ON CONFORMAL MAPPING OF NEARLY CIRCULAR REGIONS

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Introduction. A Jordan curve C in the w-plane, starshaped with respect to $w = 0$ and represented in polar coordinates by $\rho(\theta)e^{i\theta}$, will be said to satisfy an ε-condition ($\varepsilon \geqq 0$) if

$$(0.1) \qquad \begin{aligned} &\text{(i)} \quad \rho(\theta) \text{ is absolutely continuous in } \langle -\pi, +\pi \rangle \\ &\text{(ii)} \quad \left| \frac{\rho'}{\rho}(\theta) \right| \leqq \varepsilon \text{ for almost all } \theta \text{ in } \langle -\pi, +\pi \rangle . \end{aligned}$$

Sometimes the condition

$$(0.2) \qquad 1 \leqq \rho(\theta) \leqq 1 + \varepsilon \text{ for all } \theta \text{ in } \langle -\pi, +\pi \rangle$$

will be added.

Let $w = f(z)$ be the conformal mapping of $|z| < 1$ to the interior of C such that $f(0) = 0$, $f'(0) > 0$. Then one can ask: How "close" is $f(z)$ to the identity mapping z? This question has been studied by many authors, notably Marchenko [3] and, more recently, by Warschawski [9—14] and Specht [7]. For example, Marchenko stated:

THEOREM A. *If C satisfies an ε-condition and also* (0.2), *then*

$$(0.3) \qquad |f(z) - z| \leqq K \cdot \varepsilon \qquad (|z| \leqq 1)$$

for a universal constant K.

Furthermore, estimates for $M_p[f(z) - z]$ and $M_p[f'(z) - 1]$ have been given [9] where we write, for example,

$$\|f(z) - z\|_p \equiv M_p[f(z) - z] = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi}) - re^{i\varphi}|^p d\varphi \right\}^{1/p}$$
$$(p > 0; |z| = r < 1) .$$

In this connection, the theorem of M. Riesz [6] on conjugate harmonic functions is of importance.

THEOREM B. *Let $f(z) = u(z) + iv(z)$ be regular in $|z| < 1$ and $v(0) = 0$, so that $v(z)$ is a "normed conjugate" of $u(z)$. Then for every $p > 1$*

$$(0.4) \qquad M_p[v(z)] \leqq A_p M_p[u(z)] \qquad (|z| = r < 1) ,$$

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where A_p is a constant that depends on p only; one can take $A_2 = 1$, $A_p \leqq 2p\,(p \geqq 2)$ and $A_{p'} = A_p$ for $p^{-1} + p'^{-1} = 1$. If the right-hand side of (0.4) *is bounded in $0 \leqq r < 1$, then $f(re^{i\varphi})$ has radial boundary values of class L_p almost everywhere and* (0.4) *holds for $r = 1$.*

In this paper we would like to make a few remarks about Marchenko's theorem and about estimates for $M_p[f'(z) - 1]$. First, we give a new proof of Theorem A which we hope is slightly simpler than Specht's [7] while giving only a slightly larger constant K. Next we ask whether we could replace the condition (0.1.ii) by the assumption of convexity of C and still get (0.3). A counter example is constructed in I.2. Then Specht's method of proof is used to give a localized version of Theorem A, in which the ε-condition is fulfilled only for a part of C.

In the second part of the paper we obtain new estimates for $M_p[f'(z) - 1]$. Their source is a sharp and best possible estimate for $\int_0^{2\pi} [\theta'(\varphi)]^{\pm p} d\varphi$ where $\theta(\varphi) = \arg f(e^{i\varphi})$. It avoids the restriction $\varepsilon < 1$ of Warschawski [9] and gives all values of p for which $M_p[\theta'(\varphi)] < \infty$ or $M_p[f'(z)]$ remains bounded for all $r = |z| < 1$.

PART I

I.1. New proof of Marchenko's theorem. While Specht's approach to Theorem A depends on a suitable integral representation of $\theta(\varphi) - \varphi$ ([7], p. 187), and Warschawski's on an estimate of $M_2[f'(e^{i\varphi}) - 1]$ ([9], p. 566), our proof will depend on a sharp estimate of $M_2[\theta'(\varphi) - 1]$. We shall prove:

THEOREM 1. *If the Jordan curve C satisfies an ε-condition and also* (0.2) *for some $\varepsilon \geqq 0$, then $|f(z) - z| \leqq K(\varepsilon) \cdot \varepsilon$ in $|z| \leqq 1$, where $K(\varepsilon) \leqq 3.7$ for all $\varepsilon \geqq 0$ and* $\lim_{\epsilon \to 0} K(\varepsilon) = \sqrt{1 + \pi^2/3} \sim 2.1$.

Specht's proof yields another function $\bar{K}(\varepsilon)$ with $\bar{K}(\varepsilon) \leqq 3.3$ and $\lim_{\epsilon \to 0} \bar{K}(\varepsilon) = \sqrt{1 + (2 \log 2)^2} \sim 1.7$. The best possible bounds are not known.

In order to prove the theorem, we need the following

LEMMA. *Let $F(x)$ be absolutely continuous in $\langle 0, 2\pi \rangle$, periodic with 2π and $\int_0^{2\pi} F(x)\, dx = 0$, and assume $F'(x) \in L_2(0, 2\pi)$. Then for all x in $\langle 0, 2\pi \rangle$*

$$|F(x)| \leqq \frac{\pi}{\sqrt{3}} \| F'(x) \|_2 . \tag{1.1}$$

This lemma is also used in Friberg's thesis ([2], p. 14 ff). The constant $\frac{\pi}{\sqrt{3}}$ cannot be improved as $F(x) = \frac{x^2}{4} - \frac{\pi}{2}x + \frac{\pi^2}{6} (0 \leqq x \leqq 2\pi)$ shows.

