ON THE NUMBER OF PURE SUBGROUPS

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A problem due to Fuchs [3] is to determine the cardinality of the set \( \mathcal{P} \) of all pure subgroups of an abelian group. Boyer has already given a solution for nondenumerable groups \( G \) [1]; he showed that \( |\mathcal{P}| = 2^{|A|} \) if \( |G| > \aleph_0 \), where \( |A| \) denotes the cardinality of a set \( A \). Our purpose is to complement the results of [1] by determining those groups for which \( |\mathcal{P}| \) is finite, \( \aleph_0 \), and \( c = 2^{\aleph_0} \). In the following, group will mean abelian group.

**Lemma 1.** If \( G \) is a torsion group with \( |G| \leq \aleph_0 \), then \( |\mathcal{P}| = c \) unless

\[
G = p_1^{\omega} \oplus p_2^{\omega} \oplus \cdots \oplus p_n^{\omega} \oplus B,
\]

a direct sum of (at most) a finite number of groups of type \( p^{\omega} \) and a finite group, where \( p_i \neq p_j \) if \( i \neq j \). If \( G \) is of the form (1), then \( |\mathcal{P}| \) is finite.

**Proof.** The latter statements is clear, and if none of the following hold

(i) \( G \) decomposes into an infinite number of summands

(ii) \( G \) contains \( p^{\omega} \oplus p^{\omega} \) for some prime \( p \)

(iii) \( |B| = \aleph_0 \), where \( B \) is the reduced part of \( G \),

then \( G \) is of the form (1). Moreover, if (i) holds, then obviously \( |\mathcal{P}| = c \). Every automorphism of \( p^{\omega} \) determines a pure subgroup of \( p^{\omega} \oplus p^{\omega} \), and distinct automorphisms correspond to distinct subgroups. Since \( |A(p^{\omega})| = \text{automorphism group} | = c \), it follows that \( p^{\omega} \oplus p^{\omega} \) has \( c \) pure subgroups. Thus if (ii) holds, \( |\mathcal{P}| = c \) since \( p^{\omega} \oplus p^{\omega} \) is a direct summand of \( G \). Finally, if (iii) holds and if (i) does not, then the following argument shows that \( |\mathcal{P}| = c \). We may write \( B = C_1 \oplus B_1 = C_1 \oplus C_2 \oplus B_2 \), and continuing in this way define an infinite sequence \( C_* \) of cyclic groups such that no \( C_i \) is contained in the direct sum of any of the others. The direct sum of any subcollection of these cyclic groups is a pure subgroup of \( B \) and, therefore, of \( G \).

An interesting corollary is noted: there is no torsion group with exactly \( \aleph_0 \) pure subgroups.

**Lemma 2.** If \( G = F \oplus B \) is the direct sum of a torsion free group

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\(^1\) This is precisely the proof of Boyer that such a group has \( c \) subgroups [2].
$F$ of rank $r$ and a finite group $B$ with $|G| \leq \aleph_0$, then $|\mathcal{P}|$ is finite, $\aleph_0$, or $c$, depending on whether $r = 1$, $1 < r < \infty$, or $r = \infty$.

**Proof.** First, assume that $B = 0$. Let $H$ be the minimal divisible group containing $G$. The correspondence $D \to D \cap G$ is one-to-one between pure (divisible) subgroups $D$ of $H$ and pure subgroups of $G$. Thus only divisible groups $G$ need be considered, and the proof is already clear except, possibly, the relation $|\mathcal{P}| \leq \aleph_0$ for the case $1 < r < \infty$. However, let $R^*$ denote the direct sum of $r - 1$ copies of $R$, the additive rationals. Since $G = R^* \oplus R$, any pure subgroup $P$ of $G$ is a subdirect sum of a subgroup $S^*$ of $R^*$ and a subgroup $S$ of $R$. Moreover, $S^*$ and $S^* \cap P$ are pure in $R^*$; $S$ and $S \cap P$ are pure in $R$. Since $|A(R)| = \aleph_0$, it follows by induction that $|\mathcal{P}| \leq \aleph_0$.

Now consider the case $B \neq 0$. The lemma has already been proved if $r = \infty$, so assume that $r$ is finite. Any pure subgroup $P$ of $G = F \oplus B$ is a subdirect sum of a pure subgroup $E$ of $F$ and a subgroup $A$ of $B$. Since $E \cap P$ has index in $E$ which divides the order of $B$, there are only a finite number of choices of $E \cap P$ for a given $E$ (and consequently only a finite number of choice of $P$). Thus the lemma is proved.

The theorem follows almost immediately from the lemmas.

**Theorem.** For any group $G$, $|\mathcal{P}| \leq \aleph_0$ if and only if: $G = F \oplus T$ where $T$ is torsion of the form (1) and $F$ is torsion free of finite rank $r \geq 0$; further if the prime $p$ is in the collection $\pi = \{p_1, p_2, \ldots, p_n\}$ of the decomposition (1) of $T$, then $F$ has no pure subgroup which can be mapped homomorphically onto $p^\alpha$. In all other cases, $|\mathcal{P}| = 2^{[\alpha]}$. Moreover, $|\mathcal{P}|$ is finite if and only if either $r = 0$ or $r = 1$ and $T$ is finite.

**Proof.** Suppose that $|\mathcal{P}| \neq 2^{[\alpha]}$. Then $|G| \leq \aleph_0$ and the torsion part $T$ of $G$ is of the form (1). Hence $G$ splits into its torsion and torsion free components, $G = F \oplus T$. Also, $F$ is of finite rank $r \geq 0$. And there exists no homomorphism of a pure subgroup of $F$ onto $p^\alpha$ where $p \in \pi$ (since there would be $c$ such homomorphisms, each determining a pure subgroup of $G$). But suppose that $G = F \oplus T$, where $F$ and $T$ satisfy the given conditions. Let $T'$ denote the divisible part of $T$ and set $F' = F \oplus B$, where $T = T' \oplus B$. Since $B$ is finite, $|\mathcal{P}(F')| \leq \aleph_0$ is given by Lemma 2. Evidently, a pure subgroup $P$ of $G$ is the direct sum of a divisible subgroup of $T'$ and a subdirect sum of a pure subgroup of $F'$ and a finite subgroup of $T'$. Thus $|\mathcal{P}| \leq \aleph_0$.

If $r = 1$, then $|\mathcal{P}(F \oplus p^\alpha)| \geq \aleph_0$, for there are at least $\aleph_0$ homomorphisms of $F$ into $p^\alpha$, each determining a pure subgroup. In view of Lemmas 1 and 2, this completes the proof of the theorem.
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REFERENCES

3. L. Fuchs, Abelian groups, Hungarian Academy of Sciences (1958), Budapest.

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