IDEMPOTENT MEASURES ON SEMIGROUPS

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Introduction. Some interest has been shown in the problem of determining idempotent measures on topological groups and, more recently, on semigroups. Wendel [10] seems to have opened the subject in 1954 with the pleasing result that the positive idempotent measures on a compact group were precisely the (normalized) Haar measures of compact subgroups. In 1959 Rudin [6] showed that the same result held for locally compact abelian groups, and in 1960 Cohen [1] determined all idempotents (real and complex) on such groups. Glicksberg [2] (1959) showed that, on a compact abelian semigroup, to be the Haar measure of a compact subgroup was equivalent to being a positive idempotent.

In the present paper, the problem is considered for locally compact semigroups.

The problem is solved for the general locally compact group by Theorem 4.1 of §A, in which it is also shown that the support of an idempotent measure on certain types of locally compact semigroups (which include compact semigroups) is a compact kernel (definition in §B). In §B we describe the structure of compact kernels, giving results obtaing by Wallace [9] as a preliminary to describing the idempotent measures on them in §C. The relationships between invariant and idempotent measures are given in Theorems C4.1 and C5.1. Section C closes with a discussion of primitive idempotents which we see in §D are important in the structure of the semigroup of measures on a compact semigroup.

There is some slight overlap between the results given here and those published recently by Collins (Proc. Amer. Math. Soc. 13 (1962), 442–446, and Duke Math. J., 28 (1961), 387–392).

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A. Idempotent measures on locally compact semigroups.

1. The set of bounded, Borel measures on a locally compact semigroup S forms a Banach algebra when it is given the norm it acquires as the dual of $\mathfrak{K}(S)$ (the space of complex-valued continuous functions of compact support on S with the uniform norm) and when multiplication is defined by convolution: $\mu*\nu(\varphi) \equiv \int_S\int_S \varphi(xy)\,d\mu(x)\,d\nu(y)$ for $\varphi \in \mathfrak{K}$. The measure μ is said to be concentrated on a set E if, whenever the support of φ, (S_φ), is disjoint from E, $\mu(\varphi) = 0$. The support (S_μ) of μ is the smallest closed set on which it is concentrated.

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A measure μ may also be considered as a set function on the Borel sets in S; we shall make use of both definitions. We shall always suppose that μ is in the set of positive, bounded measures, $\mathfrak{S}(S)$, when $||\mu|| = \mu(S)$ and S_μ is the smallest closed set F for which $\mu(F) = ||\mu||$.

$\mathfrak{S}(S)$ forms a semigroup under convolution in which $||\mu*\nu|| = ||\mu|| \cdot ||\mu||$, whence if μ is idempotent, $||\mu|| = 1$.

The most important result on supports is

1.1 PROPOSITION. If $\mu, \nu \in \mathfrak{S}(S)$, $S_{\mu*\nu} = \overline{S_\mu \cdot S_\nu}$ (the bar denotes closure).

Wendel [10] Lemma 4 proved this result for compact groups; Glicksberg [2] Lemma 2.1 has shown it for compact abelian semigroups.

Proof. The following are evident: if $\mu_1 \geqq \mu_2$, then $S_{\mu_1} \supset S_{\mu_2}$, $\mu_1*\nu \geqq \mu_2*\nu$ and $\nu*\mu_1 \geqq \nu*\mu_2$ for each $\nu \in \mathfrak{S}$. Now Glicksberg's proof appiles verbatim to show that if μ and ν have compact supports, $S_{\mu*\nu} = S_\mu \cdot S_\nu$. In the general case, let a, b be any points in S_μ, S_ν respectively, and let U, V be any compact neighborhoods of a, b. Denoting the restriction of μ to U by μ_u we have $\mu*\nu \geqq \mu_u*\nu_v$ whence $S_{\mu*\nu} \supset S_{\mu_u} \cdot S_{\nu_v} \ni ab$. So $S_{\mu*\nu} \supset S_\mu \cdot S_\nu$, and since a support is closed, $S_{\mu*\nu} \subset \overline{S_\mu \cdot S_\nu}$.

Conversely, let U be any open set with $U \cap \overline{S_\mu \cdot S_\nu} = \emptyset$ and let $\varphi \in \mathfrak{K}(S)$ have $S_\varphi \subset U$. Then if $\mu*\nu(\varphi) = \int_S \int_S \varphi(xy) d\mu(x) d\nu(y)$ is to be nonzero, there must exist x and y which satisfy $x \in S_\mu$, $y \in S_\nu$ and $xy \in S_\varphi$ simultaneously; but this is impossible. We deduce that $S_{\mu*\nu}$ is in the complement of U, and we conclude by noting that we may take U to be the complement of $\overline{S_\mu \cdot S_\nu}$.