Proof. It suffices to estimate $F(0)$. For that we expand $F(x)$ in its Fourier series $F(x) = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ and get

$$|F(0)| = |\sum_1^{\infty} a_n| \leqq \sum_1^{\infty} |a_n| \cdot n \frac{1}{n} \leqq [\sum_1^{\infty} n^2 a_n^2]^{1/2} [\sum_1^{\infty} n^{-2}]^{1/2} .$$

The first factor is at most $[\sum_1^{\infty} n^2(a_n^2 + b_n^2)]^{1/2} = \sqrt{2}\, \|F'(x)\|_2$, by Parseval's equality, the second is $\pi/\sqrt{6}$.

Proof of the theorem. Putting $f(e^{i\varphi}) = \rho(\theta)\, e^{i\theta}$, $\theta = \theta(\varphi)$, we first estimate $|\theta(\varphi) - \varphi|$ if ε is assumed to be < 1. By the lemma, it is sufficient to estimate $\|\theta'(\varphi) - 1\|_2$. To do this, we note that $\log(f(z)/z) = u(z) + iv(z)$ is regular in $|z| < 1$, continuous in $|z| \leqq 1$, and $v(0) = 0$ since $f'(0) > 0$, so that v is a normed conjugate of u: $v = K[u]$. On $|z| = 1$ this gives

$$\theta(\varphi) - \varphi = K[\log \rho(\theta(\varphi))],\ \theta(\varphi + h) - (\varphi + h) = K[\log \rho(\theta(\varphi + h))] \tag{1.2}$$

and hence

$$\frac{\theta(\varphi + h) - \theta(\varphi)}{h} - 1 = K\Big[\frac{\log \rho(\theta(\varphi + h)) - \log \rho(\theta(\varphi))}{h}\Big] .$$

By (0.1), for all φ and $h > 0$

$$|\log \rho(\theta(\varphi + h)) - \log \rho(\theta(\varphi))| = \left|\int_{\theta(\varphi)}^{\theta(\varphi+h)} \frac{\rho'}{\rho}(t)dt\right| \leqq \varepsilon\, |\theta(\varphi + h) - \theta(\varphi)| ,$$

and therefore

$$\left\|\frac{\theta(\varphi + h) - \theta(\varphi)}{h} - 1\right\|_2 = \|[\quad]\|_2 \leqq \varepsilon \left\|\frac{\theta(\varphi + h) - \theta(\varphi)}{h}\right\|_2 . \tag{1.3}$$

Now we claim that

$$\left\|\frac{\theta(\varphi + h) - \theta(\varphi)}{h} - 1\right\|_2^2 = \left\|\frac{\theta(\varphi + h) - \theta(\varphi)}{h}\right\|_2^2 - 1 . \tag{1.4}$$

To show this,we write the left-hand side as

$$\frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{\theta(\varphi + h) - \theta(\varphi)}{h} - 1\Big]^2 d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{\theta(\varphi + h) - \theta(\varphi)}{h}\Big]^2 d\varphi + 1$$

$$- 2\frac{1}{2\pi h}\int_0^{2\pi}\{[\theta(\varphi + h) - (\varphi + h)] - [\theta(\varphi) - \varphi] + h\}d\varphi ,$$

Since $\theta(\varphi)-\varphi$ is periodic with 2π, the last term is -2, and (1.4) follows. Together with (1.3) we get $\|[\theta(\varphi+h)-\theta(\varphi)]/h-1\|_2^2 \leqq \varepsilon^2/(1-\varepsilon^2)$ for all $h>0$. But since C is rectifiable, $\theta(\varphi)$ is absolutely continuous [5] and hence $\theta'(\varphi)$ exists almost everywhere, and Fatou's lemma yields now for $h \to 0$

$$\|\theta'(\varphi)-1\|_2 \leqq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}. \tag{1.5}$$

This, incidentally, is a best possible estimate; see Theorem 6.

Now we apply our lemma to $F(\varphi)=\theta(\varphi)-\varphi$, the condition $\int_0^{2\pi} F(\varphi)d\varphi=0$ following from (1.2), and we get for all φ

$$|\theta(\varphi)-\varphi| \leqq \frac{\pi}{\sqrt{3}}\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}. \tag{1.6}$$ [2]

From this we obtain an estimate of $|f(z)-z|$. An elementary consideration gives

$$|f(z)-z|^2 \leqq \varepsilon^2+(1+\varepsilon)[\theta(\varphi)-\varphi]^2 \text{ on } |z|=1; \tag{1.7}$$

note that $1 \leqq |f(e^{i\varphi})| \leqq 1+\varepsilon$. Together with (1.6) we obtain

$$|f(z)-z| \leqq \varepsilon\left\{1+\frac{\pi^2}{3(1-\varepsilon)}\right\}^{1/2} \tag{1.8}$$

for $|z|=1$ and hence, by the maximum principle, for $|z| \leqq 1$; this is valid whenever $\varepsilon<1$. For all $\varepsilon \leqq 20/27$ the factor of ε is $\leqq 3.7$; for $\varepsilon > 20/27$ we have

$$|f(z)-z| \leqq 1+\varepsilon+1=2+\varepsilon<\frac{54}{20}\varepsilon+\varepsilon=3.7\,\varepsilon.$$

This proves $K(\varepsilon) \leqq 3.7$ for all $\varepsilon \geqq 0$, and (1.8) gives $\lim_{\varepsilon\to 0} K(\varepsilon)=\sqrt{1+\pi^2/3}$.

