ON N-HIGH SUBGROUPS OF ABELIAN GROUPS

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In a recent paper [2] the concept of high subgroups of Abelian groups was discussed. The purpose of this paper is to give further results concerning these high subgroups. All groups considered in this paper are Abelian, and our notation is essentially that of L. Fuchs in [1]. Let N be a subgroup of a group G. A subgroup H of G maximal with respect to disjointness from N will be called an *N-high* subgroup of G, or N-high in G. When $N = G^1$ (the subgroup of elements of infinite height in G), H will be called *high* in G.

After considering N-high subgroups in direct sums, we give a characterization (Theorem 3) of N-high subgroups of G in terms of a divisible hull of G. Next we show (Theorem 5) that if G is torsion, $N \subseteqq G^1$, and H is N-high in G, then H is pure and (Lemma 7) the primary components of any two N-high subgroups have the same Ulm invariants (see [3]). These results generalize the results in [2]. The concept of Σ-groups is introduced, and it is shown that any two high subgroups of torsion Σ-groups are isomorphic. Further, torsion Σ-groups are characterized in terms of their basic subgroups. Theorem 3 of [2] is generalized to show that high subgroups of arbitrary Abelian groups are pure. This leads to the solution of a more general version of Problem 4 of L. Fuchs in [1]. Finally, the question of whether any two high subgroups of a torsion group are isomorphic is considered, and a theorem in this direction is proved.

Preliminaries.

LEMMA 1. *Let M and N be subgroups of a primary group G such that M is neat in G and $M[p] \oplus N[p] = G[p]$. Then M is N-high in G.*

Proof. Suppose M is not N-high in G. Then there exists an N-high subgroup S of G properly containing M. Let $0 \neq s + M$ be in (S/M) $[p]$. By the neatness of S in G ([1], pg. 92) we may suppose that $s \in S[p]$. But this contradicts $M[p] \oplus N[p] = G[p]$, and so M is N-high in G.

As a consequence of Lemma 1, we obtain a standard

COROLLARY. ([3], pg. 24). *Let G be a primary group, and H a pure subgroup containing $G[p]$. Then $H = G$.*

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Proof. Purity implies neatness. Now put $N = 0$ in Lemma 1.

A useful generalization of Lemma 1 to torsion groups is

LEMMA 2. *Let M and N be subgroups of a torsion group G such that M is neat in G and $M[p] \oplus N[p] = G[p]$ for each relevant prime p. Then M is N-high in G.*

Proof. Use the proof of Lemma 1 with the observation that since S properly contains M, $(S/M)[p] \neq 0$ for some relevant prime p.

Concerning N-high subgroups in a direct sum, we have

THEOREM 1. *Let $G = \Sigma G_\alpha$ be an arbitrary direct sum of torsion groups, where H_α and N_α are subgroups of G_α, and where H_α is N_α-high in G_α for each α. Let $N = \Sigma N_\alpha$. Then $H = \Sigma H_\alpha$ is N-high in G.*

Proof. First notice that H is neat in G. To see this, use the neatness of H_α in G_α for each α (see [2] Lemma 10). Next observe that $G[p] = \Sigma G_\alpha[p] = \Sigma H_\alpha[p] \oplus \Sigma N_\alpha[p] = H[p] \oplus N[p]$ for each relevant prime p. Now apply Lemma 2 to get H to be N-high in G.

An interesting result concerning high subgroups (which are our main interest) in a direct sum is a corollary of Theorem 1.

THEOREM 2. *Let $G = \Sigma G_\alpha$ be an arbitrary direct sum of torsion groups where H_α is a high subgroup of G_α for each α. Then $H = \Sigma H_\alpha$ is high in G.*

Proof. By [2], Lemma 9 we have $G^1 = \Sigma(G_\alpha)^1$. Now use Theorem 1 and the definition of high subgroup.

Divisible hulls and high subgroups. Now we shall discuss the notion of a divisible hull for a group G, and the connection of such a hull with high subgroups. A group E minimal among those divisible groups containing G as a subgroup will be called a *divisible hull* of G. We need a few lemmas. The following lemma is almost obvious, and its proof is omitted.