1.2 COROLLARY. *If μ is idempotent, $\overline{S_\mu \cdot S_\mu} = S_\mu$; in particular, S_μ is a subsemigroup.*

2. We now restrict the class of semigroups we consider by insisting that they satisfy the condition:

2.1 (L) Given any two compact subsets A and B of S, there is a third $K = K(A, B)$ for which $x \notin K \Rightarrow xA \cap B = \emptyset$.

There is a corresponding condition (R) referring to multiplication on the right by x. The two are obviously equivalent if S is commutative; they are not in general, as we shall see.

Examples. Every compact semigroup satisfies both (L) and (R), for we may then always take $K = S$. They are also satisfied by groups $(K = BA^{-1})$ and by closed subsemigroups of groups $(K = S \cap BA^{-1})$. They need not be satisfied by open subsemigroups: take S to be $(0, \infty)$

with addition and the usual topology, and $A = B = [1, 2]$; then $\{x: xA \cap B \neq \emptyset\} = (0, 1]$ which is contained in no compact subset.

If $[0, 1]$ is given the discrete topology and if multiplication is defined by $x.y = x$ we find that it satisfies (L)—take $K = B$—but not (R)—take $A = B$. A dual example shows we can satisfy (R) but not (L).

Our semigroups from now on are assumed to satisfy (L); there will be dual results for those which satisfy (R). (L) ensures that certain transformations of functions in $\mathscr{C}_\infty(S)$ (the space of continuous functions which vanish at infinity) are themselves in $\mathscr{C}_\infty$ (we use the notation $f^a(x) = f(xa)$; $f_a(x) = f(ax)$):

2.2 PROPOSITION. If $f \in \mathscr{C}_\infty(S)$, $f^a \in \mathscr{C}_\infty(S)$.

Proof. Let $\varepsilon > 0$ be given. Take $A = \{a\}$, $B = \{x: |f(x)| \geqq \varepsilon\}$; then if $x \notin K(A, B)$, $|f(xa)| < \varepsilon$.

2.3 PROPOSITION. If $f \in \mathscr{C}_\infty(S)$, and $\mu \in \mathfrak{S}(S)$, $f'(x) \equiv \mu(f_x) \in \mathscr{C}_\infty(S)$.

Proof. Let $\varepsilon > 0$ be given. Then there are compact sets A and B with $\mu(A) \geqq ||\mu|| - \varepsilon$ and $|f(x)| < \varepsilon$ for $x \notin B$. Then for $x \notin K(A, B)$

$$|f'(x)| = |\mu(f_x)| \leqq \int_S |f(xy)|\, d\mu(y)$$
$$\leqq \int_{S\backslash A} |f| d\mu + \int_{S\backslash \{y: xy \in B\}} |f|\, d\mu \leqq ||f||_\infty \cdot \varepsilon + \varepsilon\, ||\mu||$$

which will yield the required result.

We deduce from (2.3) that $|f'|$ assumes its supremum on S. Then

2.4 PROPOSITION. Let μ be idempotent; let f be positive; and let f' assume its supremum at $a \in S$. Then $f'(ax) = f'(a)$ for all $x \in S_\mu$.

Proof. We have

$$f'(a) = \int_S f(ax) d\mu(x) = \int_S \int_S f(axy) d\mu(x) d\mu(y)$$
$$= \int_{S} \cdot \int_S f(axy) d\mu(y) \cdot d\mu(x) = \int_S f'(ax) d\mu(x)$$
$$\leqq f'(a)\ ,$$

whence $f'(ax) = f'(a)$ a.e. (μ) and since f' is continuous, $f'(ax) = f'(a)$ for $x \in S_\mu$.

3. We now turn to results on the supports of idempotents.

3.1 PROPOSITION. Let μ be idempotent. If for some positive $f \in \mathscr{C}_\infty$,

f' assumes its supremum at a, $\overline{aS_\mu}$ is compact.

Proof. $C = \{x : f'(x) = \sup f' = f'(a)\}$ is compact since f' vanishes at ∞; by (2.4), $aS_\mu \subset C$.

The crucial result of this section is

3.2 PROPOSITION. Let μ be idempotent. If S is compact or is locally compact and satisfies the left cancellation law, S_μ is compact.

Proof. The compact case is trivial, for S_μ is a closed subset of S.

If S is not compact, we remark that if μ is idempotent on S, its restriction to its support S_μ is again idempotent. Moreover, S_μ is a semigroup (1.2) which, being closed, is locally compact, and which clearly satisfies both (L) and the left cancellation law if S does. The above results therefore hold when S is replaced by S_μ.

This enables us to find, using (2.3) and (3.1), an $a \in S_\mu$ such that $\overline{aS_\mu}$ is compact. Now aS_μ is obviously a semigroup, and hence $\overline{aS_\mu}$ is also. We can now use Numakura's Lemma 2L [3] which states:

Let X be a compact semigroup satisfying the left cancellation law, and let B be a closed subset of x. If $p \in X$ and $pB \subset B$, then $pB = B$.

