THE SPECTRUM OF SINGULAR SELF-ADJOINT ELLIPTIC OPERATORS

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This note deals with the Dirichlet problem for the second order elliptic operator

$$L = -\frac{1}{r(x)}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right)+c(x)$$

whose coefficients are defined in a bounded domain $G \subset E^n$. We suppose the following:

(a) The $a_{ij}(x)$ are complex valued and of class C' in G; $a_{ij} - \bar{a}_{ji}$.

(b) $c(x)$ is real valued, continuous, and bounded below in G.

(c) $r(x)$ is continuous and positive in G.

(d) There exists a function $\sigma(x)$, continuous and positive in G satisfying

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\bar{\xi}_j \geqq \sigma \sum_{i=1}^{n} |\xi_i|^2$$

for all x in G and all complex n-tuples $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$.

Under these assumptions it is easy to show that L is formally self-adjoint in the Hilbert space $\mathscr{L}_r^2(G)$ of functions which satisfy

$$\int_G r\,|u|^2\,dx < \infty .$$

We denote by $C_0^\infty(G)$ the set of infinitely differentiable functions with compact support in G. The operator L defined on $C_0^\infty(G)$ is a semi-bounded symmetric operator in $\mathscr{L}_r^2(G)$ and therefore has a Friedrichs extension which is self-adjoint in $\mathscr{L}_r^2(G)$. This operator, to be denoted by $\bar{L}$, will be referred to as the Dirichlet operator associated with L on G. It is well known that $\bar{L}$ is unique, has the same lower bound as the symmetric operator L, and that in sufficiently regular cases, $\bar{L}$ can be obtained by imposing Dirichlet boundary conditions on the domain of L^*. The purpose of this note is to state a criterion for the discreteness of the spectrum of $\bar{L}$.

We shall say that the spectrum of an operator A is discrete if the spectrum of A consists of a set of isolated eigenvalues of finite multiplicity. The complex number λ belongs to the essential spectrum of A if there exists an orthonormal sequence $\{u_n\}$ it the domain of A for which $(A - \lambda I)u_n \to 0$. If A is self-adjoint, then it can be shown (see

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[2]) that λ belongs to the essential spectrum of A if and only if λ belongs to the spectrum of A and is not an isolated eigenvalue of finite multiplicity. Thus the spectrum of a self-adjoint operator is discrete if and only if the essential spectrum is empty.

In case G is bounded and the conditions (a)–(b) are satisfied in $\bar{G}$ as well as G, then it is well known that $\bar{L}$ has a discrete spectrum. Here we shall allow the possibility that $\sigma(x)$ and $r(x)$ tend to 0 or ∞ on a set $S \subset \partial G$. With this generalization the spectrum of $\bar{L}$ need not be discrete.

In order to state criteria for the discreteness of the spectrum of $\bar{L}$, it is convenient to express the problem in the canonical form where

$$\begin{aligned} &G \subset \{x \mid x_n > 0\} \\ &S \subset \{x \mid x_n = 0\} \\ &L = \frac{\partial}{\partial x_n}\left(a_{nn}\frac{\partial}{\partial x_n}\right) + \sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + c \end{aligned}$$

Mihlin [1] has shown that this canonical form can in general be attained by a change of variables. Previous criteria for discreteness derived by Mihlin [1], Wolf [2], and others depend on the behavior of a_{nn} near S. The criterion to be derived here is independent of the behavior of a_{nn}; with minor modification, the method can also be applied if G is an unbounded domain.

We define

$$\begin{aligned} G_t &= G \cap \{x \mid x_n < t\} \\ E_t &= G \cap \{x \mid x_n = t\} , \end{aligned}$$

and denote by $\bar{x}$ the coordinates $(x_1, \cdots, x_{n-1})$ in E_t. Let $\bar{L}_t$ denote the Dirichlet operator associated with L on G_t. Then the following is a special case of an invariance principle due to Wolf [2].

LEMMA 1. *For $t > 0$ the essential spectrum of $\bar{L}_t$ is identical with the essential spectrum of $\bar{L}$.*

LEMMA 2. *If $\lim_{t\to 0} \inf_{u \in C_0^\infty(G_t)} \frac{(Lu, u)}{\| u \|^2} = \infty$, then the spectrum of $\bar{L}$ is discrete.*

Proof. Suppose to the contrary that there is a $\lambda_0 < \infty$ which belongs to the essential spectrum of $\bar{L}$. We can choose $t_0 > 0$ sufficiently small so that

$$\inf_{u \in C_0^\infty(G_{t_0})} \frac{(Lu, u)}{\| u \|^2} \geqq \lambda_0 + 1 .$$

Then, by the definition of $\bar{L}_{t_0}$

$$\frac{(\bar{L}_{t_0}u, u)}{\| u \|^2} \geqq \lambda_0 + 1$$

for all u in the domain of $\bar{L}_{t_0}$, and λ_0 does not belong to the spectrum of $\bar{L}_{t_0}$. By Lemma 1 this is a contradiction.

For $t > 0$ the operator

$$T_t = -\frac{1}{r(\bar{x}, t)} \sum_{i,j=1}^{n-1} \left(a_{ij}(\bar{x}, t) \frac{\partial}{\partial x_i} \right) + c(\bar{x}, t)$$

is a nonsingular elliptic operator defined on E_t. Therefore $\bar{T}_t$, the Dirichlet operator associated with T_t on E_t, has a discrete spectrum. Let $m(t)$ denote the smallest eigenvalue of $\bar{T}_t$.