LEMMA 3. *Let E be a divisible hull of a torsion group G. Let $E_1 = \Sigma E_p$ and $G = \Sigma G_p$. Then $E_1 = E$, and E_p is the unique divisible hull of G_p in E for each relevant prime p.*

LEMMA 4. *Let D be a divisible hull of a mixed group G, and E be a divisible hull of the torsion subgroup T of G in D. Then $D = E \oplus F$ where E is torsion and F is torsion free divisible.*

Proof. Since E is divisible, it is a direct summand of D. Thus $D = E \oplus F$ for some subgroup F of D. That E is torsion follows from Lemma 3. Next we show that the torsion subgroup T_F of F is zero. To see this, consider $T_F \cap G \subset T_F \cap T \subset E \cap F = 0$ to get $T_F \cap G = 0$. Then by Kulikov's Lemma, ([1], pg. 66) applied to D, $T_F = 0$ and F is torsion free. That F is divisible is clear, and the lemma is proved.

The following theorem gives a characterization in terms of divisible hulls of N-high subgroups of a torsion group G.

THEOREM 3. *Let N be any subgroup of a torsion group G, and E be a divisible hull of G with D a divisible hull of N in E. Then the set of N-high subgroups of G is the set of intersections of G with complementary summands of D in E.*

Proof. Let $H = A \cap G$, where $A \oplus D = E$. Now by [2], Lemma 1, and [1] pg. 67, we have for each relevant prime p,

$$G[p] = E[p] = A[p] \oplus D[p] = (A \cap G)[p] \oplus N[p] = H[p] \oplus N[p] .$$

By [1] pg. 92, H is neat in G, and finally by Lemma 2, H is N-high in G. Now for the converse, suppose H is N-high in G, so that $H \cap N = 0$. Now $H \cap D = 0$. To see this, notice that $(H \cap D) \cap N = H \cap N = 0$, and by Kulikov's lemma, $H \cap D = 0$. Since D is an absolute direct summand (see [1]), there exists A containing H with $A \oplus D = E$. But $H \subset A \cap G$, and since $(A \cap G) \cap N = 0$, by the maximality of H with respect to $H \cap N = 0$, we have $H = A \cap G$.

The reader will note that in particular Theorem 3 yields a characterization of high subgroups in torsion groups.

In general, a group G may have many high subgroups. It is even possible that $H \cap K = 0$ for two high subgroups H and K of G. The following theorem indicates the extent of the non-uniqueness of N-high subgroups.

THEOREM 4. *Let G be a primary group, let N be a subgroup such that $|N[p]| = |G|$ and such that $[G[p]: N[p] = |G|$. Then there exist $2^{|G|}$ distinct N-high subgroups of G. Furthermore, there exists an independent set $\{H_\alpha\}_{\alpha \in R}$ of N-high subgroups of G such that $|R| = |G|$.*

Proof. Let H be an N-high subgroup of G. By [2], $G[p] = H[p] \oplus N[p]$. Clearly $|H[p]| = |N[p]| = |G|$. Let $H[p] = \sum_{\alpha \in S} \langle x_\alpha \rangle$ and $N[p] = \sum_{\beta \in T} \langle y_\beta \rangle$. Then $|S| = |T| = |G|$. There exists $2^{|G|}$ one-to-one mappings of S onto T. Let f be such a mapping, and let

$$P_f = \sum_{\alpha \in S} \langle x_\alpha + y_{f(\alpha)} \rangle .$$

If g is any one-to-one mapping of S onto T such that $f \neq g$, then it is easy to see that $P_f \neq P_g$. Let H_f be an N-high subgroup of G containing P_f. Then $H_f[p] = P_f$, and since $P_f \neq P_g$, it follows that $H_f \neq H_g$. Hence there exist $2^{|G|}$ N-high subgroups of G. Let $T = \bigcup_{\beta \in R} T_\beta$, where $|T_\beta| = |T|$, $|R| = |T|$, and $T_\beta \cap T_\delta = \phi$ if $\beta \neq \delta$. That is, partition T into $|T|$ subsets each of cardinal $|T|$. Let f_β be a one-to-one mapping of S onto T_β, and let H_β be an N-high subgroup containing

$$P_\beta = \sum_{\alpha \in S} \langle x_\alpha + y_{f_\beta(\alpha)} \rangle .$$

It is straightforward to verify that $\{P_\beta\}_{\beta \in R}$ is an independent set of subgroups of G, that $H_\beta[p] = P_\beta$, and hence that $\{H_\beta\}_{\beta \in R}$ is an independent set of subgroups such that $|R| = |G|$. This concludes the proof.