The conditions of this lemma are satisfied if we take $X = B = \overline{aS_\mu}$ and $p = ax$ for any $x \in S_\mu$. We deduce that $ax \cdot \overline{aS_\mu} = \overline{aS_\mu} \supset aS_\mu$, whence by the left cancellation law, $x \cdot \overline{aS_\mu} \supset S_\mu$. Since both $\{x\}$ and $\overline{aS_\mu}$ are compact, their product is compact, and so S_μ, being closed, is compact.

3.3 COROLLARY. $S_\mu \cdot S_\mu = S_\mu$.

Proof. This follows from (1.2) and the fact that the product of two compact sets is closed.

Being a compact semigroup, S_μ has a minimal (two sided) ideal (Numakura [3] Theorem 2) and in fact

3.4 PROPOSITION. Under the conditions of (3.2), S_μ is its own minimal ideal.

The proof follows closely Glicksberg's proof (Theorem 2.2) [2] for the abelian case.

Proof. Let I denote the minimal ideal of S_μ and suppose $S_\mu \backslash I \neq \emptyset$.

Then first, there exist both $z \in S_\mu$ and a positive $\varphi \in \mathfrak{K}(S)$ which vanishes on I, for which φ_z does not vanish on S_μ: for if not, for every positive $\varphi \in \mathfrak{K}$ vanishing on I and all $z \in S_\mu$, $\varphi(zx) = 0$ for all $x \in S_\mu$, so $\varphi(y) = 0$ for $y = zx \in S_\mu \cdot S_\mu = S_\mu$; i.e. if φ vanishes on I, it vanishes on

S_μ, which would imply $I = S_\mu$, a contradiction. For this φ and z, $\mu(\varphi_z) > 0$. Now $\varphi'(x) = \int_S \varphi(xy)d\mu(y)$ clearly vanishes on I. But φ' assumes its supremum at some point $a \in S$ and (2.4) on $aS_\mu \supset aI$, so there is an x_0 in $aI \subset I$ for which $\varphi'(x_0) = \sup \varphi'(x)$. Hence $0 = \varphi'(x_0) \geqq \varphi'(z) = \mu(\varphi_z) > 0$, which is a contradiction. So $S_\mu = I$.

4.

4.1 THEOREM. *A positive idempotent measure on a locally compact group is the Haar measure of a compact subgroup, and conversely.*

Proof. By (3.2), if μ is idempotent, S_μ is a compact semigroup, and since it satisfies both cancellation laws, it is a group (Numakura [3], Theorem 1).

Now, the restriction of μ to S_μ is also idempotent, and so all the above propositions hold with S replaced by S_μ. In particular, when S_μ is a group, (2.4) states merely that μ is left invariant on S_μ, i.e. is its Haar measure.

The converse is clear.

B. The structure of compact kernels.

1. The results of this section were given in a slightly more general form (see (4) below) by Wallace [9]. We give them here in order to establish our notation, and because Wallace gave no proofs.

The minimal ideal of a compact semigroup is known as its kernel. We have seen (A3.4) that the support of an idempotent measure is a compact semigroup which is itself such a kernel, and in order that we may describe these measures completely, we investigate their structure.

Let S be a compact kernel. Then Numakura [3] has shown that, if e and f are any idempotents in S, Se satisfies the right cancellation law and fS the left; eSf is a group; any two such groups are either identical or disjoint; and S is the union of all such groups. (Lemma 9 and Theorem 3)

Now let g be any idempotent in S, and let e be any idempotent in Sg. Then if $x \in Sg$, $xe^2 = xe$, whence by the right cancellation law, $xe = x$; i.e. e is a right identity for Sg. We immediately deduce that the set E of idempotents in Sg forms a subsemigroup. Similarly, the set F of idempotents in gS forms a subsemigroup, each element of which is a left identity for gS.

2. We write $G = gSg$ and we can then state our algebraic structure theorem:

2.1 THEOREM. *There is a biunique map* $S \longleftrightarrow E \times G \times F$ *which becomes an algebraic isomorphism when multiplication is defined in* $E \times G \times F$ *according to* (2.3) *below.*

This is a reformulation of Rees [4], Theorem 2.93.

Proof. First, if $e_1, e_2 \in E$ and $f_1, f_2 \in F$, $e_1Sf_1 = e_2Sf_2$ implies $e_1 = e_2$ and $f_1 = f_2$; for we have $e_1Sg = e_1Sf_1g = e_2Sf_2g = e_2Sg$; the first of these includes $e_1g = e_1$ as an idempotent, the last e_2; but since they are both the same group, there can only be one and so $e_1 = e_2$. Similarly, $f_1 = f_2$.

