A GENERAL SOLUTION OF TONELLI'S PROBLEM OF THE CALCULUS OF VARIATIONS

LIONELLO A. LOMBARDI

Summary. The *alpha-asymptotical* property is defined for integrals depending upon any number m of surfaces of any dimension in nonparametric form. The existence of an absolute minimum of any alpha-asymptotical and lower semicontinuous integral in any *regular* class of *varieties* which is closed with respect to the *m-uniform* (Tchebyshev) metric is proved.

1. DEFINITIONS. Let D_i ($i = 1, 2, \cdots, m$) be a closed bounded set of the n-dimensional Euclidean space of the variable vector $x_i \equiv x_i^j$ ($j = 1, 2, \cdots, n$), bounded by surfaces which are absolutely continuous in the sense of Tonelli [14, 16, 17], without multiple points, and let D be the cartesian product $\prod_{1i}^{m} D_i$. Let $y \equiv y_i$ ($i = 1, 2, \cdots, m$) denote a vertical m-vector, and let p denote an mxn matrix, whose row-vectors are $p_i \equiv p_i^j$ ($j = 1, 2, \cdots, n$). Let x be the mxn matrix whose row-vectors are x_i and $\phi[x, y, p]$ a real-valued function, defined for $x_i \in D_i$ ($i = 1, 2, \cdots, m$) and for any y and p, which is continuous with all its derivatives of the types

$$\frac{\partial \phi[x, y, p]}{\partial p_r^s} \text{ and } \frac{\partial^2 \phi[x, y, p]}{\partial p_r^s \partial p_r^t} \quad (r = 1, \cdots, m; \quad s, t = 1 \cdots n) .$$

Let $q \leqq m$ be a positive integer and let U_q denote a set of q distinct positive integers out of the first m; let ζ be an index ranging over U_q, and let $\mu(\zeta)$ be a mapping of U_q into the set of the first n integers. It will be assumed throughout, that for every q, every U_q and every $\mu(\zeta)$, all the partial derivatives

$$\frac{\partial^{2q} \phi[x, y, p]}{\prod_{\zeta=1}^{q} \partial x_\zeta^{\mu(\zeta)} \partial p_\zeta^{\mu(\zeta)}} \tag{1.1}$$

exist and are continuous for every $x \in D$ and for every y and p.

Let T be a real positive number. Let $y(x) \equiv y_i(x_i)$ ($i = 1, 2, \cdots, m$) denote a vector-valued function of the matrix x, such that each $y_i(x_i)$ is a function with values in $[-T, T]$, which only depends upon the row

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vector x_i. We assume that each $y_i(x_i)$ is absolutely continuous, in the sense of Tonelli [17]; we shall call *variety* V the set of m surfaces represented by $y(x)$, and the functions $y_i(x_i)$ will be called *components* of $y(x)$.

Let

$$p_i^j(x) \equiv \frac{\partial y_i(x_i)}{\partial x_i^j} \qquad (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$$

and

$$dx \equiv \prod_{i=1}^{m} dx_i \equiv \prod_{i=1}^{m} \prod_{j=1}^{n} dx_i^j .$$

The *mxn* integral

$$I_V \equiv \int_D \phi[x, y(x), p(x)]dx$$

is called a *variety integral in nonparametric form*; all the varieties V where I exists and is finite are called *ordinary*.

Let $\bar{p} \equiv \bar{p}_i \equiv \bar{p}_i^j$ denote another variable in the space of the matrix p, $\bar{y} \equiv \bar{y}_i$ another variable in the space of the vector y, $V \equiv \bar{y}(x) \equiv \bar{y}_i(x_i)$ another variety V; let

$$\bar{p}_i^j(x_i) = \frac{\partial \bar{y}_i(x_i)}{\partial x_i^j} .$$

The distance $\rho(V, \bar{V})$ between V and $\bar{V}$ is defined by the formula

$$\rho(V, \bar{V}) = \sup_{x,i} |y_i(x) - \bar{y}_i(x)| .$$

In the sequel (whenever not otherwise specified) all of the topological properties of spaces and families of varieties V will refer to this *m-uniform* (Tchebychev) metric.

