# ON AXIOMATIC HOMOLOGY THEORY

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A homology theory will be called *additive* if the homology group of any topological sum of spaces is equal to the direct sum of the homology groups of the individual spaces.

To be more precise let $H_*$ be a homology theory which satisfies the seven axioms of Eilenberg and Steenrod [1]. Let $\mathscr{A}$ be the admissible category on which $H_*$ is defined. Then we require the following.

*Additivity Axiom.* If $X$ is the disjoint union of open subsets $X_\alpha$ with inclusion maps $i_\alpha: X_\alpha \to X$, all belonging to the category $\mathscr{A}$, then the homomorphisms

$$i_{\alpha *}: H_n(X_\alpha) \to H_n(X)$$

must provide an injective representation of $H_n(X)$ as a direct sum.[1]

Similarly a cohomology theory $H^*$ will be called *additive* if the homomorphisms

$$i_\alpha^*: H^n(X) \to H^n(X_\alpha)$$

provide a projective representation of $H^n(X)$ as a direct product.

It is easily verified that the singular homology and cohomology theories are additive. Also the Čech theories based on infinite coverings are additive. On the other hand James and Whitehead [4] have given examples of homology theories which are not additive.

Let $\mathscr{W}$ denote the category consisting of all pairs $(X, A)$ such that both $X$ and $A$ have the homotopy type of a $CW$-complex; and all continuous maps between such pairs. (Compare [5].) The main object of this note is to show that there is essentially only one additive homology theory and one additive cohomology theory, with given coefficient group, on the category $\mathscr{W}$.

First consider a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots$ of $CW$-complexes with union $K$. Each $K_i$ should be a subcomplex of $K$. Let $H_*$ be an additive homology theory on the category $\mathscr{W}$.

LEMMA 1. *The homology group $H_q(K)$ is canonically isomorphic to the direct limit of the sequence*

$$H_q(K_1) \to H_q(K_2) \to H_q(K_3) \to \cdots .$$

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[1] This axiom has force only if there are infinitely many $X_\alpha$. (Compare pg. 33 of Eilenberg-Steenrod.) The corresponding assertion for pairs $(X_\alpha, A_\alpha)$ can easily be proved, making use of the given axiom, together with the "five lemma."

The corresponding lemma for cohomology is not so easy to state. It is first necessary to define the "first derived functor" of the inverse limit functor.[2] The following construction was communicated to the author by Steenrod.

Let $A_1 \xleftarrow{p} A_2 \xleftarrow{p} A_3 \longleftarrow \cdots$ be an inverse sequence of abelian groups, briefly denoted by $\{A_i\}$. Let $\Pi$ denote the direct product of the groups $A_i$, and define $d\colon \Pi \to \Pi$ by

$$d(a_1, a_2, \cdots) = (a_1 - pa_2, a_2 - pa_3, a_3 - pa_4, \cdots) .$$

The kernel of $d$ is called the inverse limit of the sequence $\{A_i\}$ and will be denoted by $\mathfrak{L}\{A_i\}$.

DEFINITION. The cokernel $\Pi/d\Pi$ of $d$ will be denoted by $\mathfrak{L}'\{A_i\}$; and $\mathfrak{L}'$ will be called the derived functor of $\mathfrak{L}$.

Now let $K_1 \subset K_2 \subset \cdots$ be $CW$-complexes with union $K$, and let $H^*$ be an additive cohomology theory on the category $\mathscr{W}$.

LEMMA 2. *The natural homomorphism* $H^n(K) \to \mathfrak{L}\{H^n(K_i)\}$ *is onto, and has kernel isomorphic to* $\mathfrak{L}'\{H^{n-1}(K_i)\}$.

REMARK. The proofs of Lemmas 1 and 2 will make no use of the dimension axiom [1 pg. 12]. This is of interest since Atiyah and others have studied "generalized cohomology theories" in which the dimension axiom is not satisfied.

*Proof of Lemma* 1. Let $[0, \infty)$ denote the $CW$-complex consisting of the nonnegative real numbers, with the integer points as vertices. Let $L$ denote the $CW$-complex

$$L = K_1 \times [0, 1] \cup K_2 \times [1, 2] \cup K_3 \times [2, 3] \cup \cdots ;$$

contained in $K \times [0, \infty)$. The projection map $L \to K$ induces isomorphisms of homotopy groups in all dimension, and therefore is a homotopy equivalence. (See Whitehead [6, Theorem 1]. Alternatively one could show directly that $L$ is a deformation retract of $K \times [0, \infty)$.)