It is easy to find examples of reduced primary groups G such that $|G^1[p]| = [G[p]: G^1[p]] = |G|$.

Purity of N-high subgroups of torsion groups. We now state and prove a generalization to N-high subgroups of torsion groups of [2] Theorem 3, namely that high subgroups of torsion groups are pure.

THEOREM 5. *Let N be a subgroup of a torsion group G with $N \subset G^1$, and let H be an N-high subgroup of G. Then H is pure in G.*

Proof. That it suffices to consider the primary case here follows from the fact H_p is N_p-high in G_p (see [2] Lemma 10 and [2] Lemma 11). So let G be primary. Now by [2] Lemma 1, we have $G[p] = H[p] \oplus N[p]$. Since $G^1[p] \supset N[p]$, then $G^1[p] = (H \cap G^1)[p] \oplus N[p]$. Now let H_1 be an $(H \cap G^1)[p]$-high subgroup of H. Since N-high subgroups are neat (see [1] c, d pg. 92) and neatness is transitive, we have that H_1 is neat in G. By [2] Lemma 1, $H[p] = H_1[p] \oplus (H \cap G^1)[p]$, so that $G[p] = H_1[p] \oplus G^1[p]$. An application of Lemma 1 yields H_1 high in G. Finally, by [2], Lemma 8, H_1 contains B basic in G, so that by [2] Lemma 2, H is pure in G as stated.

Before stating some corollaries, we would like to pose the following question: characterize all subgroups T of an Abelian group G such that T-high subgroups of G are pure. Suitable examples are easy to find which show that just any subgroup T will not do.

A couple of corollaries of Theorem 5 are

COROLLARY 1. *Let N_1 and N_2 be subgroups of a torsion group G with $N_1 \subseteqq N_2 \subseteqq G^1$. Then every N_1-high subgroup of G contains an N_2-high subgroup, and in particular every N_1-high subgroup K contains a subgroup H high in G.*

Proof. The proof is similar to the proof of Theorem 5.

COROLLARY 2. *Let N be a subgroup of G^1 in a torsion group G, and let S be an infinite subgroup of G with $S \cap N = 0$. Then there exists a subgroup K pure in G with $|K| = |S|$ and $K \cap N = 0$.*

Proof. Substitute N for G^1 in the proof of [2], Theorem 2.

This last corollary is a generalization of the solution in [2] of Fuchs' Problem 4.

Σ-groups. The following ideas arose from an investigation of the question of whether or not all high subgroups of a given group are isomorphic. A natural query in this direction is: If one of the high subgroups of a given group G is a direct sum of cyclic groups, are all of them direct sums of cyclic groups? The answer for torsion groups is yes. It is this observation that gives rise to so called Σ-groups. Before discussing this notion further, we need a few lemmas.

LEMMA 5. *Let N be a subgroup of a torsion group G with H and K both N-high subgroups of G. Then $((H \oplus N)/N)[p] = ((K \oplus N)/N)[p]$ for each relevant prime p.*

Proof. For $h \in H$ we have that $o(h+N) = p$ if and only if $o(h) = p$. Suppose $h \in H[p] \backslash (K \cap H)$. Then there exists $k \in K$, $n \in N$ with $h - k = n$, whence $o(k) = p$. Thus $h + N = (k + N) \in ((K \oplus N)/N)[p]$; and since p was arbitrary, we have by symmetry that

$$((H \oplus N)/N)[p] = ((K \oplus N)/N)[p]$$

as stated.

LEMMA 6. *Let N be a subgroup of a torsion group G with $N \subseteqq G^1$. Let H be an N-high subgroup of G. Then $((H \oplus N)/N)$ is pure in G/N.*

Proof. Suppose $m(g+N) = h+N$ for some $h \in H$, $g \in G$, m a nonzero integer. Then $mg - n = h$ for some $n \in N$, and since $n \in G^1$ and H is pure (Theorem 5), we have $h = mh_1$ for some $h_1 \in H$. Thus $h + N = m(h_1 + N)$ and the lemma is proved.

