ON TWO TAUBERIAN REMAINDER THEOREMS

MAGNUS LINDBERG

1. Introduction. In this paper we are going to treat two different tauberian problems by a common method. The first of these problems originates from a question raised by P. Erdös [6] and the second was suggested by F. Brownell [5]. Our method gives in both cases slightly better results than those hitherto known.

In connection with his work on the Prime-number-theorem Erdös came to consider the following problem:

If A is a nondecreasing function, such that

$$\int_0^x A(x-t)dA(t) = \frac{x^2}{2} + O(x)\,, \quad x \to \infty\,, \tag{1.1}$$

what can be said about the order of magnitude of $a(x) = A(x) - x$?

Erdös proved that $a(x) = o(x)$ and constructed a counter example showing that (1.1) couldn't give $a(x) = o(\sqrt{x})$.

Later on Avakumović [2] gave the result $a(x) = O(x^{\sqrt{3}-1+\varepsilon})$ for every $\varepsilon > 0$.

In 1956 Bojanić, Jurkat and Peyerimhoff [4] improved this to $a(x) = O(x^{2/3}(\log x)^{1/3})$, this being the best result known till now.

Here we will study the more general case, when the remainder in (1.1) is $O(x^q)$, $0 \leqq q < 2$. Assuming this our result is $a(x) = 0(x^{(q+1)/3})$. Thus we are able to remove the logarithm in the last estimate.

The second of our problems was presented as a research problem in the Bulletin of the American Mathematical Society [5] by F. Brownell:

"Let $F(x)$ be a real valued function of real $x \geqq 0$ which is of bounded variation over every finite interval $[0, N]$, which is continuous at $x = 0$ with $F(0) = 0$, and which has $\int_0^\infty e^{-tx}|dF(x)| < +\infty$ for real $t > 0$. With $s = t + iv$, t and v real, define g by the Lebesque-Stieltjes integral $g(s) = \int_0^\infty e^{-sx}dF(x)$, analytic in the region $t > 0$. Let F satisfy the conditions that

$$g(t) = b + 0(e^{-c/t})\,, \quad t \to 0+ \tag{1.2}$$

for some real constants $c > 0$ and b, and that

$$F(x) + Kx^\nu \tag{1.3}$$

be strictly increasing over $x \geqq 1$ for some real constants $K > 0$ and $\nu \geqq 1$.

Is it true as conjectured, that

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$$F(x) = o(x^{\nu-1/2}) \tag{1.5}$$

as $x \to \infty$ if in addition to (1.2) and (1.3) it is also assumed that $g(iv) = \lim_{t\to 0+} g(t + iv)$ exists finite for all $v \neq 0$, that the resulting $g(s)$ is continuous in $t \geqq 0$ and $s \neq 0$, and that over all such s

$$|g(s)| \leqq M_1|s|^{-\nu+\eta} + M_2 \tag{1.4}$$

for some finite constants M_1 and M_2 and $\eta > 0$?"

That

$$F(x) = O(x^{\nu-1/2}) \tag{1.6}$$

can be proved even if assumption (1.4) is excluded has been shown by several authors. In the case $\nu = 1$, this result was obtained by Avakumović [1] and the general case has been treated by Ganelius [9] and Korevaar [11], basing their works on the results of Freud [7, 8] (cf. Ganelius [10]).

The example $F(x) = \int_0^x (\sin \sqrt{t}\,)\, t^{\nu-1}\, dt$ (Korevaar [11]), for integer ν satisfies (1.2), (1.3) and (1.6) but not (1.5), thus proving (1.6) to be the best possible under (1.2) and (1.3) only. But the example also violates (1.4).

In this paper we show that (1.5) can be obtained even under the following additional assumption:

$$|g(s)| < M_1|s|^{1+2\nu+\eta} + M_2\,, \quad \cap\ Re\ s > 0 \tag{1.4)'$$

and for some finite constants M_1, M_2 and $\eta > 0$. It is also sufficient if $\nu > 1/2$. This is evidently a weaker a assumption than that suggested by Brownell.