Let us assume by convention that if η is a variable integer ranging over a set S and $\{\alpha_\eta\}$ is a sequence of numbers, then

$$\prod_{\eta \in S} \alpha_\eta = 0 , \quad \text{whenever } S \text{ is empty} .$$

Let $L[x, y, p, \bar{p}]$ denote a polynomial in the indeterminates

$$[\bar{p}_i^{(j)} - p_i^{(j)}] \tag{1.2}$$

of degree not exceeding 1 in any of the vectors $[\bar{p}_i - p_i]$, whose coefficients $W_{U_q,\mu}(x, y, p)$ are functions of (x, y, p) which are continuous together with all their derivatives of the form

$$\frac{\partial^q W_{U_q,\mu}(x, y, p)}{\prod_{\zeta \in U_q} x_\zeta^{\mu(\zeta)}} \tag{1.3}$$

$L[x, y, p, \bar{p}]$ may be written in the form

$$\sum_{q-1} \sum_{U_q} \sum_{\mu} W_{U_q,\mu}(x, y, p) \prod_{U_q} [\bar{p}_{\zeta}^{\mu(\zeta)} - p_{\zeta}^{\mu(\zeta)}] . \tag{1.4}$$

The *generalized Weierstrass function* of I_V with respect to $L[x, y, p, \bar{p}]$ is defined by the formula

$$\mathscr{E}_L(x, y, p, \bar{p}) = \phi[x, y, \bar{p}] - L[x, y, p, \bar{p}] . \tag{1.5}$$

The integral $I_V = \int \phi[x, y(x), p(x)]dx$ is said to be *positively quasi-regular with respect to L* (abbreviation: *LPQR*) if both the relations

$$\mathscr{E}_L[x, y, p, p] = 0 \tag{1.6}$$

$$\mathscr{E}_L[x, y, p, \bar{p}] \geqq 0 \tag{1.7}$$

hold for every $x \in D$ and for every $y, p, \bar{p}$.

We say that I_V is *positively quasi-regular* (abbreviation: *PQR*) if there exists at least one function $L[x, y, p, \bar{p}]$ such that I_V is *LPQR*.

REMARK 1.1. If I_V is *PQR*, then its value at every nonordinary variety is $+\infty$.

Suppose now that $I_V \equiv \int_D \phi[x, y(x), p(x)]dx$ is *PQR* and let $L[x, y, p, \bar{p}]$ be one of the functions such that I_V is *LPQR*.

The function

$$\bar{\phi}[x, y, p] = \mathscr{E}_L[x, y, \Omega, p] , \tag{1.8}$$

where Ω is an *mxn* matrix whose elements are all 0, is never negative. Furthermore,

$$\bar{I}_V = \int_D \bar{\phi}[x, y(x), p(x)]\, dx \tag{1.9}$$

is *LPQR*, where

$$\bar{L}[x, y, p, \bar{p}] = L[x, y, p, \bar{p}] - L[x, y, \Omega, \bar{p}] . \tag{1.10}$$

By (1.6), the equation

$$\bar{\phi}[x, y, \Omega] = 0$$

holds for every $X \in D$ and every y.

Let R denote a positive real number and let $\varphi^R[x, y, p]$ denote a function such that the following conditions are satisfied:

I. $\varphi^R[x, y, p]$ is continuous with all its partial derivatives of any of the forms

$$\frac{\partial \varphi^R[x, y, p]}{\partial p_r^s}, \quad \frac{\partial^2 \varphi^R[x, y, p]}{\partial p_r^s \partial p_r^t}, \quad \frac{\partial^{2q} \varphi^R[x, y, p]}{\prod_{1\zeta}^{q} \partial x_\zeta^{\mu(\zeta)} \partial p_\zeta^{\mu(\zeta)}}$$

II. The integral

$$Y_V = \int_D \varphi^R[x, y(x), p(x)]dx \tag{1.11}$$

is PQR.