THEOREM. *If* $\lim_{t \to 0} m(t) = \infty$, *then the spectrum of* $\bar{L}$ *is discrete.*

Proof. If $u \in C_0^\infty(G)$, then clearly $u(\bar{x}, t) \in C_0^\infty(E_t)$. Thus for all $u \in C_0^\infty(G)$

$$\begin{aligned} m(t)\int_{E_t} | u |^2 r d\bar{x} &\leqq \int_{E_t} \left[\sum_{i,j=1}^{n-1} a_{ij} \frac{\partial u}{\partial \bar{x}_i} \frac{\partial \bar{u}}{\partial x_j} + rc | u |^2 \right] d\bar{x} \\ &\leqq \int_{E_t} \left[a_{nn} \left| \frac{\partial u}{\partial x_n} \right|^2 + \sum_{i,j=1}^{n-1} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + rc | u |^2 \right] dx \ . \end{aligned}$$

Defining $\bar{m}(t) = \inf_{\tau \leqq t} m(\tau)$ and integrating both sides from $x_n = 0$ to $x_n = t$ we obtain

$$\bar{m}(t)\int_{G_t} | u |^2 r dx \leqq \int_{G_t} \left[a_{nn} \left| \frac{\partial u}{\partial x_n} \right|^2 + \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + rc | u |^2 \right] dx \ .$$

Since $\lim_{t \to 0} \bar{m}(t) = \infty$ we have

$$\lim_{t \to 0} \inf_{u \in C_0^\infty(G_t)} \frac{(Lu, u)}{\| u \|^2} = \infty \ .$$

The desired result now follows from Lemma 2.

We give two simple applications of the preceding theorem.

COROLLARY 1. *Let* V_t *denote the* $(n-1)$*-dimensional Lebesgue measure of* E_t. *Let* $\phi(t)$ *and* $\rho(t)$ *be continuous positive functions satisfying*

(i) $\rho(t) \geqq r(\bar{x}, t)$

(ii) $\phi(t) \sum_{i=1}^{n-1} | \xi_i |^2 \leqq \sum_{i,j=1}^{n-1} a_{ij}(x, t) \xi_i \xi_j$ for all $\vec{\xi} = (\xi_1, \cdots, \xi_{n-1})$.

If $\lim_{t\to 0} \phi(t)/\rho(t) V_t^{2/n-1} = \infty$, then the spectrum of $\bar{L}$ is discrete.

Proof. Let $\mu(t)$ denote the smallest eigenvalue of the Dirichlet operator associated with $-\Delta = -\sum_{i=1}^{n-1}\partial^2/\partial x_i^2$ on E_t. By (i) and (ii) $m(t) \geqq \phi(t)\mu(t)/\rho(t)$. It is well known that $\mu(t)$ is minimized if E_t is a $(n-1)$-dimensional sphere of volume V_t and that then $\mu(t) = C/V_t^{2/n-1}$, C being a constant. Therefore $m(t) \geqq C\phi(t)/\rho(t) V_t^{2/n-1}$ and the result follows from the preceding theorem.

The preceding corollary made no use of the shape of E_t. The following corollary gives stronger results in case E_t becomes "narrow" in the proper sense.

COROLLARY 2. *Suppose we can find functions* $\alpha_1(x_n), \cdots, \alpha_{n-1}(x_n)$, $\gamma(x_n)$ *and* $\rho(x_n)$ *which satisfy conditions* (a)–(d) *and*

(i) $\sum_{i=1}^{n-1} \alpha_i(x_n) \,|\, \xi_i \,|^2 \leqq \sum_{i,j=1}^{n-1} a_{ij}\xi_i\bar{\xi}_j$ *for all* $\xi = (\xi_1, \cdots, \xi_{n-1})$ *and all* x *in* G.

(ii) $\gamma(x_n) \leqq c(x)$ *for all* x *in* G.

(iii) $\rho(x_n) \geqq r(x)$ *for all* x *in* G.

Suppose also that we can enclose G *in a region*

$$\Gamma = \{x \mid f_i(x_n) < x_i < g_i(x_n),\, i = 1, \cdots, n-1;\, 0 < x_n < b < \infty\} .$$

If for some $i < n$

$$\lim_{t\to 0} \frac{\alpha_i(t)}{\rho(t)[g_i(t) - f_i(t)]^2} + \gamma(t) = \infty$$

then the spectrum of $\bar{L}$ *is discrete.*

Proof. Denote by $\mu(t)$ the smallest eigenvalue of the Dirichlet operator associated with

$$\tau(t) = -\frac{1}{\rho(t)} \sum_{i=1}^{n-1} \alpha_i(t) \frac{\partial^2}{\partial x_i^2} + \gamma(t)$$

on $\Gamma \cap \{x \mid x_n = t\}$. By classical variational principles $\mu(t) \leqq m(t)$. Since we can compute

$$\mu(t) = \pi^2 \sum_{i=1}^{n-1} \frac{\alpha_i(t)}{\rho(t)[g_i(t) - f_i(t)]^2} + \gamma(t) ,$$

the discreteness of the spectrum of $\bar{L}$ follows from the preceding theorem.

BIBLIOGRAPHY

1. S. G. Mihlin, *Degenerate Elliptic Equations*, Vestnik Lengrad Univ. **8** (1954), 19–48.
2. F. Wolf, *On the essential spectrum of partial differential boundary problems*, Comm. on Pure and App. Math., **12** (1959), no. 2.

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