COROLLARY. *Let N be a subgroup of a reduced torsion group G with $N \subseteqq G^1 \neq 0$, and let H be an N-high subgroup of G. Then H is not closed.*

Proof. This follows easily from a theorem of Kulikov and Papp ([1] pg. 117).

LEMMA 7. *Let H and K be any two N-high subgroups of a primary group G with N a subgroup of G^1. Then for all positive integers n*

(a) *p^nH is N-high in p^nG,*

(b) *p^nH is pure in p^nG,*

(c) $(p^n((H \oplus N)/N))[p] = (p^n((K \oplus N)/N))[p]$,

(d) *H, K, and G have the same* nth *Ulm invariants* (see [3]).

Proof. (a) Use the proof of Theorem 5 (e) in [2] and the fact that $N \subseteqq p^nG$ for all n.

(b) Use (a), $N \subseteqq (p^nG)^1$, and Theorem 5.

(c) First notice that $p^n((H \oplus N)/N) = (p^nH \oplus N)/N$. Now (c) follows immediately from Lemma 5 applied to the right sides of this equation and the corresponding one for K.

(d) The proof is similar to that of Theorem 6 in [2].

The following theorem is a slight generalization of the fact that any two high subgroups of a countable group G are isomorphic. (See [2], Theorem 5 (u).)

THEOREM 6. *Let N be a subgroup of a countable torsion group G with $N \subseteqq G^1$, and G^1 elementary. Then any two N-high subgroups H and K of G are isomorphic.*

Proof. Write $G = \Sigma G_p$, $H = \Sigma H_p$, $K = \Sigma K_p$. Now H_p and K_p are both N_p high in G_p, and $N_p \subseteqq G_p^1$ for each relevant prime p. Let $\bar{H}_p = (H_p \oplus N_p)/N_p$ and $\bar{K}_p = (K_p \oplus N_p)/N_p$. Using Lemmas 5 and 6 and the fact that G^1 is elementary, we get immediately that $(p^\omega \bar{H}_p)[p] = (p^\omega \bar{K}_p)[p]$, and that $(p^{\omega+1}\bar{H}_p)[p] = (p^{\omega+1}\bar{K}_p)[p] = 0$. Thus for $\alpha \geqq \omega$, the αth Ulm invariants of $\bar{H}_p$ and $\bar{K}_p$ are the same. Since $\bar{H}_p \cong H_p$ and $\bar{K}_p \cong K_p$, Lemma 7 (d) implies that H_p and K_p have the same Ulm invariants. Since G^1 is elementary, H and K are reduced, and Ulm's theorem yields $H \cong K$.

REMARK. In Theorem 6, if we take N to be a subgroup of G^1 such that $N[p] \neq G^1[p]$, then neither H nor K will be a direct sum of cyclic groups as is easily seen.

A *Σ-group* is any group G all of whose high subgroups are direct sums of cyclic groups. This means that in a torsion Σ-group every high subgroup is basic. This implies further that in Σ-groups, every high subgroup is an endomorphic image. Examples of Σ-groups are very to easy to find. For instance, direct sums of countable groups turn out to be Σ-groups. Also, any group G such that G/G^1 is a direct sum of cyclic groups is a Σ-group. (See the proof of Theorem 7.) For a non-Σ-group, see [2], Theorem 5(t).

THEOREM 7. *Let H and K be high subgroups of a torsion group G. Then if H is a direct sum of cyclic groups, so is K. Moreover $H \cong K$.*

Proof. Let $\tilde{S}$ be the image of S under the natural homomorphism of G onto G/G^1. Now $\tilde{H} \cong H$, $\tilde{K} \cong K$, and by Lemma 5 we have $\tilde{K}[p] = \tilde{H}[p]$ for each relevant prime p. By [3], Theorem 12, we have that $\tilde{H}[p]$ is the union of a sequence $\tilde{P}_n$ of subgroups of bounded height in $\tilde{H}$. The purity of $\tilde{H}$ (Theorem 5) tells us that $\tilde{P}_n$ has bounded height in $\tilde{K}$ for each n. Hence by [3], Theorem 12, each $\tilde{K}_p$ is a direct sum of cyclic groups, so that $\tilde{K}$ is a direct sum of cyclic groups. Thus K, which is isomorphic to $\tilde{K}$, is a direct sum of cyclic groups. Since H and K are both basic in G, we have $H \cong K$. Thus we have shown that in a torsion group, if one high subgroup is a direct sum of cyclic groups, they all are, and they are all isomorphic.