III. The relation

$$0 \leqq \varphi^R[x, y, p] \leqq \bar{\phi}[x, y, p] \tag{1.12}$$

holds for every y, p and for every $x \in D$; furthermore,

$$\varphi^R[x, y, p] = \bar{\phi}[x, y, p], \quad \text{whenever } \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i^j)^2 \leqq R . \tag{1.13}$$

IV. There exists at least one function $\Lambda[x, y, p, \bar{p}]$, such that Y_V is ΛPQR, and such that there exists a number Q, for which the following condition is satisfied:

Let q, U_q, ζ, $\mu(\zeta)$ be defined as they were above; let $\bar{U}_q$ denote the complement of U_q with respect to the set of the first m positive integers, and let $\bar{\zeta}$ be an index ranging over $\bar{U}_q$. Then the inequality

$$| W^R_{U_q,\mu}[x, y, p]| < Q\Big(1 + \prod_{\zeta \in \bar{U}_q} |p_{\bar{\zeta}}^{\mu(\bar{\zeta})}|\Big)$$

where $W^R_{U_q,\mu}[x, y, p]$ denotes the coefficient of the element

$$\prod_{\zeta \in U_q} [\bar{p}_\zeta^{\mu(\zeta)} - p_\zeta^{\mu(\zeta)}]$$

of the function $\Lambda[x, y, p, \bar{p}]$, holds for every q, U_q, p, for every $x \in D$, for every μ and for every y, such that

$$|y_i| \leqq T , \qquad (i = 1, 2, \cdots, n) .$$

The integral I_V is said to be *asymptotically evaluable* [8, 12] (abbreviation: AE) when there exists a function $L[x, y, p, \bar{p}]$ such that I_V is $LPQR$ and such that for every positive R there exists a function $\varphi^R[x, y, p]$ as described above. By using continuity lemmas proved in [11], in [12] this author proved the following:

Semicontinuity Theorem 1.1. *An integral I_V which is PQR and AE is lower semicontinuous at every variety V, ordinary or not.* (See also [8, 13, 14]).

REMARK 1.2. If V is a nonordinary variety and I_V is PQR, the lower semicontinuity of I_V at V denotes the fact that if $\{V_i\}$ is any sequence of varieties such that

$$\lim_{i\to\infty} V_i = V ,$$

then

$$\lim_{i\to\infty} I_{V_i} = +\infty .$$

DEFINITION. *We shall say that an integral* $I_V \equiv \int_D \phi[x, y, p]dx$ *is alpha-asymptotic if there exist two real numbers* $\alpha > 0$ *and* $Y > 1$ *such that*

$$\phi(x, y, p) > \prod_{i=1}^{m} \sum_{j=1} |p_j^i|^{n+\alpha}$$

whenever

$$\prod_{i=1}^{m} \sum_{j=1} |p_i^j| > Y$$

and if $\phi[x, y, p]$ *is bounded below by a number* H.

We shall say that a set F of varieties has the *property A* if there exist two real numbers $\beta > 0$ and M such that for each $V \equiv y(x) \in F$,

$$\int_D \prod_{i=1}^{m} \sum_{j=1}^{n} |p_i^j(x_i)|^{n+\beta} dx \leqq M .$$

We shall say that F has the *property B if* there exist two real numbers $\gamma > 0$ and N such that, for each $V \equiv y(x) \in F$,

$$\int_D \sum_{j=1}^{n} |p_i^j(x_i)|^{n+\gamma} \leqq N , \qquad (i + 1, 2, \cdots, m) .$$

DEFINITION. *A set* $\mathscr{F}$ *of varieties is called a* regular class *if each set* $F \subset \mathscr{F}$ *which has* property A *has also* property B.

REMARK. For $m = 1$, every set of varieties is a regular class. If all the m pointsets D_i and the m components $y_i(x_i)$ of a variety V are identical to each other, we shall say that V is *symmetric*. If $\mathscr{F}$ consists only of symmetric varieties it is clearly a regular class: however, sets of symmetric varieties are only a very special case of regular classes. Clarifying examples of regular classes and regularity criteria were given in [9] for the case $m = 2$, $n = 1$ (Fubini-Tonelli integrals).

DEFINITION. *A set* $\mathscr{F}$ *of ordinary varieties which contains all the ordinary varieties belonging to the boundary of* $\mathscr{F}$ *is called a* complete class.