Specht ([7], p. 188) obtains $|\theta(\varphi)-\varphi| \leqq \varepsilon(2\log 2+\varepsilon)$. Combining this for $\varepsilon \leqq 0.9$ with $|f(z)-z| \leqq \varepsilon+|\theta(\varphi)-\varphi|$ ($|z|=1$) and taking $|f(z)-z| \leqq 2+\varepsilon$ for $\varepsilon>0.9$, one obtains $\bar{K}(\varepsilon) \leqq 3.3$ for all $\varepsilon>0$; for $\varepsilon \to 0$ use (1.7).

I.2. Convex regions. Our next problem is to decide whether Marchenko's theorem remains valid if the condition $|\rho'/\rho| \leqq \varepsilon$ is replaced by the convexity of C. To study a suitable counter example, it will be convenient to use the following localization theorem.

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[1] This also follows directly from $\|\theta'\|_2 \leqq (1-\varepsilon^2)^{-1/2}$([8], p. 26) and $\|\theta'-1\|_2^2=\|\theta'\|_2^2-1$, but we wanted to give an independent proof of (1.5).

[2] The application of Warschawski's inequality ([8], p. 18) would have given a slightly larger bound for K in Theorem 1.

THEOREM 2. *Let C: $\rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w = 0$ and contained in $1 \leqq |w| \leqq 1 + \varepsilon$, and let $w = f(z)$ with $f(0) = 0$, $f'(0) > 0$ map $|z| < 1$ conformally to the interior of C; put $\theta(\varphi) = arg\ f(e^{i\varphi})$.*

Then to every δ, $0 < \delta < \pi$, corresponds a constant $D = D(\delta)$ such that

$$(1.9) \qquad \left| [\theta(\varphi) - \varphi] - \frac{1}{2\pi}\int_{\varphi-\delta}^{\varphi+\delta} \log \rho(\theta(t)) \operatorname{ctg} \frac{\varphi - t}{2}\, dt \right| \leqq D \cdot \varepsilon$$

for all φ, the integral being a Cauchy principal value.

Proof. Since $\theta(\varphi) - \varphi$ is a normed conjugate of $\log \rho(\theta(\varphi))$ (see (1.2)), we have

$$\theta(\varphi) - \varphi = \frac{1}{2\pi}\int_{\varphi-\pi}^{\varphi+\pi} \log \rho(\theta(t)) \operatorname{ctg} \frac{\varphi - t}{2} dt = \frac{1}{2\pi}\int_{\varphi-\delta}^{\varphi+\delta} + \frac{1}{2\pi}\left[\int_{\varphi-\pi}^{\varphi-\delta} + \int_{\varphi+\delta}^{\varphi+\pi}\right].$$

In the last term $|\operatorname{ctg}[(\varphi - t)/2]|$ is bounded by $\operatorname{ctg}[\delta/2]$ while $0 \leqq \log \rho(\theta(t)) \leqq \varepsilon$. This proves (1.9) with $D(\delta) = \operatorname{ctg}[\delta/2]$.

Furthermore, we shall use another theorem of Marchenko ([3], p. 289) which, in the generalization by Warschawski ([10], p. 343), reads as follows. Let R be a simply connected region containing $w = 0$ whose boundary is contained in $1 \leqq |w| \leqq 1 + \varepsilon$. Let λ be such that any two points in R with distance $< \varepsilon$ may be connected in R by an arc of diameter $< \lambda$. *If $f(z)$ is the normalized mapping of $|z| < 1$ to R, then*

$$(1.10) \qquad |f(z) - z| \leqq M\varepsilon |\log \varepsilon| + M_1\lambda$$

for two absolute constants M and M_1. Ferrand ([1], p. 133) states without proof that one can take $M = 1/\pi$ as the best possible constant; note that in her paper the boundary is assumed to be in $1 - \varepsilon \leqq |w| \leqq 1 + \varepsilon$.-Obviously $\lambda \leqq 3\varepsilon$ if R is starshaped with respect to $w = 0$.

Now we shall study the following family of conformal maps. Let the Jordan curve $C = C(\varepsilon)(0 < \varepsilon < 1/2)$ be defined as follows:

$$|w| = 1 \qquad \text{if } -\pi \leqq \arg w \leqq 0\,,$$

$$|w| = 1 + \varepsilon \quad \text{if } 0 < \theta_2 \leqq \arg w \leqq \frac{\pi}{2} + \kappa, \text{ where } 0 < \kappa < \frac{\pi}{2} \text{ and}$$

$$\sin \kappa = 1/(1 + \varepsilon)\,,$$

and where these two circular arcs are connected by straight line segments. The angle θ_2 will also depend on ε and is subject to

(1.11) $$\theta_2 \to 0 \text{ and } \frac{\theta_2}{\varepsilon\,|\log\varepsilon|} \to +\infty \quad (\varepsilon \to 0)\,.$$

Let $w = f(z)$ map $|z| < 1$ to the interior of C with $f(0) = 0$, $f(0) > 0$ and let

$$f(e^{i\varphi_1}) = 1 = e^{i\theta_1},\ f(e^{i\varphi_2}) = (1+\varepsilon)e^{i\theta_2}\,.$$

