ITERATIONS OF GENERALIZED EULER FUNCTIONS

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1. Introduction. In this paper p and q will denote primes. We recall that a function $f(n)$ of an integral variable $n \geqq 1$ is said to be multiplicative, if

$$f(mn) = f(m)f(n) \tag{1}$$

whenever $(m, n) = 1$, and additive, if

$$f(mn) = f(m) + f(n) \tag{2}$$

whenever $(m, n) = 1$. If however $f(n)$ satisfies (2) for all integers $m \geqq 1$, $n \geqq 1$ we shall say that $f(n)$ is *completely additive*. Consider a multiplicative integral-valued function $\psi(n) > 0$ and put

$$\psi_0(n) = n,\ \psi_1(n) = \psi(n),\ \cdots,\ \psi_r(n) = \psi[\psi_{r-1}(n)],\ \cdots . \tag{3}$$

We shall say that $\psi(n)$ is of finite index if, to each $n > 1$, there is an integer $C = C(n)$ such that

$$\psi_r(n)\begin{cases} > 1 \text{ for } r \leqq C \\ = 1 \text{ for } r > C\ , \end{cases} \tag{4}$$

in which case we put $C(1) = 0$.

The familiar Euler function

$$\varphi(n) = \sum_{\substack{m \leqq n \\ (m,n)=1}} 1 = n \prod_{p/n} \left(1 - \frac{1}{p}\right) \tag{5}$$

is an example of such a function, since $\varphi(n) < n$. For this case $(\psi = \varphi)$, properties of the corresponding function $C(n)$ were investigated by Pillai [1], who attributes the problem to Vaidyanathaswami. Later, Shapiro [2, 3, 4] observed that this particular $C(n)$ satisfied the condition

$$C(mn) = C(m) + C(n) + \begin{cases} 1 \text{ for } m, n \text{ both even} \\ 0 \text{ otherwise}\ , \end{cases} \tag{6}$$

and went on to obtain, inter alia, a certain class (S) of multiplicative functions $\psi(n)$ of finite index satisfying (6). In a restricted sense, (S) consists of functions similar in form to $\varphi(n)$; for example they satisfy

$$\psi(x^n)[\psi(x)]^{n-2} = [\psi(x^2)]^{n-1}$$

for all positive integers x, n.

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Our first purpose is to impose mild conditions on $\psi(n)$ to ensure that it has a finite index, the characterization of all such functions being an unsolved problem.

THEOREM 1. *Let $\psi(n)$ be any multiplicative integral-valued function satisfying*

$$(7) \qquad \text{(i)} \quad q/\psi(p^t) \Rightarrow q \leqq p \quad \textit{for all } p, q \textit{ and all } t \geqq 1\,,$$

$$(8) \qquad \text{(ii)} \quad p^t \nmid \psi(p^t) \quad \textit{for any } p \textit{ or any } t \geqq 1\,.$$

Then $\psi(n)$ is of finite index.

We shall refer to the class of functions $\psi(n)$ admitted by (7) and (8) by the letter (W) if, by analogy with the Euler function, they also satisfy[1]

$$(9) \qquad \psi(n) \equiv 0 \pmod 2 \qquad \text{for} \quad n > 2\,.$$

It is evident that not all members of (W) satisfy (6); for example

$$(10) \qquad \psi(n) = 2^{\delta(n)}$$

where $\delta(n)$ is the number of different odd prime factors of n, and $C(3) = C(5) = C(15) = 1$. Our main purpose is to isolate the members of (W) which *do* satisfy (6), thereby enlarging the class (S) obtained by Shapiro (*loc. cit.* 3). Theorem 2 does, in fact, prescribe necessary and sufficient conditions, but before stating it we need some further notation. Our calculations are a little simpler if we introduce the function $c(n)$, where

$$(11) \qquad c(n) = \begin{cases} C(n) + 1 & \text{if } n \text{ is even} \\ C(n) & \text{if } n \text{ is odd}\,, \end{cases}$$

for then, by (6), $c(n)$ is completely additive.[2] By (7) and the multiplicative property of ψ, we have

$$(12) \qquad \psi(n) = \prod_{p \leqq n} p^{\lambda(p,n)}$$

for some $\lambda(p, n) \geqq 0$ defined for all $n \geqq 2$ and all $p \leqq n$. Then, (7), (8) and (9) may be expressed alternatively as

$$(13) \qquad \lambda(q, p^t) = 0 \quad \text{for all } q > p\,,$$

$$(14) \qquad \lambda(p, p^t) < t\,,$$

$$(15) \qquad \lambda(2, n) > 0 \quad \text{for } n > 2.$$

Assigning arbitrary values to $\psi(p)$, subject only to conditions (7), (8)

[1] We remark that condition (9) may be generalized, if (6) and (11) are reformulated.

