A NOTE ON WEAK SEQUENTIAL CONVERGENCE

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1. Let X be a real Banach space, J_X the canonical mapping from X into X^{**}, and $K(X)$ the set of all elements F in X^{**} which are X^*-limits of sequences in $J_X X$. Thus $F \in K(X)$ if and only if there exists a sequence $\{x_n\}$ in X such that

$$F(f) = \lim_n f(x_n) \tag{1.1}$$

for all $f \in X^*$. While the closure of $J_X X$ in the X^*-topology is X^{**} [4, p. 229], it is not true in general that $K(X) = X^{**}$. By using properties of the space of continuous real functions defined on a real interval, we shall prove that the subspace $K(X)$ is norm-closed in X^{**}.

2. If x is a bounded real function defined on a closed interval $[a, b]$, let $||x|| = \sup\{|x(s)| : a \leqq s \leqq b\}$. If x is a bounded Baire function of the first class, then there exists a sequence $\{x_n\} \subset \mathscr{C}[a, b]$ such that $x(s) = \lim_n x_n(s)$ for all $s \in [a, b]$ and $||x_n|| = ||x||$ for all n [2, p. 138]. However, if a bounded function x is the pointwise limit of an unbounded sequence of elements of a subspace X of $\mathscr{C}$, then it is not necessarily true that x is the pointwise limit of a bounded sequence in X.

LEMMA 1. *Lex X be a subspace of $\mathscr{C}$, and let x be a real function which is the pointwise limit of a bounded sequence in X. Then there exists a sequence $\{x_n\}$ in X such that x is the pointwise limit of $\{x_n\}$ and $||x_n|| = ||x||$ for all n.*

Proof. If $\{y_n\}$ is a sequence in X which converges pointwise to x, with $\sup_n ||y_n|| = M < \infty$, let continuous functions $\varphi, \varphi_1, \varphi_2, \cdots$ be defined by

$$\begin{cases} \varphi(s) \equiv ||x|| \\ \varphi_n(s) = \max(y_n(s),\ ||x||) \end{cases} \tag{2.1}$$

for all $s \in [a, b]$. Then $\{\varphi_n\}$ converges to φ in the $\mathscr{C}^*$-topology of $\mathscr{C}$ [1, p. 224], and hence [3, p. 36] for each positive integer n there exist nonnegative numbers $a_{n1}, \cdots, a_{nk_n}$ such that

$$\sum_{k=1}^{k_n} a_{nk} = 1\ , \quad \left|\left|\sum_{k=1}^{k_n} a_{nk}\,\varphi_{n+k} - \varphi\right|\right| < n^{-1}\ . \tag{2.2}$$

Define $\{z_n\} \subset X$ by

$$z_n = \sum_{k=1}^{k_n} a_{nk}\, y_{n+k}\ . \tag{2.3}$$

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Then $\{z_n\}$ converges pointwise to x, and $-M \leqq z_n(s) \leqq ||x|| + n^{-1}$ for each n.

If a sequence $\{\psi_n\}$ is now defined in $\mathscr{C}$ by $\psi_n = \min(z_n, -\varphi)$, an argument like that used with $\{\varphi_n\}$ shows that there exist, for each n, nonnegative numbers $b_{n1}, \cdots, b_{nj_n}$ such that

$$(2.4) \qquad \sum_{j=1}^{j_n} b_{nj} = 1 , \qquad \left|\left|\sum_{j=1}^{j_n} b_{nj}\,\psi_{n+j} + \varphi\right|\right| < n^{-1} .$$

If $\{u_n\} \subset X$ is defined by

$$(2.5) \qquad u_n = \sum_{j=1}^{j_n} b_{nj}\, z_{n+j} ,$$

then x is the pointwise limit of $\{u_n\}$, and $||u_n|| \to ||x||$ as $n \to \infty$. Since it may be assumed that $||u_n|| \neq 0$ for each n, the desired sequence $\{x_n\}$ is obtained by defining $x_n = (||x||/||u_n||)\, u_n$.