By (1.10) we have for all φ and ε

(1.12) $$|\theta(\varphi) - \varphi| \leqq M\varepsilon\,|\log\varepsilon| + O(\varepsilon)\,,$$

in particular $\varphi_1 \to 0$, $\varphi_2 \to 0(\varepsilon \to 0)$. We therefore get from Theorem 2

$$\theta(\varphi_1) - \varphi_1 = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} \log \rho(\theta(t)) \operatorname{ctg} \frac{\varphi_1 - t}{2}\, dt + O(\varepsilon)\,,$$

$$|\theta(\varphi_1) - \varphi_1| = \frac{1}{2\pi}\int_{\varphi_1}^{+\pi/2} \log \rho(\theta(t)) \operatorname{ctg} \frac{t - \varphi_1}{2}\, dt + O(\varepsilon) > \frac{1}{2\pi}\int_{\varphi_2}^{+\pi/2} + O(\varepsilon)\,;$$

note that $\rho(\theta(t)) = 1$ for t in $< \pi/2, \varphi_1 >$. The last integral is equal to

$$\log(1+\varepsilon)\int_{\varphi_2}^{+\pi/2} \operatorname{ctg} \frac{t - \varphi_1}{2}\, dt = 2\,|\log(\varphi_2 - \varphi_1)|\,\varepsilon + O(\varepsilon)\,.$$

Here we have by (1.11) and (1.12)

$$\varphi_2 - \varphi_1 = \theta_2 - \theta_1 + O(\varepsilon\,|\log\varepsilon|) = (\theta_2 - \theta_1)(1 + o(1)) = \theta_2(1 + o(1))\,,$$

so that altogether we obtain

(1.13) $$|\theta(\varphi_1) - \varphi_1| > \frac{|\log\theta_2|}{\pi}\,\varepsilon + O(\varepsilon) \qquad (\varepsilon \to 0)\,.$$

Before we specialize (1.13), we remark that for the regions considered here

(1.14) $$|f(z) - z| = |\theta(\varphi) - \varphi| + O(\varepsilon) \qquad (z = e^{i\varphi})\,.$$

We have namely on $|z| = 1$

$$2\sin\frac{\theta(\varphi) - \varphi}{2} \leqq |f(z) - z| \leqq 2\sin\frac{\theta(\varphi) - \varphi}{2} + (|f(z)| - 1)\,.$$

By (1.12), $|\theta(\varphi) - \varphi| = O(\varepsilon\,|\log\varepsilon|)$ and (1.14) follows.

We now make two special choices of $\theta_2 = \theta_2(\varepsilon)$, always subject to (1.11). For our first choice $\theta_2(\varepsilon) = \varepsilon\,|\log\varepsilon|^2$ we obtain from (1.13) and (1.14)

$$|f(z) - z| \geqq \frac{\varepsilon\,|\log \varepsilon|}{\pi}(1 + o(1)) \qquad (z = e^{i\varphi_1};\ \varepsilon \to 0)\ .$$

Thus we proved that the best constant M in (1.10) *must satisfy $M \geqq 1/\pi$*, in agreement with Ferrand.

Next we choose θ_2 such that $1 = (1 + \varepsilon)\cos\theta_2$, which makes $C(\varepsilon)$ convex. If we insert $\theta_2 = \sqrt{2\varepsilon} + O(\varepsilon)$ in (1.13), we obtain

$$|f(z) - z| \geqq \frac{\varepsilon\,|\log \varepsilon|}{2\pi}(1 + o(1)) \qquad (z = e^{i\varphi_1};\ \varepsilon \to 0)\ .$$

THEOREM 3. *If $C(\varepsilon)$ is the family of convex curves defined by cos $\theta_2(\varepsilon) = [1/1 + \varepsilon]$, we have*

$$\max_{|z|=1} |f(z) - z| \geqq \frac{\varepsilon\,|\log \varepsilon|}{2\pi}(1 + o(1)) \qquad (\varepsilon \to 0)\ .$$

In particular, Theorem A does not hold if the condition $|\rho'(\theta)/\rho(\theta)| \leqq \varepsilon$ is replaced by the convexity of C.

I.3. Localization of the theorem of Marchenko. In I.1 we have seen that Theorem A can be proved with a quite satisfactory constant K by a "global" method, a method involving means rather than the function itself. Nevertheless, Specht's proof of Theorem A, directly aiming at $|\theta(\varphi) - \varphi|$, has besides giving a slightly better constant the advantage of being useful to obtain a localization of Theorem A, where $|[\rho'/\rho]| \leqq \varepsilon$ is known only for a part of C.

We begin with the following localization of Specht's representation theorem ([7], p. 187).

THEOREM 4. *Let $C : \rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w = 0$ which satisfies*:

(i) $1 \leqq \rho(\theta) \leqq 1 + \varepsilon$ *for all θ and some $\varepsilon \geqq 0$*;

(ii) *$\rho(\theta)$ has bounded difference quotients for θ in $\langle a, b\rangle$.*

Then to every $\delta > 0$ corresponds an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a constant $N(\delta)$ such that for $\varepsilon < \varepsilon_0$ we have

$$\left|[\theta(\varphi) - \varphi] - \frac{1}{\pi}\int_a^b \log\left|\sin\frac{t(\theta) - \varphi}{2}\right| \frac{\rho'(\theta)}{\rho(\theta)}\,d\theta\right| \leqq N(\delta)\cdot\varepsilon\ , \tag{1.15}$$

for all φ in $\langle a + \delta,\ b - \delta\rangle$ for which $\theta'(\varphi)$ and $\rho'(\theta(\varphi))$ exist and $\theta'(\varphi) \neq 0$, i.e. for almost all φ in $\langle a + \delta,\ b - \delta\rangle$.

Here $t = t(\theta)$ is the inverse function of $\theta(t)$, and the integral exists as a Lebesgue integral.