The common treatment of these problems was suggested to me by Professor T. Ganelius. I would like to thank him for his help and valuable advice.

2. A lemma on Laplace-transforms.

In order to sum up the common properties of the proofs of the two theorems we state the following lemma:

LEMMA. *Let F be a real valued function on* $[0, \infty)$, *such that $F(x) + Kx^\alpha$ is nondecreasing over $x > x_0$ for some positive constants K and α:*

Define the function f by

$$f(s) = \int_0^\infty e^{-sx}\, F(x)dx \tag{2.1}$$

where the integral is supposed to be absolutely convergent for $Re\ s =$

$t > 0$.

Then if

$$\int_{-\infty}^{+\infty} |f(t+iv)|\, v^{-2}\omega^{-1}\sin^2(\omega v)dv = O(t^{\beta}), \tag{2.2}$$

$t \to 0+$, *where* $\omega = t^{(\alpha-1+\beta)/2} = t^{\gamma}$ *and* $\alpha + \beta + 1 > 0$ *it follows that*

$$F(x) = O(x^{(\alpha-\beta-1)/2}), \quad x \to +\infty . \tag{2.3}$$

Proof. Let G_ω be the function defined by

$$G_\omega(x) = \begin{cases} 1 - (2\omega)^{-1}|x| & \text{if } |x| < 2\omega \\ 0 & \text{if } |x| \geqq 2\omega . \end{cases}$$

The Fourier transform of G_ω is the function H_ω, defined by $H_\omega(y) = 2\omega^{-1}y^{-2}\sin^2(\omega y)$ and Parseval's formula gives

$$\begin{aligned} &(2\pi)^{-1}\int_{-\infty}^{+\infty} f(t+iv)e^{iTv}H_\omega(v)dv \\ &= \int_{T-2\omega}^{T+2\omega} e^{-tx}F(x)\,(1-(2\omega)^{-1}|\,T-x\,|)\,dx . \end{aligned}$$

Taking the absolute value of both sides we get

$$\begin{aligned} &\left|\int_{T-2\omega}^{T+2\omega} e^{-tx}F(x)\,(1-(2\omega)^{-1}|\,T-x\,|)dx\right| \\ &\leqq \pi^{-1}\int_{-\infty}^{+\infty} |f(t+iv)|\,\omega^{-1}v^{-2}\sin^2(\omega v)\;dv = o(t^{\beta}) . \end{aligned} \tag{2.4}$$

If we in formula (2.4) put $T = t^{-1} + 2t^{\gamma}$ we may conclude, $F(x) + Kx^{\alpha}$ being nondecreasing, that for $T - 2t^{\gamma} \leqq \tau \leqq T + 2t^{\gamma}$

$$\begin{aligned} F(\tau) &\geqq F(t^{-1}) - K((t^{-1}+4t^{\gamma})^{\alpha} - t^{-\alpha}) \\ &= F(t^{-1}) - 4\,\alpha\,Kt^{\gamma}(t^{-1}+4\theta t^{\gamma})^{\alpha-1} \geqq F(t^{-1}) - K_1t^{\gamma-\alpha+1} . \end{aligned} \tag{2.5}$$

Suppose that $F(t^{-1}) \geqq 0$. By aid of (2.5) we infer from (2.4) that

$$\begin{aligned} K_2t^{\beta} &\geqq \int_{t^{-1}}^{t^{-1}+4t^{\gamma}} e^{-tx}F(x)\,(1-(2t^{\gamma})^{-1}|\,T-x\,|)\,dx \\ &\geqq \int_{t^{-1}}^{t^{-1}+4t^{\gamma}} e^{-tx}(F(t^{-1}) - K_1t^{\gamma-\alpha+1})\,(1-(2t^{\gamma})^{-1}|\,T-x\,|)dx \\ &\geqq e^{-5}F(t^{-1})2t^{\gamma} - K_1e^{-1}t^{\gamma-\alpha+1}2t^{\gamma} \geqq 2e^{-5}F(t^{-1})t^{\gamma} - K_1t^{\beta} . \end{aligned}$$

Consequently $F(t^{-1}) \leqq K_3t^{(\beta-\alpha+1)/2}$.