EXAMPLES. Any closed set of varieties is a complete class. Both the set of all varieties and the set of all ordinary varieties are also complete classes.

2. The existence of the minimum. (a) The Problem of Tonelli consists of giving conditions on $\phi[x, y, p]$ which yield the existence of the minimum of I_V in any regular complete class $\mathscr{F}$ of varieties.

Only conditions on $\phi[x, y, p]$ are allowed. In fact, several types of conditions on $\mathscr{F}$ (such as uniform bounds of the *m-gradients* $p(x)$ of the varieties of $\mathscr{F}$ or of their integrals) would easily yield compactness of $\mathscr{F}$ itself and a fortiori the existence of the minimum. However, no such condition is consistent with the physical phenomena that can be described mathematically as minima of integrals of the calculus of variation or with the use of solutions of Tonelli's problem as tools for investigating the existence of solutions of partial differential and integro-differential equations [5, 18]. Our purpose in this study is to discuss the existence of a minimum of I_V in classes $\mathscr{F}$ of varieties which *are not* a priori compact, imposing conditions only on $\phi[x, y, p]$.

(b) LEMMA 2.b.1. *If I_V is alpha-asymptotic and bounded above on a set F of varieties, then F has property A.*

Let $V \equiv y(x) \in F$, and let K be the upper bound of I_V over F.
Let D_1 denote the set of points x of D such that

$$\prod_{i=1}^{m} \sum_{j=1}^{n} |p_i^j(x_i)| \leqq Y \tag{2.1}$$

and D_2 the complement of D_1 with respect to D. Then

$$\int_D \prod_{i=1}^{m} \sum_{j=1}^{n} |p_i^j(x_i)|^{n+\alpha} dx \leqq \int_{D_1} \prod_{i=1}^{m} \sum_{j=1}^{n} |p_i^j(x_i)|^{n+\alpha} dx$$
$$+ \int_{D_2} \phi[x, y(x), p(x)]dx \leqq (Y^{n+\alpha} + |H|)\mu(D) + K$$

where $\mu(D)$ denotes the measure of D.
The lemma is then proved simply by posing

$$M \equiv (Y^{n+\alpha} + |H|)\mu(D) + K \quad \text{and} \quad \beta \equiv \alpha .$$

Let now $\{V_s^{(r)}\}$ $(s = 0, 1, 2, \cdots, m)$ denote sequences $(V_s^{(1)}, V_s^{(2)}, V_s^{(3)}, \cdots)$ of varieties and $y_{i,s}^{(r)}(x_i)$ the ith component of $V_s^{(r)}$. Let us prove the following

Compactness Theorem 2.b.2. If I_V is alpha-asymptotic and bounded above on $\{V_0^{(r)}\}$ and $\{V_0^{(r)}\}$ is contained in a regular class $\mathscr{F}$, then there exists a variety $V^{(\infty)} \equiv y_i^{(\infty)}$ $(i = 1, 2, \cdots, m)$ of accumulation for

$\{V_0^{(r)}\}$.

For Lemma 2.b.1 $\{V_0^{(r)}\}$ has the *property A*; since $\{V_0^{(r)}\} \subset \mathscr{F}$ and $\mathscr{F}$ is a regular class, $\{V_0^{(r)}\}$ has also *property B*. Consequently, for each i $(i = 1, 2, \cdots, m)$ the ith components $y_{i,i-1}^{(r)}(x_i)$ of the varieties $V_{i-1}^{(r)}$ are equiabsolutely continuous functions of the n variables x_i^j, and form a compact set with respect to the uniform (Tchebychev) metric, i.e., there is a sequence $\{V_i^{(r)}\} \subset \{V_{i-1}^{(r)}\}$ and an absolutely continuous function $y_i^{(\infty)}(x_i)$ such that for each $\varepsilon > 0$ there is an $\bar{r}$ such that

$$| y_i^{(\infty)}(x_i) - y_i^{(r)}(x_i) | < \varepsilon$$

whenever $x_i \in D_i$ and $r < \bar{r}$. Consequently, in the metric of the space of varieties, the variety $V^{(\infty)} \equiv y_i^{(\infty)}$ $(i = 1, 2, \cdots, m)$ is a variety of accumulation for $\{V_m^{(r)}\}$ and, since $\{V_m^{(r)}\} \subset \{V_0^{(r)}\}$, also of $\{V_0^{(r)}\}$. The theorem is thus proved.