Proof. For our fixed $\delta > 0$, choose $\varepsilon_0(\delta)$ such that $\alpha = \theta^{-1}(a)$ and $\beta = \theta^{-1}(b)$ satisfy $|\alpha - a| < \delta/2, |\beta - b| < \delta/2$; this is asserted by (1.10) or (1.12) as soon as $\varepsilon < \varepsilon_0$. Then we can write

$$\theta(\varphi) - \varphi = \frac{1}{2\pi}\int_{\varphi-\pi}^{\varphi+\pi}[\log \rho(\theta(t)) - \log \rho(\theta(\varphi))] \operatorname{ctg}\frac{\varphi - t}{2}\,dt$$
$$= \frac{1}{2\pi}\int_{\alpha}^{\beta}[\quad] \operatorname{ctg}\frac{\varphi - t}{2}\,dt + O(\varepsilon);$$

compare Theorem 2. Now one applies partial integration to the integral as in Specht's proof, and (1.15) follows.

Now we can prove the following localization of Theorem A.

THEOREM 5. *Let $C : \rho(\theta)e^{i\theta}$ be a rectifiable Jordan curve, starshaped with respect to $w = 0$ which satisfies*

(i) $1 \leqq \rho(\theta) \leqq 1 + \varepsilon$ *for all* θ *and some* $\varepsilon \geqq 0$,
(ii) $|\rho(\theta + \tau) - \rho(\theta)| \leqq \rho(\theta)|\tau|\varepsilon$ *for all* θ *in* $\langle a, b\rangle$ *and all real* τ.
Then to every $\delta > 0$ corresponds a constant $K_1(\delta)$ such that

$$(1.16)\qquad |f(z) - z| \leqq K_1(\delta)\cdot\varepsilon \text{ for } z = e^{i\varphi},\ \varphi \text{ in } \langle a + \delta,\ b - \delta\rangle\,.$$

Proof. It suffices to prove this for small ε. Condition (ii) implies that we can estimate the integral term in (1.15) by

$$(1.17)\qquad \varepsilon\left|\frac{1}{\pi}\int_a^b \log\left|\sin\frac{t(\theta) - \varphi}{2}\right|d\theta\right| \leqq -\varepsilon\frac{1}{\pi}\int_{\varphi-\pi}^{\varphi+\pi}\log\left|\sin\frac{t - \varphi}{2}\right|\theta'(t)dt$$
$$\leqq \varepsilon(2\log 2 + \varepsilon)$$

(see [7], p. 188). Hence $|\theta(\varphi) - \varphi| \leqq K_2(\delta)\cdot\varepsilon$ for almost all φ in $\langle a + \delta,\ b - \delta\rangle$. By continuity, this holds for all φ in $\langle a + \delta,\ b - \delta\rangle$, and (1.16) follows.

REMARK. By a simple approximation argument it is seen that the rectifiability of C, needed for the last inequality in (1.17), is not necessary for the validity of Theorem 5.

PART II

II. 1. Sharp estimates for the means of $\theta'(\varphi)$. Our aim is now to give an estimate for $M_p[f'(z) - 1]$ if C satisfies an ε-condition. As a first step we prove the following

THEOREM 6. *Let $C : \rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w = 0$, which satisfies an ε-condition for some $\varepsilon \geqq 0$, and* *let* $w = f(z)$ *with* $f(0) = 0$ *map* $|z| < 1$ *conformally to the interior of* C. *Then* $\theta(\varphi) = \arg f(e^{i\varphi})$ *satisfies*

$$\int_0^{2\pi} [\theta'(\varphi)]^{\pm p} d\varphi < \infty \text{ if } 0 \leqq p < \frac{\pi}{2 \operatorname{arctg} \varepsilon}. \tag{2.1}$$

More precisely, we have

$$\frac{1}{2\pi}\int_0^{2\pi} [\theta'(\varphi)]^p \, d\varphi \begin{cases} \leqq \dfrac{(\cos \operatorname{arctg} \varepsilon)^p}{\cos(p \operatorname{arctg} \varepsilon)} & \text{if } 1 \leqq p < \dfrac{\pi}{2 \operatorname{arctg} \varepsilon} \\ \leqq 1 & \text{if } 0 \leqq p \leqq 1 \end{cases} \tag{2.2}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} [\theta'(\varphi)]^{-p} \, d\varphi \leqq \frac{1}{(\cos \operatorname{arctg} \varepsilon)^p \cos(p \operatorname{arctg} \varepsilon)} \quad \text{if } 0 \leqq p < \frac{\pi}{2 \operatorname{arctg} \varepsilon}. \tag{2.3}$$

Moreover, the bounds in (2.2) *and* (2.3), *as well as the upper bound for* p *in* (2.1), *are best possible.*

REMARKS. a. It easily follows from F. Riesz's theorem ([5], p. 95), that not only $\theta = \theta(\varphi)$ but also its inverse $\varphi = \varphi(\theta)$ is an absolutely continuous and monotonically increasing function, whenever C satisfies an ε-condition for some $\varepsilon \geqq 0$. The substitution $\varphi = \varphi(\theta)$ in (2.1) is therefore permissible[3] and gives

$$\frac{1}{2\pi}\int_0^{2\pi} [\theta'(\varphi)]^{\pm p} d\varphi = \frac{1}{2\pi}\int_0^{2\pi} [\varphi'(\theta)]^{1 \mp p} d\theta \text{ if } 0 \leqq p < \frac{\pi}{2 \operatorname{arctg} \varepsilon},$$

so that (2.2) and (2.3) contain also estimates for the means of $\varphi'(\theta)$. In particular, since $\pi/(2 \operatorname{arctg} \varepsilon) > 1$, (2.3) is always applicable for $p = 1$, and *we obtain that* $\varphi'(\theta) \in L_2$ *whenever* C *satisfies an* ε*-condition for some* $\varepsilon \geqq 0$. (For $\theta'(\varphi) \in L_2$ we need $\varepsilon < 1$.)

b. For $p = 2$ the bounds in (2.2) and (2.3) become $(1 - \varepsilon^2)^{-1}$ (see [8], p. 26) and $(1 + \varepsilon^2)^2/(1 - \varepsilon^2)$.