If $F(t^{-1}) < 0$ we can in the same way show that $F(t^{-1}) \geqq -K_4t^{(\beta-\alpha+1)/2}$, choosing $T = t^{-1} - 2t^{\gamma}$. Hence our lemma is proved.

3. On a nonlinear Tauberian theorem. We shall now apply our lemma to problem of Erdös, mentioned in the introduction.

THEOREM 1. *Let A be a nondecreasing function, defined on* $[0, \infty)$ *and such that*

$$h(x) = \int_0^x A(x-t)\,dA(t) = \frac{x^2}{2} + 0(x^q)\ , \quad x \to +\infty \tag{3.1}$$

where q is some real number, $0 \leqq q < 2$.
Then

$$A(x) = x + O(x^{(q+1)/3})\ , \quad x \to +\infty\ . \tag{3.2}$$

Proof. We are going to prove the theorem in the case when A is a normalized function, i.e. $A(0) = 0$ and $A(x) = 2^{-1}(A(x-0) + A(x+0))$ for $x > 0$. Owing to the nondecreasing of A this is not a restriction, since formula (3.2) will be correct for any nondecreasing function A, if it is so for the corresponding normalized function. We also assume the function h to be normalized, because normalization is possible at the countable set of points, where it may not be defined (see [12] p. 84).

Our problem is now reduced to getting an estimate of the type (2.2). For this purpose we use some of the results of Bojanić, Jurkat and Peyerimhoff [4].

Since

$$\begin{aligned} A^2\left(\frac{x}{2}\right) = A\left(\frac{x}{2}\right)\int_0^{x/2} dA(u) &\leqq \int_0^{x/2} A(x-u)dA(u) \\ &\leqq \int_0^x A(x-u)dA(u) = 0(x^2)\ , \end{aligned}$$

we get $A(x) = O(x)$, which implies that the integral ($s = t + iv$)

$$f(s) = \int_0^\infty e^{-su}dA(u) \tag{3.3}$$

is absolutely convergent for $t > 0$, and that

$$f^2(s) = \int_0^\infty e^{-su}dh(u) = s\int_0^\infty e^{-su}h(u)\ du \ \textit{for}\ t > 0 \quad ([12]\ \text{p. } 91)\ . \tag{3.4}$$

According to the assumptions on h, we have for the function g, defined by $g(x) = h(x) - x^2/2$, the estimate

$$|g(x)| \leqq K_1(x^q + 1) \text{ for some } K_1 \text{ and every } x \geqq 0\ . \tag{3.5}$$

Putting the function g into (3.4) we get

$$f^2(s) = s\int_0^\infty e^{-su}\left(\frac{u^2}{2} + g(u)\right)du = s^{-2} + s\int_0^\infty e^{-su}g(u)du\ . \tag{3.6}$$

We now restrict t to the interval (0, 1) and take the absolute of (3.6)

$$(3.7)\qquad \begin{aligned} |f^2(s) - s^{-2}| &\leqq |s| \int_0^\infty K_1(u^q + 1)\, e^{-tu} du \\ &= K_1 \Gamma(q+1)\, |s|\, t^{-(q+1)} + K_1 |s|\, t^{-1} \leqq K_2 |s|\, t^{-(q+1)} \,. \end{aligned}$$

This formula may also be expressed

$$(3.8)\qquad f^2(s) = s^{-2}(1 + r_1(s)\, s^3\, t^{-(q+1)}) \,, \quad \text{where } |r_1(s)| \leqq K_2 \,.$$