Secondly, every group $e'Sf'(e', f' \in S)$ is identical with a group $eSf(e \in E, f \in F)$; for let $e = e'xg$, $f = gyf'$ be the identities of the groups $e'Sg$, gSf' respectively: then $eSf = e'xgSgyf' \subset e'Sf'$ and since therefore $eSf \cap e'Sf' \neq \phi$, $eSf = e'Sf'$.

Now, since $S = \cup \{e'Sf' : e', f' \in S\} = \cup \{eSf : e \in E, f \in F\}$ every element $x \in S$ has a unique expression in the form $x = exf\,(e \in E, f \in F)$. Then $x = eg \cdot x \cdot gf = e \cdot gxg \cdot f$. We write $\bar{x} = gxg$ and define our map

2.2 $$x \longleftrightarrow (e, \bar{x}, f)\ .$$

Then $x_1x_2 = e_1x_1f_1 \cdot e_2x_2f_2 = e_1 \cdot gx_1g \cdot gf_1e_2g \cdot gx_2g \cdot f_2$. We write $gf_1e_2g = [f_1e_2]$ and then

2.3 $$(e_1, \bar{x}_1, f_1)\,(e_2, \bar{x}_2, f_2) = (e_1, \bar{x}_1\,[f_1e_2]\,\bar{x}_2, f_2)$$

defines a multiplication in $E \times G \times F$ which makes (2.2) an isomorphism.

2.4. We notice that $[fg] = g \cdot fg \cdot g = g$ and that $[ge] = g$ for all $e \in E$, $f \in F$.

We have the evident

2.5 COROLLARY. *S is the direct product of E, F and G if and only if* $[fe] = g$ *for all* $e \in E$, $f \in F$.

We denote $\{[fe] : f \in F' \subset F, e \in E' \subset E\}$ by $[F'E']$, so the condition reads, $[FE] = \{g\}$.

A semigroup is said to be left-simple if $Se = S$ for all idempotents $e \in S$. In this case, $gS = gSg$ contains only one idempotent g, and so $F = \{g\}$. Then $[FE] = [gE] = \{g\}$ (2.4) and so (2.5):

2.6 PROPOSITION. A left-simple semigroup S is a direct product $E \times G \times \{g\}$ (which is clearly isomorphic with $E \times G$).

3. We give E, G, and F the topologies induced on them by S, and then state

3.1 THEOREM. *S is homeomorphic with* $E \times G \times F$ *under the map*

(2.2).

Proof. $(e, \bar{x}, f) \to e\bar{x}f$ is continuous by continuity of multiplication in S.

To prove $x \to (e, \bar{x}, f)$ is continuous, we show that the maps into each of the axes are continuous. Continuity of multiplication again ensures that $x \to \bar{x} = gxg$ is continuous.

We now show that the projection onto E is continuous. Since gSg is a compact group, $gxg \to (gxg)^{-1}$ is continuous, and so $x \to (gxg)^{-1}$ is continuous. Then $x \to (x, (gxg)^{-1})$ of $S \to S \times gSg$ with the product topology is continuous. We use continuity of multiplication in S again to show that $(x, (gxg)^{-1}) \to x\cdot(gxg)^{-1} = ex\cdot g(gxg)^{-1} = eg\cdot x\cdot g(gxg)^{-1} = e\cdot(gxg)\cdot(gxg)^{-1} = eg = e$ is continuous, and we combine these to conclude that $x \to e$ of $S \to E$ is continuous. Similarly, $S \to F$ is continuous, and the result is proved.

3.2 COROLLARY. *E, G, and F are compact.*

Proof. They are continuous images of S.

3.3. Since multiplication is continuous in $E \times G \times F$ and since $(g, g, f)(e, g, g) = (g, [fe], g)$ we have

COROLLARY. *The map $(f, e) \to [fe]$ of $F \times E \to G$ is continuous.*

4. The decomposition of S described in (2) and (3) is known as the canonical decomposition. If we take g_1 and g_2 to be any two idempotents in S, E' to be the set of idempotents in Sg_1 and F' in g_2S, and $G' = g_1Sg_2$ we can show that S is isomorphic and homeomorphic with $E' \times G' \times F'$ using the same proofs, though the details are more complicated. The results have the same form excepting (2.4) where all simplicity is lost. (See Wallace [9]).

5. We now show that (3.2), (3.3) and (2.3) are sufficient to characterize compact kernels.

5.1 THEOREM. *Let E and F be any two compact sets, and let G be any compact group; let $(f, e) \to [fe]$ be any continuous map of $F \times E \to G$ (there exist at least the trivial ones, $[fe] = x$ for all (f, e) and some fixed $x \in G$) and let a product be defined in $E \times G \times F$ by (2.3). Then $E \times G \times F$ is a compact kernel.*

Proof. $E \times G \times F$ is obviously compact and a semigroup, and multiplication is easily seen to be continuous.