From Theorem 7 we see that if in a torsion group G there exist two non-isomorphic high subgroups, then no high subgroup is a direct sum of cyclic groups. The next theorem shows that torsion Σ-groups are closed under direct sums.

THEOREM 8. *For torsion groups it is true that a direct sum of Σ-groups is a Σ-group.*

Proof. By Theorem 2, a direct sum of highs is high. But such a direct sum is basic. An application of Theorem 7 completes the proof.

COROLLARY. *A direct sum of countable torsion group is a Σ-group.*

Proof. It suffices by Theorem 8 to verify that a countable torsion group is a Σ-group, and this is very easy.

REMARK. Examples exist of torsion groups G such that G/G^1 is a direct sum of cyclic groups, but such that G is not a direct sum of countable groups. Therefore, we see that the class of torsion Σ-groups properly contains the class of all torsion groups that are the direct sum of countable groups.

The next theorem gives an interesting characterization of torsion Σ-groups.

THEOREM 9. *A torsion group G is a Σ-group if and only if G*

contains a maximal basic subgroup.

Proof. If G is a Σ-group, then any high subgroup will be a maximal basic subgroup of G. Now suppose G contains a maximal basic subgroup B. Let H be a high subgroup containing B, and suppose $B \neq H$. By [1] pg. 114, there exists B_1 basic in H with $B_1 > B$. Since H is pure and G/H is divisible, B_1 is basic in G, a contradiction. Therefore $B = H$, and G is a Σ-group by Theorem 7.

The next theorem is a result concerning the Σ-groups of a torsion group.

THEOREM 10. *Every torsion group G contains a Σ-subgroup R pure in G such that $R^1 = G^1$.*

Proof. First if $G^1 = 0$, put $R = B$ basic in G. Also if G is a Σ-group, but $R = G$. So suppose that $G^1 \neq 0$ and G is not a Σ-group. Let B be a basic subgroup of G. Embed B in a high subgroup H of G. By Theorem 8 and the assumptions on G, $H/B \neq 0$. B is basic in H so that $G/B = H/B \oplus R/B$, where the divisibility of H/B guarantees that R/B may be chosen to contain $(G^1 \oplus B)/B$. Hence R contains G^1. The purity of R/B in G/B gives us that R is pure in G. Hence $R^1 = G^1$. Now $H \cap R = B$, so that $G[p] = H[p] \oplus G^1[p]$ and

$$R[p] = (R \cap H)[p] \oplus G^1[p] = B[p] \oplus R^1[p] .$$

By Lemma 2, B is high in R so that by Theorem 7, R is a Σ-group, and the proof is complete.

We do not know whether every subgroup of a Σ-group is a Σ-group. However, every pure subgroup of a torsion Σ-group is a Σ-group. In fact, we have

THEOREM 11. *Every subgroup L of a torsion Σ-group G with $L^1 = L \cap G^1$ is a Σ-group.*

Proof. Embed a high subgroup H_L of L in a high subgroup H G. Since G is a Σ-group, H is a direct sum of cyclic groups and hence so is H_L. Now apply Theorem 7 to L to get that L is a Σ-group.

COROLLARY. *Every pure subgroup R of a torsion Σ-group G is a Σ-group.*

Proof. $R^1 = R \cap G^1$, and Theorem 11 then yields the desired result.

COROLLARY. *Every pure subgroup of a direct sum of countable torsion groups is a Σ-group.*

High subgroups in mixed groups. In this section we will discuss some properties of high subgroups of arbitrary Abelian groups and generalize some of our results for the torsion case. A lemma which is useful is

LEMMA 8. *Let S be any subgroup of an Abelian group G with $S \cap G^1 = 0$. Then for any subgroup T with $(T/S) \cap (G/S)^1 = 0$, we have $T \cap G^1 = 0$.*

Proof. Suppose $T \cap G^1 \neq 0$. Then $(T/S) \cap (G/S)^1 \neq 0$.