We are now in a position to prove the following

THEOREM 2.b.3 *of Existence of the Minimum: If the integral I_V is positively quasi-regular, asymptotically evaluable and alpha asymptotic, then it has an absolute minimum in every regular complete class $\mathscr{F}$ of ordinary varieties.*

Let i denote the lower bound of I_V over $\mathscr{F}$. Let us show that i is finite: indeed, if it were that

$$i = -\infty \tag{2.2}$$

there would exist a sequence $\{Y^{(r)}\} \subset \mathscr{F}$ of varieties such that

$$I_V(r) < -r\ ; \tag{2.3}$$

by the compactness Theorem 2.b.2 there would exist a variety $V^{(\infty)}$ of accumulation for $\{V^{(\infty)}\}$. Since I_V is positively quasi-regular by hypothesis, I_V^∞ exists, and

$$I_{V(\infty)} > -\infty \tag{2.4}$$

(see Remark 1.1).

But by the semicontinuity Theorem 1.1, I_V is lower semicontinuous at $V^{(\infty)}$; (2.4) then contradicts (2.3): consequently (2.2) is false.

Let now $\{V^{(r)}\}$ $(b = 1, 2, \cdots)$ be a sequence of varieties of $\mathscr{F}$ such that

$$I_V(r) < i + \frac{1}{r}\,.$$

By the compactness Theorem 2.b.2 there exists a variety $V^{(\infty)}$ of accumulation for $\{V^{(r)}\}$; semicontinuity Theorem 1.1 implies that $V^{(\infty)}$ is ordinary, because I_V is upper bounded on $\{V^{(r)}\}$; hence, since $\mathscr{F}$ is a complete class, $V^{(\infty)} \in \mathscr{F}$. Semicontinuity Theorem 1.1 also yields

$$I_{V(\infty)} \leqq \min \lim_{r \to \infty} I_V(r) ;$$

consequently

$$I_{V(\infty)} = i$$

which completely proves Theorem 2.b.3.

REMARK. Particular cases of the theorem of existence of the minimum have been proved previously, by Tonelli [**13, 15, 16,**] in the case $m = 1$, $n = 1$ (simple integrals) and $m = 1$, $n = 2$ (double integrals), and by this author [**9**] in the case $m = 2$, $n = 1$ (Fubini-Tonelli integrals). However this new general proof is considerably simpler than all those given in [**9, 13, 15, 16**].

(c) The concept of *regular class* needs discussion. In fact, this concept can only be accepted after proving that semicontinuous alpha-asymptotic integrals may not have any minimum in some complete classes of ordinary varieties which are not regular. This can be shown by means of the following example (see also [**9**]):

Let $m = 2$, $n = 1$, $D_1 \equiv D_2 \equiv [0, \pi]$, $\phi[x, y, p] \equiv (p_1^1 \cdot p_2^1)^2$. I_V is lower semicontinuous and alpha asymptotic for any $\alpha \leqq 1$. Consider the sequence $V^{(r)}$, $(r = 1, 2, \cdots)$ of varieties whose components are

$$y_1(x) = \sin(rx_1) \quad \text{and} \quad y_2(x) = r^{-3} \sin(rx_2) ,$$

respectively; $\{V^{(r)}\}$ is a complete class, because there are no varieties of accumulation. Each $V^{(r)}$ is ordinary, and

$$I_V(r) = \int_0^\pi \int_0^\pi r^{-2} \cos^2(rx_1) \cos^2(rx_2) dx_1 dx_2 = \left(\frac{\pi}{r}\right)^2 ;$$

the lower bound of I_V over $V^{(r)}$ is 0, but I_V never vanishes in $\{V^{(r)}\}$.

The fact that $\{V^{(r)}\}$ is not a regular class can also be shown directly without difficulty.

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UNIVERSITY OF CALIFORNIA AT LOS ANGELES