[2] Note that $C(n)$ is additive, but not completely. Note also that $c(1) = C(1) = 0$, while (9) and (11) imply that $c(n) > 0$ for $n > 1$.

and (9), the $\lambda(q, p)$ are then determined uniquely by (12), for all $q < p$ and p. We define inductively a new function $\Gamma(p)$ over the primes, by

$$\Gamma(p) = \begin{cases} 1 \text{ if } p = 2\,, \\ \sum_{q<p} \lambda(q, p)\Gamma(q) \quad \text{if } p > 2\,. \end{cases} \tag{16}$$

For $n \geqq 1$ and odd p, we introduce the linear relations

$$\lambda(2, p^n) + \sum_{3 \leqq q \leqq p} \Gamma(q)\lambda(q, p^n) = n\Gamma(p) \tag{17}$$

which represents, for each $n > 1$, a restriction on the values of $\lambda(2, p^n)$, $\lambda(3, p^n), \cdots, \lambda(p, p^n)$. Note that (17) is an identity for $n = 1$, while for $n > 1$ it possesses at least one solution, namely

$$\lambda(q, p^n) = \begin{cases} n\Gamma(p) \text{ if } q = 2\,, \\ \;0 \quad \text{ if } q > 2\,. \end{cases} \tag{18}$$

For $p = 2$, we set

$$\psi(2^n) = 2^{n-1} \quad \text{for } n \geqq 1 \tag{19}$$

We are now in a position to state our main theorem:

THEOREM 2. *Then let $\psi(n)$ be any multiplicative function satisfying* (7), (8) *and* (9).

(i) *If $c(n)$ is completely additive, $c(p) = \Gamma(p)$.*

(ii) *$c(n)$ is completely additive if, and only if, $\psi(n)$ satisfies* (17) *and* (19).

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2. Proof of Theorem 1. Suppose $n > 1$. If we express $n = \prod_i p_i^{\nu_i}$ then $\psi(n) = \prod_i [\psi(p_i^{\nu_i})]$, by the multiplicative property. Let p_{n_0} denote the greatest prime factor of n. Then no prime $p > p_{n_0}$ can divide $\psi(n)$ and $p_{n_0}^{\nu_{n_0}} \nmid \psi(n)$. Hence no prime $p > p_{n_0}$ can divide any $\psi_r(n)[r = 0, 1, \cdots]$ and the greatest power of p_{n_0} dividing $\psi_r(n)$, if not zero, exceeds by at least one the greatest power of p_{n_0} dividing $\psi_{r+1}(n)$. Hence there is an integer r_0 such that $p_{n_0} \nmid \psi_{r_0}(n)$. Then either $\psi_{r_0}(n) = 1$ or the greatest prime factor of $\psi_{r_0}(n)$ is $p_{n_1} < p_{n_0}$. If $\psi_{r_0}(n) \neq 1$, we can repeat the process and determine an integer r_1, such that $p_{n_1} \nmid \psi_{r_1}(n)$. Hence either $\psi_{r_1}(n) = 1$ or the greatest prime factor of $\psi_{r_1}(n)$ is $p_{n_2} < p_{n_1}$. In this way, we obtain a decreasing sequence of primes $p_{n_0} > p_{n_1} > p_{n_2} > \cdots$ which clearly terminates at, say p_{n_s}, when $\psi_{r_s}(n) = 1$. Since $\psi(1) = 1$, $C = r_s - 1$ has the desired property.

3. The main lemma. We use the following property of the function $c(n)$:

$$c[\psi(n)] = \begin{cases} c(n) - 1 & \text{if } n \text{ is even,} \\ c(n) & \text{if } n \text{ is odd,} \end{cases} \tag{20}$$

which follows immediately from (4), (9) and (11). For any p, let

$$S(p) = \{n : q/n \Rightarrow q < p\}\,, \qquad (n > 0)\,. \tag{21}$$

Then $S(p)$, being the set of all positive integers whose prime factors are $< p$, is closed under multiplication. Moreover, if $c(mn) = c(m) + c(n)$ for all m, n in $S(p)$, then

$$c(1) = 0 \tag{22}$$

and

$$c[\prod_{q<p} q^{\nu}] = \sum_{q<p} \nu c(p)\,. \tag{23}$$

The lemma which follows will provide an important step in the induction proof of Theorem 2.