The following theorem was proved in [2] for the torsion case. The fact that G^1 is divisible in the torsion free case makes this case easy, so we proceed directly to the general case.

THEOREM 12. *Let H and K be any two high subgroups of a group G. Then*

(a) *G/H is divisible*

(b) *G/H is a divisible hull of $(G^1 \oplus H)/H \cong G^1$*

(c) *$G/H \cong G/K$.*

Proof. (a) Let T/H be the torsion subgroup of G/H. Now T/H is divisible, for if not, T/H would have a non-zero cyclic direct summand L/H. But L/H would be a direct summand of G/H since T/H is pure in G/H. Hence $(L/H) \cap (G/H)^1 = 0$, and Lemma 8 gives us that $L \cap G^1 = 0$. Consequently H is not high in G, a contradiction. Thus we have $G/H = T/H \oplus F/H$, where F/H is torsion free. This means that $(F/H)^1$ is divisible, whence $F/H = (F/H)^1 \oplus R/H$. Now clearly, $(R/H) \cap (G/H)^1 = 0$, so that by Lemma 8, $R = H$, and G/H is divisible as stated.

(b) As a divisible group, G/H must contain a divisible hull D/H of $(G^1 \oplus H)/H$. Put $G/H = D/H \oplus L/H$. Clearly $L \cap G^1 = 0$, hence $L/H = 0$ and (b) is proved.

(c) This follows from $(G^1 \oplus H)/H \cong G^1 \cong (G^1 \oplus K)/K$ and the fact that divisible hulls of isomorphic groups are isomorphic. Thus we see that G/H is a structural invariant of G.

We shall now discuss a generalization to arbirary Abelian groups of a theorem proved in [2] for the torsion case. Here again, the torsion free case is easy (G^1 is divisible), and for a torsion free group G we see easily that all high subgroups are isomorphic. First we need

THEOREM 13. *Let T be the torsion subgroup of an Abelian group G, H be a high subgroup of G, and T_H be the torsion subgroup of H. Then T_H is high in T.*

Proof. We need only consider the case of a mixed group G. Let E be a divisible hull of G with D_H and D_{G^1} divisible hulls in E of H and G^1. Then $E = D_H \oplus D_{G^1}$ (see [2]). Next let D_{T_H} and $D_{T_{G^1}}$ be divisible hulls of T_H and T_{G^1} in D_H and D_{G^1} respectively. Then

$$E = D_H \oplus D_{G^1} = D_{T_H} \oplus D_1 \oplus D_{T_{G^1}} \oplus D_2 = D_{T_H} \oplus D_{T_{G^1}} \oplus D_1 \oplus D_2 .$$

Applying Lemmas 3 and 4 to D_H and D_{G^1}, we have that $T_E = D_{T_H} \oplus T_{T_{G^1}}$ is the torsion subgroup of E, and $D_1 \oplus D_2$ is torsion free. Clearly (Lemma 3) T_E is a divisible hull of T in E. Now by Theorem 3, it remains to verify that $T \cap D_{T_H} = T_H$. To this end put $L = T \cap D_{T_H}$. That $T_H \subset L$ is clear. By Lemma 4, $H \cap D_{T_H} = T_H$. So suppose there exists $t \in (G \backslash H) \cap L$. Then by the definition of H, there exists $h \in H$ with $h + mt = g_1 \neq 0$, where $g_1 \in G^1$. But since L is torsion we have that $h \in T_H$. Hence $(h + mt) \in D_{T_H}$, and $h + mt = g_1 \neq 0$ together with $D_{T_H} \subset D_H$ contradict $D_H \cap D_{G^1} = 0$, and T_H is high in T as desired.

COROLLARY. *Let H be a high subgroup of G, and let T_H be the torsion subgroup of H. Then T_H is pure in G.*

Proof. By Theorem 13, T_H is high in T, and consequently pure in T. Since T is pure in G, it follows that T_H is pure in G.