Let $L_1 \subset L$ denote the union of all of the $K_i \times [i-1, i]$ with $i$ odd. Similarly let $L_2$ be the union of all $K_i \times [i-1, i]$ with $i$ even. The additivity axiom, together with the homotopy axiom, clearly implies that

$$H_*(L_1) \approx H_*(K_1) \oplus H_*(K_3) \oplus H_*(K_5) \oplus \cdots$$

with a similar assertion for $L_2$, and similar assertions for cohomology. On the other hand $L_1 \cap L_2$ is the disjoint union of the $K_i \times [i]$, and

[2] This derived functor has been studied in the thesis of *Z-Z.* Yeh, Princeton University 1959; and by Jan-Eric Roos [8].

therefore

$$H_*(L_1 \cup L_2) \approx H_*(K_1) \oplus H_*(K_2) \oplus H_*(K_3) \oplus \cdots .$$

Note that the triad $(L; L_1, L_2)$ is proper. In fact each set $K_i \times [i-1, i]$ used in the construction can be thickened, by adding on $K_{i-1} \times [i-3/2, i-1]$, without altering its homotopy type. Hence this triad $(L; L_1, L_2)$ has a Mayer-Vietoris sequence. The homomorphism

$$\psi\colon H_*(L_1 \cap L_2) \to H_*(L_1) \oplus H_*(L_2)$$

in this sequence is readily computed, and turns out to be:

$$\begin{aligned}&\psi(h_1, h_2, \cdots, 0, 0, \cdots)\\ &\qquad = (h_1, ph_2 + h_3, ph_4 + h_5, \cdots) \oplus (-ph_1 - h_2, -ph_3 - h_4, \cdots) ;\end{aligned}$$

where $h_i$ denotes a generic element of $H_*(K_i)$, and $p\colon H_*(K_i) \to H_*(K_{i+1})$ denotes the inclusion homomorphism.

It will be convenient to precede $\psi$ by the automorphism $\alpha$ of $H_*(L_1 \cap L_2)$ which multiplies each $h_i$ by $(-1)^{i+1}$. After shuffling the terms on the right hand side of the formula above, we obtain

$$\psi\alpha(h_1, h_2, \cdots) = (h_1, h_2 - ph_1, h_3 - ph_2, h_4 - ph_3, \cdots) .$$

From this expression it becomes clear that $\psi$ has kernel zero, and has cokernel isomorphic to the direct limit of the sequence $\{H_*(K_i)\}$. Now the Mayer-Vietoris sequence

$$0 \longrightarrow H_*(L_1 \cap L_2) \xrightarrow{\psi} H_*(L_1) \oplus H_*(L_2) \longrightarrow H_*(L) \longrightarrow 0$$

completes the proof of Lemma 1.

The proof of Lemma 2 is completely analogous. The only essential difference is that the dual homomorphism

$$H^*(L_1 \cap L_2) \xleftarrow{\psi} H^*(L_1) \oplus H^*(L_2)$$

is not onto, in general. Its cokernel gives rise to the term $\mathfrak{L}'\{H^{n-1}(K_i)\}$ in Lemma 2.

Now let $K$ be a possibly infinite formal simplicial complex with subcomplex $L$, and let $|K|$ denote the underlying topological space in the weak (=fine) topology. (Compare [1 pg. 75]) Let $H_*$ denote an additive homology theory with coefficient group $H_0(\text{Point}) = G$.

LEMMA 3. *There exists a natural isomorphism between $H_q(|K|, |L|)$ and the formally defined homology group $\boldsymbol{H}_q(K, L; G)$ of the simplicial pair.*

*Proof.* If $K$ is a finite dimensional complex then the proof given on pages 76–100 of Eilenberg-Steenrod applies without essential change. Now let $K$ be infinite dimensional with $n$-skeleton $K^n$. It follows from this remark that the inclusion homomorphism

$$H_q(|K^n|) \to H_q(|K^{n+1}|)$$

is an isomorphism for $n > q$. Applying Lemma 1, it follows that the inclusion

$$H_q(|K^n|) \to H_q(|K|)$$

is also an isomorphism. Therefore the inclusion

$$H_q(|K^n|, |L^n|) \to H_q(|K|, |L|)$$

is an isomorphism for $n > q$. Together with the first remark this completes the proof of Lemma 3.

The corresponding lemma for cohomology groups can be proved in the same way. The extra term in Lemma 2 does not complicate the proof since $\mathfrak{L}' = 0$ for an inverse sequence of isomorphisms.

Uniqueness Theorem. *Let $H_*$ be an additive homology theory on the category $\mathscr{W}$ (see introduction) with coefficient group $G$. Then for each $(X, A)$ in $\mathscr{W}$ there is a natural isomorphism between $H_q(X, A)$ and the* $q$th *singular homology group of $(X, A)$ with coefficients in $G$.*

*Proof.* Let $|SX|$ denote the geometric realization of the total singular complex of $X$, as defined by Giever, Hu, or Whitehead. (References [2, 3, 7].) Recall that the second barycentric subdivision $S''X$ is a simplicial complex. Since $X$ has the homotopy type of a $CW$-complex, the natural projection

$$|SX| = |S''X| \to X$$

is a homotopy equivalence. (Compare [7, Theorem 23]). Using the five lemma it follows that the induced homomorphism

$$H_*(|S''X|, |S''A|) \to H_*(X, A)$$

is an isomorphism. But the first group, by Lemma 3, is isomorphic to

$$\mathbf{H}_*(S''X, S''A; G) \approx \mathbf{H}_*(SX, SA; G)\ ,$$

which by definition is the singular homology group of the pair $X, A$.

It is easily verified that the resulting isomorphism

$$\mathbf{H}_*(SX, SA; G) \to H_*(X, A)$$

commutes with mappings and boundary homomorphisms. (Compare pp. 100–101 of [1] for precise statements.) This completes the proof of the

Uniqueness Theorem.

The corresponding theorem for cohomology groups can be proved in the same way.

## References

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