Proof. We begin with three preliminary remarks. First, we have $|\rho'(\theta)/\rho(\theta)| \leqq \varepsilon$ for *all* θ, for which $\rho'(\theta)$ exists. For by (0.1)

$$|\log \rho(\theta + h) - \log \rho(\theta)| = \left|\int_\theta^{\theta+h} \frac{\rho'(t)}{\rho(t)} dt\right| \leqq \varepsilon |h|,$$

for all θ and $h \neq 0$; this implies our proposition.

[4] See C. Caratheodory, Vorlesungen über reelze Funktionen, Leipzig und Berlin, 1927, pages 563 and 556.

Next, since C is rectifiable, we know by F. Riesz's theorem ([5], p. 95; see also [16], p. 157 ff.) that

(i) $f(e^{i\varphi})$ is absolutely continuous, so that $[df(e^{i\varphi})/de^{i\varphi}]$ exists almost everywhere and is integrable; furthermore

(ii) $f'(z) \in H_1$, i.e. $\int_0^{2\pi} |f'(re^{i\varphi})|\, d\varphi \leqq A < \infty$ for all $r < 1$.

We claim that

$$f'(re^{i\varphi}) \to \frac{df(e^{i\varphi})}{de^{i\varphi}} \text{ as } r \to 1, \text{ for almost all } \varphi . \tag{2.4}$$

To prove this, let $f'(re^{i\varphi}) \to h(e^{i\varphi})(r \to 1$, almost all $\varphi)$, so that by (ii) $h(e^{i\varphi})$ is integrable and $\int_0^{2\pi} |f'(re^{i\varphi}) - h(e^{i\varphi})|\, d\varphi \to 0 (r \to 1)$. Therefore, for any fixed φ_0,

$$[f(re^{i\varphi_0}) - f(r)] - r\int_0^{\varphi_0} h(e^{i\varphi}) ie^{i\varphi} d\varphi = r\int_0^{\varphi_0} [f'(re^{i\varphi}) - h(e^{i\varphi})] ie^{i\varphi} d\varphi \to 0 \ (r \to 1)$$

that is

$$f(e^{i\varphi_0}) = f(1) + \int_0^{\varphi_0} h(e^{i\varphi}) ie^{i\varphi} d\varphi .$$

Differentiation yields $[df(e^{i\varphi})/de^{i\varphi}] = h(e^{i\varphi})$ almost everywhere, which is (2.4). From now on we shall put $[df(e^{i\varphi})/de^{i\varphi}] = f'(e^{i\varphi})$ whenever this exists.

Finally, since $f' \in H_1$ and $f' \not\equiv 0$, one knows (see, e.g., [4], p. 56) that $f'(e^{i\varphi})$ vanishes only on a null set.

To start the proof of theorem, let M be the set of all φ in $\langle 0, 2\pi \rangle$ for which (i) $f'(e^{i\varphi})$ exists and is $\neq 0$ and (ii) $\lim_{r \to 1} f'(re^{i\varphi}) = f'(e^{i\varphi})$; by our above remarks, M is of measure 2π.

We consider now the function $g(z) = zf'(z)/f(z)$, regular and $\neq 0$ in $|z| < 1$, $g(0) = 1$, and put

$$F(z) = \log\, g(z) = \log\, |g(z)| + i \arg\, g(z) = u(z) + iv(z) ,$$

which is regular in $|z| < 1$ and vanishes at $z = 0$. We study $u(z)$, $v(z)$ foy $|z| \to 1$.

(a) Since

$$\frac{zf'(z)}{f(z)} = \left\{1 - i\,\frac{\rho'(\theta(\varphi))}{\rho\,(\theta(\varphi))}\right\} \theta'(\varphi) \qquad (z = e^{i\varphi},\ \varphi \in M) \tag{2.5}$$

we have $\theta'(\varphi) \neq 0 (\varphi \in M)$ and furthermore

$$|g(re^{i\varphi})| \to \theta'(\varphi) \left|1 + \frac{\rho'}{\rho} i \right| = \frac{\theta'(\varphi)}{\cos \beta(\theta(\varphi))} \qquad (r \to 1,\ \varphi \in M)$$

where $\beta(\theta)$ denotes the angle between $\arg w = \theta$ and the normal to C at $(\rho(\theta), \theta)$. Hence

$$u(re^{i\varphi}) \to \log \frac{\theta'(\varphi)}{\cos \beta(\theta(\varphi))} = u(e^{i\varphi}) \qquad (r \to 1,\ \varphi \in M) .$$

(b) On the other hand we have for $v(z)$

$$\begin{aligned} v(re^{i\varphi}) &= \arg g(re^{i\varphi}) \to \arg f'(e^{i\varphi}) + \varphi - \arg f(e^{i\varphi}) \\ &= \beta(\theta(\varphi)) = v(e^{i\varphi}) \qquad (r \to 1,\ \varphi \in M) . \end{aligned}$$

In particular, $\beta(\theta)$ exists for $\theta = \theta(\varphi)$, $\varphi \in M$, and hence by our first remark $|\beta(\theta(\varphi))| \leqq \operatorname{arctg} \varepsilon(\varphi \in M)$.