3. The conjugate space $\mathscr{C}^*$ of $\mathscr{C}$ is equivalent with the space of all finite regular signed Borel measures on $[a, b]$, under a mapping U such that if $f \in \mathscr{C}^*$ and $\mu_f = Uf$, then

$$(3.1) \qquad f(x) = \int_a^b x d\mu_f$$

for all $x \in \mathscr{C}$ [4, p. 397]. It follows that if X is a closed subspace of $\mathscr{C}$ and $F \in X^{**}$, then $F \in K(X)$ if and only if there exists a bounded, pointwise-convergent sequence $\{y_n\}$ in X with the property that

$$(3.2) \qquad F(f\,|\,X) = \int_a^b (\lim y_n)\, d\mu_f$$

for all $f \in \mathscr{C}^*$.

LEMMA 2. *If X is a real Banach space and $F \in K(X)$, then there exists a sequence $\{x_n\}$ in X such that F is the X^*-limit of $\{J_X x_n\}$ and $||x_n|| = ||F||$ for all n.*

Proof. Case 1. If X is a closed subspace of $\mathscr{C}$ and $F \in K(X)$, there must be a bounded, pointwise-convergent sequence $\{y_n\} \subset X$ such that (3.2) holds for all $f \in \mathscr{C}^*$. If $x(s) = \lim_n y_n(s)$ for $a \leqq s \leqq b$, then by Lemma 1 there exists a sequence $\{x_n\}$ in X such that x is the pointwise limit of $\{x_n\}$ and $||x_n|| = ||x||$ for all n. Thus F is the X^*-limit of $\{J_X x_n\}$ and it is easily verified that $||F|| = ||x_n||$ for each n.

Case 2. If X is an arbitrary real Banach space and $F \in K(X)$, then there is a sequence $\{y_n\}$ in X such that F is the X^*-limit of $\{J_X y_n\}$. If Y is the closed subspace of X generated by $\{y_n\}$, we can define

$G \in Y^{**}$ by

$$G(f \mid Y) = F(f) \text{ for all } f \in X^* , \tag{3.3}$$

and this definition is unambiguous since F is the X^*-limit of a sequence in $J_X Y$. It is easy to verify that $G \in K(Y)$ and $\|G\| = \|F\|$. Since Y is separable, Y is equivalent with a closed subspace of $\mathscr{C}$ [1, p. 185], and hence by Case 1, there is a sequence $\{x_n\}$ in Y such that G is the Y^*-limit of $\{J_Y x_n\}$ and $\|x_n\| = \|G\| = \|F\|$ for all n. Finally, if $f \in X^*$, then

$$F(f) = G(f \mid Y) = \lim_n f(x_n) , \tag{3.4}$$

so F is the X^*-limit of $\{J_X x_n\}$, and the lemma is proved.

4. THEOREM. *If X is a real Banach space, then $K(X)$ is norm-closed in X^{**}.*

Proof. If $F \in \overline{K(X)}$, then there is a sequence $\{F_n\}$ in $K(X)$ such that $F_n \to F$ in norm, and $\|F_n - F_{n-1}\| < 2^{-n}$ for each $n > 1$. If we let $F_0 = 0$, then by Lemma 2 there exists, for each $n \geqq 1$, a sequence $\{x_{nk}\}$ in X such that $\|x_{nk}\| = \|F_n - F_{n-1}\|$ for all k and such that $F_n - F_{n-1}$ is the X^*-limit of $\{J_X x_{nk}\}$.

For each k the series $\sum_{n=1}^{\infty} x_{nk}$ converges to an element $x_k \in X$ such that

$$\left\| x_k - \sum_{n=1}^{j} x_{nk} \right\| < 2^{-j} \text{ for each } j.$$

Given $0 \neq f \in X^*$ and $\varepsilon > 0$, there exist positive integers J and K such that $2^{-J} < \varepsilon/(3\|f\|)$ and $|F_J(f) - f(\sum_{n=1}^{J} x_{nk})| < \varepsilon/3$ for all $k \geqq K$. Hence for $k \geqq K$,

$$\begin{aligned} |F(f) - f(x_k)| \leqq |(F - F_J)(f)| &+ \left| F_J(f) - f(\sum_{n=1}^{J} x_{nk}) \right| \\ &+ \left| f(\sum_{n=1}^{J} x_{nk} - x_k) \right| < \varepsilon , \end{aligned} \tag{4.1}$$

so that F is the X^*-limit of $\{J_X x_k\}$.

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