It remains to show that it is a kernel, i.e. has no proper ideals. Now (Numakura [3] Lemma 5) a compact semigroup S is a kernel if and only if $SxS = S$ for all $x \in S$. We have here, for each $(e, x, f) \in (E, G, F)$,

$$(E, G, F)(e, x, f)(E, G, F) = (E, G\cdot[Fe]\cdot x, f)(E, G, F)$$
$$= (E, G, f)(E, G, F) = (E, G, F)\ ,$$

whence the result.

6. Let S' be a subkernel (a subsemigroup which is its own kernel) of S. If we take an idempotent $g \in S'$ and use it to form a canonical decomposition of S as $E \times G \times F$ we simultaneously decompose S' canonically as $E' \times G' \times F'$ where $E' \subset E$, $F' \subset F$ and G' is a compact subgroup of G. Since S' is a semigroup we have $(g, g, F')(E', g, g) = (g, [F'E'], g) \subset S'$, whence $[F'E'] \subset G'$.

Conversely, if $S = E \times G \times F$ and E' and F' are compact subsets of E and F respectively and if G' is a compact subgroup of G for which $[F'E'] \subset G'$, $E' \times G' \times F'$ is a compact subkernel of S.

C. Idempotent measures on compact kernels.

1. According to (A3.4) an idempotent measure on a semigroup of the types considered in §A has as its support a compact kernel. To describe all idempotent measures on such semigroups we must be able to state which kernels can act as supports, and to give the structure of an idempotent on such a kernel. To these ends (and, in particular, the latter) we investigate the idempotent measures on a general kernel S.

In this section, μ will denote a positive, idempotent measure on S. Then the restriction of μ to S_μ (which we again denote by μ) is also idempotent. S_μ is itself a kernel (A3.4) and so may be expressed in canonical form as $S_\mu = E_\mu \times G_\mu \times F_\mu$ (see B3); g will denote the identity of G_μ. Then

1.1 LEMMA. *Let μ be idempotent on S, $\varphi \in \mathscr{C}(S_\mu)$ and $\varphi \geqq 0$. Then $\varphi'(e_0, x, f)$ is a constant for each $e_0 \in E_\mu$. (φ' is defined in* (A2.3).)

Proof. Let $e_0 \in E$. Take some $x_1, x_2 \in G_\mu$, $f_1, f_2 \in F_\mu$. Then since φ' is continuous, given $\varepsilon > 0$, there is a neighborhood E_0 of e_0 in E_μ such that

$$|\varphi'(e_0, x_1, f_1) - \varphi'(e, x_1, f_1)| < \varepsilon/2$$

and
$$|\varphi'(e_0, x_2, f_2) - \varphi'(e, x_2, f_2)| < \varepsilon/2 \tag{i}$$

for all $e \in E_0$.

Now let χ_E be some continuous function on E_μ satisfying $\chi_E(e_0) = 1$, $0 \leqq \chi_E \leqq 1$, $\chi_E(e) = 0\ (e \notin E_0)$. Then $\chi(e, x, f) \equiv \chi_E(e)$ for all e, x, f is a continuous function on S_μ, and moreover

$$\begin{aligned}\chi'(e, x, f) &= \int_{S_\mu} \chi(e, x[fe']x', f')\, d_\mu(e', x', f') \\ &= \int_{S_\mu} \chi_E(e)\, d\mu(e', x', f') = \chi_E(e) = \chi(e, x, f) .\end{aligned}$$

Write $\psi = k\chi + \varphi$ where the constant k is chosen so large that $\psi' = k\chi' + \varphi'$ assumes its supremum on $E_0 \times G_\mu \times F_\mu$, say at (e', x', f'). Then by (A2.4) we have for all $(e, x, f) \in S_\mu$,

$$\psi'(e', x', f') = \psi'(e', x'[f'e]x, f) .$$

If we take $e = g$, $x = x'^{-1}x_c$, $f = f_c$ for $c = 1, 2$ we get

$$\psi'(e', x_1, f_1) = \psi'(e', x_2, f_2) .$$

Now also $\chi'(e', x_1, f_1) = \chi'(e', x_2, f_2)$ whence we deduce

$$\varphi'(e', x_1, f_1) = \varphi'(e', x_2, f_2) .$$

But $e' \in E_0$, and so using inequalities (i),

$$|\varphi'(e_0, x_1, f_1) - \varphi'(e_0, x_2, f_2)| < \varepsilon ,$$

and the result follows since ε is arbitrary.

