrence of ξ in κ by a variable from the set $\xi_1, \cdots, \xi_p$, so that in $\kappa_\beta(\xi_1, \cdots, \xi_p)$, no variable occurs more than once. Applying the ϕ-condition to $\kappa_\beta(\xi_1, \cdots, \xi_p)$ with respect to A, there $\psi_1, \cdots, \psi_p$ such that

$$\kappa_\beta(\psi_1, \cdots, \psi_p) = a(\xi)(\mathfrak{A})\ . \tag{3.7}$$

By Theorem 2.10, there exists a set of expressions $\{\phi_i\}$ with

$$\phi_i = \psi_i(\mathfrak{A}) \text{ and } \phi_1 = \phi_2 = \cdots = \phi_p(\mathfrak{B})\ . \tag{3.8}$$

Then

$$\kappa_\beta(\phi_1, \cdots, \phi_p) = a(\xi)(\mathfrak{A})\ , \tag{3.9}$$

by (3.7), but

$$\kappa_\beta(\phi_1, \cdots, \phi_p) = \kappa_\beta(\phi_1(\xi))(\mathfrak{B}) = \beta(\mathfrak{B})\ . \tag{3.10}$$

LEMMA 3.4. *Let $\mathfrak{A} = (A, o_1, \cdots, o_m)$ be a primal algebra of species Sp in which every primitive operation o_i is a transformation onto A. Let $\mathfrak{B}$ be a two-element primal algebra of the some species Sp, with no constant primitive operations. Then if $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, they are strictly independent.*

Proof. The operations of $\mathfrak{B}$ are transformations onto B, since they are non-constant and B has only two elements. Moreover, $\mathfrak{B}$ is primal. Therefore Theorem 2.9 applies; the ϕ-condition holds in algebra $\mathfrak{B}$. $\mathfrak{A}$ is also primal with the same kind of primitive operations; hence, by Theorem 2.9, the ϕ-condition holds for $\mathfrak{A}$ too.

Since $\mathfrak{A}$ is primal, there exists an expression, $\Sigma(\xi, \zeta)$, and an element $o \in A$ such that

$$\Sigma(\xi, o) = \xi \, , \quad \Sigma(o, \zeta) = \zeta \, . \tag{3.11}$$

Let p be the number of occurrences of ξ in Σ and q the number of occurrences of ζ. Replace each occurrence of ξ or ζ by a different variable from the set $(\zeta_1, \cdots, \xi_p)$ or $(\zeta_1, \cdots, \zeta_q)$ respectively. Let the resulting expression be denoted $\Sigma(\xi_1, \cdots, \xi_p, \zeta_1, \cdots, \zeta_q)$. By Lemma 3.2, there exist a strict $\mathfrak{B}$-function $\psi_j(\zeta)$ and constant $\mathfrak{B}$-functions such that

$$\Sigma(\gamma_1, \cdots, \gamma_{j-1}, \psi_j(\zeta), \beta_{j+1}, \cdots, \beta_{p+q}) = \zeta(\mathfrak{B}) \, . \tag{3.12}$$

Suppose $j \leqq p$, then by Theorem 2.10, there are $\phi_i(\zeta)$ such that

$$\phi_i = \begin{cases} 0(\mathfrak{A}) & (i = 1, \cdots, p) \\ \gamma_i(\mathfrak{B}) & (i = 1, \cdots, j-1) \\ \psi_j(\xi) & (\mathfrak{B}) \ i = j \\ \beta_i(\mathfrak{B}) & (i = j+1, \cdots, p) \end{cases} \tag{3.13}$$

and by Lemma 3.3, $\phi_i(\xi)$ such that

$$\phi_i = \begin{cases} \xi(\mathfrak{A}) & (i = p+1, \cdots, p+q) \\ \beta_i(\mathfrak{B}) & (i = p+1, \cdots, p+q) \, . \end{cases} \tag{3.14}$$

Thus,

$$\Sigma(\phi_1, \cdots, \phi_{p+q}) = \begin{cases} \Sigma(0, \xi) = \xi(\mathfrak{A}) \\ \Sigma(\gamma_1, \cdots, \gamma_{j-1}, \psi_j(\zeta), \beta_{j+1}, \cdots, \beta p_+ q) = \zeta(\mathfrak{B}) \, . \end{cases} \tag{3.15}$$

An exactly similar argument shows the construction if $p < j$. Therefore, it is always possible to find an expression χ such that

$$\chi(\xi, \zeta) = \begin{cases} \xi(\mathfrak{A}) \\ \zeta(\mathfrak{B}) \, , \end{cases} \tag{3.16}$$

and the two algebras are strictly independent by Definition 2.5.

We now return to the proof of Theorem 3.1.

Proof. Each algebra is primal, and every primitive operation is an onto transformation because none is constant and each algebra has but two elements. Therefore, by Theorem 2.9, the ϕ-condition holds in each algebra. Moreover, by Lemma 3.4, each pair of algebras is independent. Therefore, by Theorem 2.11, every finite subset of $\{\mathfrak{A}_1, \cdots\}$ is independent, and $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_n, \cdots\}$ is a primal cluster.

BIBLIOGRAPHY

1. A. L. Foster, *Generalized "Boolean" theory of universal algebras*, Part 1: *Subdirect sums and Normal represention theorem*, Math. Zeit., **58** (1953), 306-336.

2. ———, *Generalized "Boolean" theory of universal algebras*, Part II: *Identities and subdirect sums of functionally complete algebras*, Math. Zeit., **59** (1953), 191-199.

3. ———, *The identities of—and unique subdirect factorization within—Classes of universal algebras*, Math. Zeit., **62** (1955), 171-188.

4. ———, *Ideals and their structure in classes of operational Algebras*, Math. Zeit., **65** (1956), 70-75.

5. E. S. O'Keefe, *On the independence of primal algebras*, Math. Zeit., **73** (1960), 79-94.

6. E. L. Post, *The two-valued iterative systems of mathematical logic*, Annals of Math Studies, Princeton, 1941.