THEOREM 14. *Let H be a high subgroup of an Abelian group G. Then H is pure in G.*

Proof. Let the notation be the same as in Theorem 13. Now by Theorem 13 T/T_H is divisible, so that $G/T_H = T/T_H \oplus R/T_H$, where R is chosen such that R/T_H contains H/T_H. Since T_H is pure in G, R is pure in G, and since H is neat in G, H/T_H is neat in R/T_H. But R/T_H is torsion free and since a neat subgroup of a torsion free group is pure we have that H/T_H is pure in R/T_H. Thus H is pure in R, so that H is pure in G, and the proof is complete.

The following embedding theorem is a generalization to arbitrary Abelian groups of the solution to Fuchs' Problem 4 (see [1]).

THEOREM 15. *Let S be any infinite subgroup of an Abelian group G with $S \cap G^1 = 0$. Then there exists a subgroup K pure in G with $S \subset K$, $K \cap G^1 = 0$, and $|K| = |S|$.*

Proof. Embed S in H high in G. By [1] pg. 78 N, there exists a pure subgroup K of H with $S \subset K$ and $|S| = |K|$. The purity of

H implies the purity of K in G, and $K \cap G^1 \subset H \cap G^1 = 0$, so that $K \cap G^1 = 0$, completing the proof.

An unsolved problem. To conclude the present paper we shall make a few remarks concerning the question of whether all high subgroups of an Abelian torsion group are isomorphic. The reader may have observed, from the proof of Theorem 6, that this question is a special case of the more general open question: Given two pure subgroups A and B of a primary group G with $A[p] = B[p]$, is it true in general that $A \cong B$? The authors feel that an affirmative answer to this question would have important consequences in the theory of Abelian torsion groups. A step in this direction is

THEOREM 16. *Let A and B be pure subgroups of primary group G with $A[p] = B[p]$. Then $G = A \oplus C$ implies $G = B \oplus C$ and $A \cong B$.*

Proof. Let $G = A \oplus C$. Then $G[p] = A[p] \oplus C[p] = B[p] \oplus C[p]$. We will show that $G = B \oplus C$. First notice that $A[p] = B[p]$ gives us that $B \cap C = 0$. To prove $G = B \oplus C$, it suffices to verify that $G \subset B \oplus C$. For this purpose it is sufficient that $G[p^n] \subset B \oplus C$ for each n. But this is true if and only if $A[p^n] \subset B \oplus C$ for each n. Now we use induction to show that $G[p^n] \subset B \oplus C$ for each n. First, $G[p] \subset B \oplus C$ by hypothesis. Next suppose that $G[p^n] \subset B \oplus C$, and let $a \in A$ with $o(a) = p^{n+1}$. Then $p^n a = b \in A[p] = B[p]$. By the purity of B, $p^n a = p^n b_1$ with $b_1 \in B$, and $p^n(a - b_1) = 0$, so that $a - b_1 \in B \oplus C$ by the induction hypothesis. Hence $a \in B \oplus C$, therefore

$$A[p^{n+1}] \subset B \oplus C ,$$

which means that $G[p^{n+1}] \subset B \oplus C$. Thus $G = B \oplus C$. Finally $A \cong G/C \cong B$, and the proof is complete.

The foregoing theorem suggests the following generalization.

THEOREM 17. *Let G be a direct sum of torsion groups, $G = \sum_{\alpha \in A} G_\alpha$, and let $\{T_\alpha\}_{\alpha \in A}$ be a family of subgroups pure in G with $T_\alpha[p] = G_\alpha[p]$ for each relevant prime p and each $\alpha \in A$. Then for any subfamily $\{T_\alpha\}_{\alpha \in S}$, $G = \sum_{\alpha \in S} T_\alpha \oplus \sum_{\alpha \notin S} G_\alpha$. In particular, $G = \sum_{\alpha \in A} T_\alpha$ and $G_\alpha \cong T_\alpha$ for each $\alpha \in A$.*

Proof. Put $T = \sum_{\alpha \in A} T_\alpha$. It suffices to show that $G = T$. We show as before that for each n we have $G[p^n] \subset T$. This is true if for each $\alpha \in A$ we have for the primary components G_{α_p}, that $G_{\alpha_p}[p^n] \subset T$ for each n. This is accomplished as in the proof of Theorem 16. Finally, that $T_\alpha \cong G_\alpha$ for each α follows as before.

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