We shall require in particular the formula $\varphi'(e_0, g, g) = \varphi'(e_0, x, f)$ for all $x \in G_\mu$, $f \in F_\mu$ for each $e_0 \in E_\mu$, or in full

1.2.
$$\begin{aligned}&\int_{S_\mu} \varphi(e_0, x', f')\, d\mu(e', x', f') \\ &= \int_{S_\mu} \varphi((e_0, x, f)(e', x', f'))\, d\mu(e', x', f') .\end{aligned}$$

Since a compact semigroup S_μ satisfies [R] as well as [L] (see A2), a symmetric result also holds:

1.3.
$$\begin{aligned}&\int_{S_\mu} \varphi(e', x', f_0)\, d\mu(e', x', f') \\ &= \int_{S_\mu} \varphi((e', x', f')(e, x, f_0))\, d\mu(e', x', f') ,\end{aligned}$$

for all $e \in E_\mu$, $x \in G_\mu$ and for each $f_0 \in E_\mu$.

2. We now simultaneously express $S = E \times G \times F$ and $S_\mu = E_\mu \times G_\mu \times F_\mu$ in canonical form using the idempotent $g \in S_\mu$ (B6). We are going to express μ in terms of its projections on E, G, and F:

2.1 Definitions. Let $\varphi_E(e)$ be any continuous function on E. Then $\varphi(e, x, f) \equiv \varphi_E(e)$ for all x, f is a continuous function on S. The measure

μ_E on E defined by $\mu_E(\varphi_E) = \mu(\varphi)$ is said to be the projection of μ on E. It is clear that its support is E_μ.

Similar definitions give μ_G and μ_F.

2.2 PROPOSITION. μ_G is the Haar measure of G_μ.

Proof. Let $\varphi_G(x)$ be any positive continuous function on G_μ, and define φ on S_μ by $\varphi(e, x, f) \equiv \varphi_G(x)$. Then if we take f to be g in (1.2) we get $\int_{G_\mu} \varphi_G(x')d\mu_G(x') = \int_{G_\mu} \varphi(xx')d\mu_G(x')$ for all $x \in G_\mu$, i.e. μ_G is left invariant.

Since all subgroups $gS_\mu g$ are isomorphic, the μ_G are isomorphic measures for different canonical decompositions.

2.3 PROPOSITION. $\mu = \mu_E \times \mu_G \times \mu_F$.

Proof. Let φ be any positive continuous function on S; then

$$\begin{aligned}
\mu(\varphi) &= \mu*\mu*\mu(\varphi) \\
&= \int_{S_\mu}\int_{S_\mu}\int_{S_\mu} \varphi((e_1, x_1, f_1)(e_2, x_2, f_2)(e_3, x_3, f_3))\, d\mu_1 d\mu_2 d\mu_3 \\
&\qquad \text{(where } d\mu_c \text{ means } d\mu(e_c, x_c, f_c),\ c = 1, 2, 3) \\
&= \int_{S_\mu}\int_{S_\mu}\int_{S_\mu} \varphi((e_1, x_2, f_2)(e_3, x_3, f_3))\, d\mu_1 d\mu_2 d\mu_3 \\
&\qquad \text{(using (1.2) on the integration w.r.t. } \mu_2) \\
&= \int_{S_\mu}\int_{S_\mu}\int_{S_\mu} \varphi(e_1, x_2, f_3)\, d\mu_1 d\mu_2 d\mu_3 \\
&\qquad \text{(using (1.3) on the integration w.r.t. } \mu_2) \\
&= \int_E\int_G\int_F \varphi(e, x, f)\, d\mu_E(e)\, d\mu_G(g)\, d\mu_F(f) \\
&= \mu_E \times \mu_G \times \mu_F(\varphi)\ .
\end{aligned}$$

We notice that for a given decomposition $S = E \times G \times F$, μ_E, μ_G and μ_F are unique.

All measures of this form are idempotent:

2.4 PROPOSITION. Let μ_E and μ_F be any positive normalized measures with supports $E_\mu \subset E$, $F_\mu \subset F$ respectively, and let μ_G be the Haar measure of any compact subgroup G_μ of G for which $[F_\mu E_\mu] \subset G_\mu$. Then $\mu_E \times \mu_G \times \mu_F$ is idempotent.

The proof is straightforward; we remark only that $[F_\mu E_\mu] \subset G_\mu$ is necessary for $E_\mu \times G_\mu \times F_\mu$ to be a semigroup.