(c) This implies that $|v(re^{i\varphi})| \leqq \operatorname{arctg} \varepsilon$ for $r < 1$. For $v(re^{i\varphi})$ is harmonic in $r < 1$ and clearly represents the angle between $\arg w = \theta$ and the normal to the level curve corresponding to $|z| = r$, which is again starshaped. Thus $|v(re^{i\varphi})| < \pi/2$, and $v(re^{i\varphi})$ can therefore be represented by its Poisson integral in $r < 1$. Since the boundary values are $\leqq \operatorname{arctg} \varepsilon$, also $|v(re^{i\varphi})| \leqq \operatorname{arctg} \varepsilon(r < 1)$.

For the main part of the proof, we apply a method of Zygmund ([15], p. 286). Let $p > 0$ and consider

$$(2.6) \quad 1 = e^{\pm pF(0)} = \frac{1}{2\pi i} \int_{|z|=r<1} \frac{e^{\pm pF(z)}}{z} dz = \frac{1}{2\pi} \int_{|z|=r<1} e^{\pm pu(z)} \cos[pv(z)] d\varphi .$$

By (c) and our assumption on p, we have $|pv(z)| \leqq p \operatorname{arctg} \varepsilon < \pi/2$, so that the integrand in the last integral is positive for all $r < 1$ and φ. Recalling (a) and (b), an application of Fatou's lemma yields

$$\frac{1}{2\pi} \int_M e^{\pm pu(e^{i\varphi})} \cos[pv(e^{i\varphi})] d\varphi \leqq 1$$

that is

$$(2.7) \qquad \frac{1}{2\pi} \int_M [\theta'(\varphi)]^{\pm p} \frac{\cos[p\beta(\theta(\varphi))]}{[\cos \beta(\theta(\varphi))]^{\pm p}} d\varphi \leqq 1 .$$

Now we note that $|\beta(\theta(\varphi))| \leqq \operatorname{arctg} \varepsilon(\varphi \in M)$, and the fact that

$$\frac{\cos px}{(\cos x)^p} \text{ is monotonically } \begin{array}{l} \text{decreasing in } 0 \leqq x < \pi/2p \text{ if } p > 1 \\ \text{increasing in } 0 \leqq x < \pi/2 \text{ if } 0 < p < 1 . \end{array}$$

This proves our estimates (2.2) and (2.3)

We now show that our bounds are best possible. More precisely: *For every $\varepsilon \geqq 0$ and for every p with $0 \leqq p < \pi/(2 \operatorname{arctg} \varepsilon)$, there exists a curve C such that Theorem 6 holds with equality in (2.2) and (2.3), respectively.*

For $\varepsilon = 0$, and for $\varepsilon > 0$, $0 \leqq p \leqq 1$ in (2.2), we simply let C be

$|w| = 1$, $\theta'(\varphi) \equiv 1$. For $\varepsilon > 0$ and the other two cases in (2.2) and (2.3) we consider the curve $C : \rho(\theta) = e^{\varepsilon|\theta|}(|\theta|) \leqq \pi)$, which is composed of two pieces of logarithmic spirals that meet in $w = 1$ and $w = -e^{\varepsilon\pi}$. Let $f(z)$ be such that $f(1) = 1$ and $f(-1) = -e^{\varepsilon\pi}$.

We claim that for this mapping we have equality in (2.7) whenever $0 \leqq p < \pi/(2 \operatorname{arctg} \varepsilon)$. Since $tg\, \beta(\theta(\varphi)) = \pm \varepsilon$ for all $\varphi \neq 0, \pi$, this would immediately give equality in (2.2) and (2.3).

To prove equality in (2.7), we study the behaviour of $f'(z)$ in $|z| < 1$. The curve C is composed of two analytic arcs meeting at angles $\alpha_1\pi$ and $\alpha_2\pi$ with $\alpha_1 = 1 + [2/\pi] \operatorname{arctg} \varepsilon$ and $\alpha_2 = 1 - [2/\pi] \operatorname{arctg} \varepsilon$. By a theorem of Warschawski ([13], p. 835), we have therefore

$$(2.8) \qquad f'(z)(z-1)^{-(2/\pi)\operatorname{arctg}\varepsilon} \to C_1 \neq 0 (z \to 1)$$
$$\text{and } f'(z)(z+1)^{+(2/\pi)\operatorname{arctg}\varepsilon} \to C_2 \neq 0 (z \to -1)\,,$$

for unrestricted approach within $|z| < 1$. Thus,

$$f'(z)(z+1)^{(2/\pi)\operatorname{arctg}\varepsilon} \text{ and } [f'(z)]^{-1}(z-1)^{(2/\pi)\operatorname{arctg}\varepsilon}$$

are continuous in $|z| \leqq 1$, and we have for $re^{i\varphi}$, $0 \leqq r < 1$, $0 < |\varphi| < \pi$,

$$|f'(re^{i\varphi})| \leqq \frac{\text{const}}{(\pi - |\varphi|)^{(2/\pi)\operatorname{arctg}\varepsilon}} \text{ and } |f'(re^{i\varphi})|^{-1} \leqq \frac{\text{const}}{|\varphi|^{(2/\pi)\operatorname{arctg}\varepsilon}}\,.$$

Therefore, if $2p \operatorname{arctg} \varepsilon < \pi$, $\exp\{\pm pu(re^{i\varphi})\} = |g(re^{i\varphi})|^{\pm p}$ is bounded by an integrable function, uniformly for all r in $0 \leqq r < 1$, so that Lebesgue's convergence theorem can be applied to (2.6) for $r \to 1$, giving equality in (2.7).