If $|s|^3 t^{-(q+1)} < (2K_2)^{-1}$ we conclude from (3.8) that

$$f(s) = s^{-1}(1 + r_2(s)\, s^3\, t^{-(q+1)}) \,, \quad \text{where } \ |r_2(s)| \leqq K_2 \,.$$

Hence

$$(3.9)\qquad |f(s) - s^{-1}| \leqq K_2 |s|^2\, t^{-(q+1)} \leqq \sqrt{K_2/2}\, |s|^{1/2}\, t^{-(q+1)/2} \,.$$

If $|s|^3 t^{-(q+1)} \geqq (2K_2)^{-1}$ we get from (3.8) the estimate $|f(s)| \leqq |s|^{-1}(1 + |s|^3 t^{-(q+1)} K_2)^{1/2} \leqq \sqrt{3K_2}\, |s|^{1/2}\, t^{-(q+1)/2}$ and

$$(3.10)\qquad \begin{aligned} |f(s) - s^{-1}| &\leqq |s|^{1/2}\, t^{-(q+1)/2} (\sqrt{3K_2} + \sqrt{2K_2}) \\ &\leqq |s|^2\, t^{-(q+1)/2}\, K_2(\sqrt{6} + 2) \,. \end{aligned}$$

We now define a new function a by

$$a(u) = \begin{cases} A(u) - u & \text{if } u \geqq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

and a function L by

$$L(s) = s^{-1}(f(s) - s^{-1}) = \int_0^\infty e^{-su}(A(u) - u) du \quad \text{for } t > 0 \,.$$

The estimates (3.9) and (3.10) give us

$$(3.11)\qquad |L(s)| \leqq K_3\, |s|\, t^{-(q+1)} \quad \text{and} \quad |L(s)| \leqq K_3\, |s|^{-1/2}\, t^{-(q+1)/2} \,.$$

These are the estimates corresponding to those given by Bojanić, Jurkat and Peyerimhoff in [4].

We are now going to use our lemma with a as the function F, and L as the function f in formula (2.1). The number α is 1 and we need an estimate of the type (2.2):

$$(3.12)\qquad \begin{aligned} &\int_{-\infty}^{+\infty} |L(t + iv)|\, \omega^{-1} v^{-2} \quad \sin^2(\omega v)\, dv \\ &\leqq 2K_3 \int_0^{t^{-(q+1)/3}} |s|^2\, t^{-(q+1)}\, \omega\, dv + 2K_3 \int_{t^{-(q+1)/3}}^{+\infty} \omega^{-1}\, v^{-2}\, |s|^{-1/2} t^{-(q+1)/2}\, dv \\ &= 2K_3\, \omega\, t^{1-q} \int_0^{t^{(q-2)/3}} (1 + u^2)^{1/2}\, du \end{aligned}$$

$$
\begin{aligned}
&+ 2K_3\, \omega^{-1}\, t^{-2-(q/2)} \int_{t^{(q-2)/3}}^{+\infty} u^{-2}(1+u^2)^{-1/4}\, du \\
&\leqq 2K_3(\sqrt{2}\; t^{1-q}\, \omega + t^{1-q}\, t^{(2q-4)/3}\, \omega + (2/3)\, \omega^{-1} t^{-2-(q/2)} t^{-(3/2)(q-2)/3}) \\
&\leqq 2K_3(3t^{-(q+1)/3}\, \omega + \omega^{-1}\, t^{-(q+1)}) = 8K_3\, t^{-(2/3)(q+1)}
\end{aligned}
$$

for

$$w = t^{\gamma} = t^{(1-(2/3)(q+1)-1)/2} = t^{-(q+1)/3} .$$

With $\alpha = 1, \beta = -(2/3)(q+1)$ and $\alpha + \beta = (1/3)(1-2q) > -1$ we apply the lemma and get (3.2):

$$a(x) = O(x^{(q+1)/3}) .$$

Putting $q = 1$ we obtain that $h(x) = x^2/2 + O(x)$ implies $A(x) = x + O(x^{2/3})$.