3. If we identify the sets E, G, F with the subsets (E, g, g),

(g, G, g), (g, g, F) of S respectively, we may regard the measures μ_E, μ_G, μ_F as measures on S with supports in those subsets, and as such they have a convolution product:

3.1 PROPOSITION. $\mu = \mu_E * \mu_G * \mu_F$.

Proof. Let φ be positive and continuous on S. Then, from (2.3)

$$\begin{aligned}
\mu(\varphi) &= \int_E \int_G \int_F \varphi(e_1, x_2, f_3) d\mu_E(e_1) d\mu_G(x_2) d\mu_F(f_3) \\
&= \int_{(E\ g, g)} \int_{(g, G, g)} \int_{(g, g, F)} \varphi((e_1, g, g)(g, x_2, g)(g, g, f_3)) \\
&\qquad \cdot d\mu_E(e_1, g, g)\ d\mu_G(g, x_2\, g)\ d\mu_F(g, g, f_3) \\
&= \int_S \int_S \int_S \varphi((e_1, x_1, f_1)(e_2, x_2, f_2)(e_3, x_3, f_3)) d\mu_E(e_1, x_1, f_1) \\
&\qquad \cdot d\mu_G(e_2, x_2, f_2)\ d\mu_F(e_3, x_3, f_3) \\
&= \mu_E * \mu_G * \mu_F(\varphi)\ .
\end{aligned}$$

4. We recall (B2.6) that a left-simple kernel is the direct product of its subsemigroup of idempotents E and any of its maximal subgroups G. An idempotent μ whose support is such a kernel is of the form $\mu_E \times \mu_G = \mu_E * \mu_G$ from (2) and (3) above. We also have

4.1 THEOREM. *On a left-simple kernel, idempotent measures are characterized by being right invariant on their supports.*

Proof. We remark that if S is left-simple, so is the subkernel S_μ. We find that (1.3) for left-simple kernels reads

$$\int_{S_\mu} \varphi(e', x')\, d\mu(e', x') = \int_{S_\mu} \varphi((e', x')(e, x))\, d\mu(e', x')$$

for all $(e, x) \in S_\mu$, i.e. μ is right invariant.

Conversely, it is trivial that a normalized measure which is right invariant on its support, is idempotent.

Since every semigroup of the form $E \times G$ is left-simple, we have

4.2 COROLLARY. *Let $S = E \times G \times F$ be a compact kernel; let μ_E be any normalized measure on E, and let μ_G be the Haar measure of some subgroup of G. Then $\mu_E * \mu_G$ is right invariant on its support.*

We note that there are corresponding results for right-simple kernels.

5. Although in general positive idempotent measures are not invariant, there is still a close connection:

5.1 THEOREM. *A measure μ is idempotent if and only if it is the convolution product of a right invariant measure on any minimal left ideal of its support, and a left invariant measure on any minimal right ideal of its support.*

Proof. Let S_μ be a compact kernel; then any minimal left ideal of S_μ is of the form $S_\mu \cdot e$ for some idempotent $e \in S_\mu$, and this ideal is a left-simple semigroup. It intersects the right ideal fS_μ in the group $fS_\mu e$, which contains one idempotent, say g. Then $S_\mu \cdot g = S_\mu \cdot e$; $gS_\mu g = fS_\mu e$; $gS_\mu = fS_\mu$, and when we form the canonical decomposition with respect to g, $S_\mu = E_\mu \times G_\mu \times F_\mu$. We have also $S_\mu g = E_\mu \times G_\mu \times \{g\}$ and $gS_\mu = \{g\} \times G_\mu \times F_\mu$ (These results follow easily from §B).

Now if μ is idempotent $\mu = \mu_E * \mu_G * \mu_F$; μ_G being the Haar measure of a compact group is idempotent and so $\mu = (\mu_E * \mu_G)*(\mu_G * \mu_F)$. Then $(\mu_E * \mu_G)$ has support $E_\mu \times G_\mu \times \{g\} = S_\mu g$ and is right invariant from (4), and similarly for $(\mu_G * \mu_F)$.

Conversely, if there are right and left invariant measures with supports $S_\mu g$ and gS_μ respectively, they must be of the forms $(\mu_E * \mu_G)$ and $(\mu_G * \mu_F)$ by (4) and then $\mu = (\mu_E * \mu_G)*(\mu_G * \mu_F) = \mu_E * \mu_G * \mu_F$ is idempotent with support S_μ by (2.4).

5.2 PROPOSITION. The invariant measure on the ideal $S_\mu g$ corresponding to the idempotent μ, is $\mu * \varepsilon_g$ where ε_g is the unit point mass at g.

Proof. Canonically decompose S_μ with respect to g. Then $\mu * \varepsilon_g = \mu_E * \mu_G * \mu_F * \varepsilon_g$. It is straightforward to show both that $\mu_F * \varepsilon_g = \varepsilon_g$, and that $\mu_G * \varepsilon_g = \mu_G$, so that $\mu * \varepsilon_g = \mu_E * \mu_G$, which was to be shown.

6.

6.1 DEFINITION. A nonzero idempotent e is said to be primitive if the relations $ef = fe = f$ for some nonzero idempotent f, imply $e = f$.

In order to avoid anomalies arising from the fact that if the minimal ideal of S is a compact group its Haar measure is a zero in $\mathfrak{S}(S)$ and therefore not primitive, when we say that μ is primitive we shall mean primitive in $\mathfrak{S}(S)$ with zero adjoined.