Finally, also the bound on p is best possible. For this we simply note that by (2.5) and (2.8)

$$[\theta'(\varphi)]^{-1}|\varphi|^{+(2/\pi)\operatorname{arctg}\varepsilon} \geqq D_1 > 0 \text{ near } \varphi = 0$$
$$\text{and } \theta'(\varphi) \cdot (\pi - |\varphi|)^{+(2/\pi)\operatorname{arctg}\varepsilon} \geqq D_2 > 0$$

near $\varphi = \pi$, so that for $p = \pi/(2 \operatorname{arctg} \varepsilon)$ the functions $[\theta'(\varphi)]^p$ and $[\theta'(\varphi)]^{-p}$ are not integrable.

COROLLARY. *Under the assumptions of Theorem* 6, *we have for* $0 \leqq r < 1$

$$(2.9) \quad \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\varphi})|^{\pm p} d\varphi \leqq \frac{\max [\rho(\theta)]^{\pm p}}{\cos (p \operatorname{arctg} \varepsilon)} \; \textit{if } 0 \leqq p < \frac{\pi}{2 \operatorname{arctg} \varepsilon}\text{[4]}.$$

For $p = \pi/(2 \operatorname{arctg} \varepsilon)$, *the left-hand side need not be uniformly bounded in* $0 \leqq r < 1$.

[4] See also a similar estimate for smooth curves ([**11**], p. 254).

For the proof we note that by (2.6)

$$1=\frac{1}{2\pi}\int_{|z|=r<1}|g(z)|^{\pm p}\cos[p\,v(z)]\,d\varphi\,.$$

Recalling $|p\,v(z)|\leqq p\operatorname{arctg}\varepsilon$ and that $|z/f(z)|^{\pm p}$ assumes its minimum for $|z|=1$, we arrive at (2.9).

II.2. An estimate for $M_p[f'(z)-1]$. Theorem 6 enables us to derive an estimate for the mean of $f'(z)-1$, which is small for small ε.

THEOREM 7. *Let $C:\rho(\theta)e^{i\theta}$ be a Jordan curve, starshaped with respect to $w=0$, which satisfies an ε-condition and which lies in the ring $1\leqq|w|\leqq1+\varepsilon$ for some $\varepsilon\geqq0$. Let $w=f(z)$ with $f(0)=0$, $f'(0)>0$ map $|z|<1$ conformally to the interior of C. Then we have for all r with $0\leqq r<1$*

$$\text{(2.10)}\quad M_p[f'(re^{i\varphi})-1]\leqq\left\{(1+\varepsilon)\frac{\cos\operatorname{arctg}\varepsilon}{[\cos(p\operatorname{arctg}\varepsilon)]^{1/p}}+e^{\varepsilon}\right\}(1+A_p)\cdot\varepsilon$$
$$\textit{if}\ 1<p<\frac{\pi}{2\operatorname{arctg}\varepsilon}\,,$$

where A_p denotes the constant in Riesz's Theorem B. The upper bound for p is best possible[5].

This improves a theorem of Warschawski ([9], p. 566) with respect to the restrictions on ε and p.

Proof. We first estimate $M_p[\{z\,f'(z)/f(z)\}-1]$ (see [9], p. 565) and write by (2.5)

$$\frac{z\,f'(z)}{f(z)}-1=(\theta'(\varphi)-1)-i\,\frac{\rho'(\theta)}{\rho(\theta)}\,\theta'(\varphi)\qquad(z=e^{i\varphi})\,.$$

Since the left-hand side vanishes at $z=0$, Riesz's theorem gives

$$M_p[\theta'(\varphi)-1)\leqq A_pM_p\left[\frac{\rho'(\theta(\varphi))}{\rho(\theta(\varphi))}\,\theta'(\varphi)\right]\leqq A_pM_p[\theta'(\varphi)]\cdot\varepsilon\,.$$

With (2.2) and Minkowski's inequality we obtain

$$\text{(2.11)}\qquad M_p\left[\frac{e^{i\varphi}f'(e^{i\varphi})}{f(e^{i\varphi})}-1\right]\leqq(1+A_p)M_p[\theta'(\varphi)]\cdot\varepsilon$$
$$\leqq(1+A_p)\frac{\cos\operatorname{arctg}\varepsilon}{[\cos(p\operatorname{arctg}\varepsilon)]^{1/p}}\cdot\varepsilon\,.$$

[5] For $0<p\leqq1$ an estimate can be obtained by an application of Hölder's inequality.

Next, we use the estimate

$$M_p[f'(z) - 1] \leqq (1 + \varepsilon)M_p\left[\frac{zf'(z)}{f(z)} - 1\right] + M_p\left[\frac{f(z)}{z} - 1\right] \quad (|z| = r < 1),$$

where the last term is $\leqq (1 + A_p)e^{\varepsilon}\ \varepsilon$; see [9], p. 564–566. Combining this with (2.11) and using the monotonicity of $M_p[\{zf'(z)/f(z)\} - 1]$ with respect to r, we arrive at (2.10).

For $p = \pi/(2 \operatorname{arctg} \varepsilon)$, $M_p[f'(re^{i\varphi}) - 1]$ need not be uniformly bounded in $0 \leqq r < 1$. To see this, one has to modify our example in II.1 slightly in an obvious way so that it satisfies also $1 \leqq \rho(\theta) \leqq 1 + \varepsilon$; note that only the angle $\pi\alpha_2$ is of importance.

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