LEMMA 1. *Suppose that $\psi(n)$ satisfies (17) for all odd p and all $n \geqq 1$. Let $p_1 < p_2 < \cdots$ denote the odd primes. Suppose also that, for some $k \geqq 1$,*

$$c(p) = \Gamma(p) \quad \textit{for all} \quad p \in S(p_k)\,. \tag{24}$$

and

$$c(mn) = c(m) + c(n) \quad \textit{for all } m, n \textit{ in } S(p_k) \tag{25}$$

Then

$$\text{(i)} \quad c(p) = \Gamma(p) \quad \textit{for all} \quad p \in S(p_{k+1}) \tag{26}$$

$$\text{(ii)} \quad c(p^t n) = c(p^t) + c(n) \quad \textit{if} \quad p = p_k,\ t \geqq 0,\ n \in S(p) \tag{27}$$

$$\text{(iii)} \quad c(p^t) = tc(p) \quad \textit{if} \quad p = p_k,\ t \geqq 0 \tag{28}$$

$$\text{(iv)} \quad c(mn) = c(m) + c(n) \quad \textit{for all} \quad m, n \textit{ in } S(p_{k+1}) \tag{29}$$

Proof. (i) By (24), it suffices to prove that $c(p_k) = \Gamma(p_k)$. But, with $p = p_k$, we have

$$c(p) = c[\psi(p)] = c[\prod_{q \leqq p} q^{\lambda(q,p)}] = \sum_{q<p} \lambda(q, p)c(q)$$

by (20), (12), (14), (23) and noting that $\psi(p) \in S(p)$. By (24), $c(q) = \Gamma(q)$ for all $q < p$ and so $c(p) = \Gamma(p)$, by (16).

(ii) The case $t = 0$ is obvious. Proceeding by induction on t, assume that

$$c(p^s n) = c(p^s) + c(n) \quad \text{for all } s < t \text{ and all } n \in S(p) .$$

Since $\psi(p^t) = mp^r$ for some $m \in S(p)$ and some $r < t$, by (13) and (14), we have

$$\begin{aligned} c[\psi(p^t n)] &= c[\psi(p^t)\psi(n)] \\ &= c[mp^r\psi(n)] \\ &= c(p^r) + c[m\psi(n)] , && \text{by our induction hypothesis} \\ &= c(p^r) + c(m) + c[\psi(n)] , && \text{by (25)} \\ &= c(p^r m) + c[\psi(n)] , && \text{(on using the hypothesis again!)} \\ &= c[\psi(p^t)] + c[\psi(n)] . \end{aligned}$$

Hence, by (20), $c(p^t n) = c(p^t) + c(n)$, and (ii) follows directly.

(iii) The cases $t = 0, 1$ are obvious. By induction on t, we assume that

$$c(p^s) = sc(p) \quad \text{for all } s < t .$$

Then, by (20) and (ii),

$$\begin{aligned} c(p^t) &= c[\psi(p^t)] \\ &= c[p^{\lambda(p,p^t)} \prod_{q<p} q^{\lambda(q,p^t)}] \\ &= c[p^{\lambda(p,p^t)}] + c[\prod_{q<p} q^{\lambda(q,p^t)}] . \end{aligned}$$

Since $\lambda(p, p^t) < t$ by (14), we can apply our inductive hypothesis to the first term. Hence

$$c(p^t) = \lambda(p, p^t)c(p) + \sum_{q<p} \lambda(q, p^t)c(q) ,$$

on using (25) on the second term. By (i), $c(q) = \Gamma(q)$ for $q \leqq p$, so that

$$\begin{aligned} c(p^t) &= \sum_{q \leqq p} \lambda(q, p^t)\Gamma(q) , \\ &= t\Gamma(p) \\ &= tc(p) \end{aligned}$$

by (17), and (iii) is immediate.

(iv) Let $m = p^\mu m_1$, $n = p^\nu n_1$, where $p = p_k$ and m_1, n_1 are in $S(p)$. Then

$$\begin{aligned} c(mn) &= c[p^{\mu+\nu}m_1n_1] = c(p^{\mu+\nu}) + c(m_1n_1)\ , && \text{by (ii)} \\ &= (\mu+\nu)c(p) + c(m_1) + c(n_1)\ , && \text{by (iii) and (25)} \\ &= \{\mu c(p) + c(m_1)\} + \{\nu c(p) + c(n_1)\}\ , \\ &= \{c(p^\mu) + c(m_1)\} + \{c(p^\nu) + c(n_1)\}\ , && \text{by (iii)} \\ &= c(m) + c(n)\ , && \text{by (ii)}\ . \end{aligned}$$

This completes the proof of (iv), and so of Lemma 1.