6.2 PROPOSITION. μ is primitive idempotent on $S = E \times G \times F$ if and only if μ_G is the Haar measure of G.

Proof. By the remark which concludes (2.2) if the result holds for one decomposition, it holds for all.

Let μ be any idempotent on S and let $S = E \times G \times F$ be a canonical decomposition of S for which $\mu = \mu_E \times \mu_G \times \mu_F$. Let $\nu = \nu_E \times \nu_G \times \nu_F$ be any idempotent with ν_G the Haar measure of G, (such measures certainly exist). Then

$$\mu*\nu = \mu_E*\mu_G*\mu_F*\nu_E*\nu_G*\nu_F = \mu_E*\nu_G*\nu_F$$

since (A1.1) $S_{\mu_F*\nu_E} = S_{\mu_F}\cdot S_{\nu_E} \subset (g, g, F)(E, g, g) \subset G$, whence $\mu_G*(\mu_F*\nu_E)$ has support in G and so is annihilated by ν_G. Similarly, $\nu*\mu = \nu_E*\nu_G*\mu_F$. From these relations and the definition the result follows.

6.3 COROLLARY. *μ is primitive on S_μ*

Proof. Immediate from the proposition and (2.2)

6.4 PROPOSITION. On a left-simple semigroup S, an idempotent measure μ is primitive if and only if it is right invariant.

The point of this proposition is that μ is right invariant on the whole of S and not just on S_μ, as in (4.1).

Proof. If μ is right invariant, μ_G must be right invariant on the whole of G, and so must be the Haar measure of G.

Conversely, we have, for $(e, x) \in S = E \times G$,

$$\begin{aligned}\int_S \varphi((e', x')(e, x))\, d\mu(e', x') &= \int_{S_\mu} \varphi((e', x')(e, x))\, d\mu(e', x') \\ &= \int_{E_\mu}\int_G \varphi(e', x'x)\, d\mu_E(e')\, d\mu_G(x') \\ &= \int_S \varphi(e', x')\, d\mu(e', x')\end{aligned}$$

since, μ being primitive, μ_G is the Haar measure of G (6.2).

As a corollary, we can get a characterization of primitive idempotents on any kernel corresponding to (5.1).

By slightly altering the proof of (6.4) to conform more nearly to the proof of Schwarz's Theorem 5.1 [7], we find we have generalized the whole of his §5; in particular, we have found (in the form $\mu_E \times \mu_G = \mu_E*\mu_G$ where μ_G is the Haar measure of G) all invariant measures on the semigroups he considers.

D. On compact semigroups $\mathfrak{S}_1(S)$.

This section improves and generalizes from the finite to the compact case many of the results of Schwarz [8].

1. If S is compact, the set $\mathfrak{S}_1(S)$ of positive measures of total mass 1, becomes a compact semigroup when it is given the weak* topology.

It therefore has a minimal ideal, k say, and if $\mu \in k$, $S_\mu \subset K$, the minimal ideal of S. This follows from the facts that if ν is arbitrary and μ has its support in K, $S_{\nu*\mu} = S_\nu \cdot S_\mu \subset S_\nu \cdot K \subset K$, and similarly $S_{\mu*\nu} \subset K$. From this we also deduce that the set Π of primitive idempotents in $\mathfrak{S}_1(S)$ is just the set of idempotents primitive on K.

1.1 LEMMA. *Let* $\pi \in \Pi$, $\nu \in \mathfrak{S}_1$. *Then* $\pi*\nu*\pi = \pi$.

Proof. $\pi*\nu$ has its support in K. We decompose K canonically, and then

$$\pi*\nu*\pi = \pi*\pi*\nu*\pi = (\pi_E*\pi_G*\pi_F)*(\pi*\nu)*(\pi_E*\pi_G*\pi_F) .$$

But now $\pi_F*(\pi*\nu)*\pi_E$ has support in $F \cdot K \cdot E = G$ and so is annihilated by π_G, whence the result.

1.2 PROPOSITION. $\nu*\pi$ and $\pi*\nu$ are primitive idempotents.

Proof. We have immediately from (1.1) that $\nu*\pi$ and $\pi*\nu$ are idempotent. Now suppose there is an idempotent such that (i) $(\nu*\pi)*\mu = \mu$, and (ii) $\mu*(\nu*\pi) = \mu$. Then, from (ii), $\pi*\mu = \pi*(\mu*\nu)*\pi = \pi$ by (1.1), whence in (i), $\nu*\pi = \mu$ so that $\nu*\pi$ is primitive by definition (C6.1). Similarly $\pi*\nu$ is primitive.

1.3 THEOREM. $k = \Pi$.

Proof. By (1.2) Π is an ideal. By (1.1) $\Pi*\pi*\Pi \supset \Pi$ for each $\pi \in \Pi$, and so Π has no proper sub-ideals.

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