4. Proof of Theorem 2. Suppose that $\psi(n)$ satisfies (7), (8), (9), (17) and (19); we will deduce that $c(n)$ is completely additive (and incidentally that $c(p) = \Gamma(p)$). Consider the hypotheses of Lemma 1 in the case $k = 1$, when $S(3)$ consists of all powers of 2. Since $\psi(2^t) = 2^{t-1}$ for $t \geqq 1$, we have

(30) $$c(2^t) = 1 + C(2^t) = t\ ,$$

whence

(31) $$c(2) = 1 = \Gamma(2)\ ,$$

by (16). By definition $c(1) = 0$, so that for any integers $s \geqq 0, t \geqq 0$, we have

(32) $$c(2^s \cdot 2^t) = c(2^{s+t}) = s + t = c(2^s) + c(2^t)\ .$$

Thus the hypotheses (24) and (25) of Lemma 1 are valid for the particular case $k = 1$ and we conclude that

(33) $$c(p) = \Gamma(p), c(mn) = c(m) + c(n)$$

hold for all p, m, n in $S(5)$; which permits up to repeat the argument. Proceeding by induction on k we deduce, finally, that (33) holds for all primes p and all positive integers m, n.

Conversely, we suppose now that $c(n)$ is completely additive, and $\psi(n)$ satisfies (7), (8) and (9). We prove now that $\psi(n)$ satisfies (17) and (19) and that $c(p) = \Gamma(p)$. By (20) and the completely additive property of $c(n)$ we have

(34) $$c(p) = c[\psi(p)] = \sum_{q<p} \lambda(q, p)c(q) = \Gamma(p)\ ,$$

(35) $$c[\psi(p^t)] = c(p^t) = tc(p) = t\Gamma(p)\ ,$$

(36) $$c[\psi(p^t)] = \sum_{q \leqq p} \lambda(q, p^t)c(q)$$

for all odd p and all $t \geqq 1$. By (7) and (8), $\psi(2) = 1$, and so from (11) and (16),

$$c(2) = 1 = \Gamma(2) .$$

We may combine this result with (34) to replace $c(q)$ by $\Gamma(q)$ in (36). Then (35) and (36) together imply (17). By (7), with $p = 2$,

$$\psi(2^t) = 2^u , \quad \text{for some integer } u \geqq 0 .$$

Hence, using $c(2) = 1$ and (20), we have

$$u = c(2^u) = c[\psi(2^t)] = c(2^t) - 1 = t - 1 ,$$

which implies (19). Thus, Theorem 2 is established.

5. **Remarks.** (1) We remark that our subclass of W (whose $c(n)$ is completely additive) admits functions $\psi(n)$ of the type

$$\psi(p^t) = \begin{cases} 2^{t-1} & \text{if } p = 2 , \\ p^{t-l}[\psi(p)]^l & \text{if } p > 2 , \end{cases}$$

where $t \geqq 1$ and $l = l(p^t)$ is any integer between 1 and t. Note, in particular, that the special case $l(p^t) = 1$ includes the Euler function.

(2) In passing, it is worthy of notice that a converse problem, (where given any completely additive $c(n)$ with $c(n) > 0$ for $n > 1$ we seek the set of all multiplicative functions $\psi(n)$ satisfying (7), (8) and (9) and having this $c(n)$ as their counting function), is a direct consequence of Theorem 2. The solution may be expressed in the form

$$\psi(p^t) = \begin{cases} 2^{t-1} & \text{if } p = 2 \\ 2^{tc(p)} \prod\limits_{3 \leqq q \leqq p} [q2^{-c(q)}]^{\lambda(q, p^t)} & \text{if } p > 2 , \end{cases}$$

provided that $\psi(p^t) \equiv 0 \pmod 2$ for $p > 2$. Inspection of relations (17) and (18) shows that our set is never empty.

(3) Given any multiplicative $\psi(n)$ satisfying (7), (8) and (9) and having a completely additive $c(n)$, it is evident that the relation $c(p) = \Gamma(p)$ provides an alternative method for evaluating $c(n)$, for each n.

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