As an example that the conclusion $A(x) = x + o(x^{q/2})$ is false, we give

$$A(x) = \begin{cases} 10^n \text{ if } 10^n - 4^{-1}\, 10^{nq/2} < x < 10^n + 4^{-1}\, 10^{nq/2} , & n = 1, 2, \cdots \\ x \quad \text{everywhere else} \end{cases}$$

REMARK. If the assumption (3.1) is formulated

$$h(x) = x^{2p}\, \Gamma^2(p+1)\, \Gamma^{-1}(2p+1) + O(x^q) \qquad (p > 0,\ 0 \leqq q < 2p)$$

we can with the same method as above prove

$$A(x) = x^p + O(x^d) , \quad \text{where } d = (4p^2 + p + pq + 4q)/(6p+9) , \quad x \to \infty .$$

$p = q$ gives for instance $d = 5(p^2 + p)/(3(2p+3)) \sim (5/6)p$ as $p \to \infty$.

4. On Brownell's conjecture.

THEOREM 2. *Let F be a real valued function on $[0, \infty)$, which is of bounded variation over every finite interval $[0, N]$, which is continuous at $x = 0$ with $F(0) = 0$, and for which $\int_0^\infty e^{-tx}\, |dF(x)| < +\infty$ for real $t > 0$. With t and v real and $s = t + iv$ we define g by the Lesbesques-Stieltjes integral $g(s) = \int_0^\infty e^{-sx}\, dF(x)$, analytic in the region $t > 0$. Let F satisfy the three conditions that*

(4.1) $g(t) = b + O(e^{-c/t})$, $t \to 0+$, *for some real constants $c > 0$ and b,*
(4.2) $F(x) + Kx^{\alpha}$ *is nondecreasing over $x \geqq 1$ for some real constants $K > 0$ and $\alpha > 1/2$,*
(4.3) $|g(s)| \leqq M_1\, |s|^{1-2\alpha+\eta} + M_2$ *for $t > 0$ and some real constants M_1, M_2 and $1 > \eta > 0$, $\eta < \alpha - 1/2$.*

Then as $x \to +\infty$

4.4) $F(x) = o(x^{\alpha-1/2})$, *and actually is* $O^{(\alpha-1/2-\varepsilon)}$ *for some* $\varepsilon > 0$.

Proof. Define $\tilde{g}(s) = g(s) - b$, so $\tilde{g}(t) = O(e^{-c/t})$ in (4.1). Also $\tilde{g}(s) = \int_0^\infty e^{-sx} dF(x) - b = s\int_0^\infty F(x)e^{-sx}\,dx - b = s\int_0^\infty [F(x) - b]\,e^{-sx}\,dx$.

From (4.3) we get $|\tilde{g}(s)| \leqq K_1 |s|^{1-2\alpha+\eta}$ if $|s| \leqq 2$ and $|\tilde{g}(s)| \leqq K_1$ if $|s| \geqq 1$ for some real constant K_1.

We now define a function h by $h(s) = K_1^{-1} s^{2\alpha-1-\eta}\,\tilde{g}(s)$.

This function evidently satisfies the following inequalities:

$$|h(s)| \leqq 1 \text{ if } |s| \leqq 2 \text{ and } |h(s)| \leqq |s|^{2\alpha-1-\eta} \text{ if } |s| \geqq 1\,. \tag{4.5}$$

In order to use our lemma, our problem is now to estimate the integral

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\tilde{g}(t+iv)|\,|t+iv|^{-1}\,\omega^{-1}\,v^{-2}\,\sin^2(\omega v)\,dv \\
&\qquad = K_1\int_{-\infty}^{+\infty} |h(t+iv)|\,|t+iv|^{\eta-2\alpha}\,\omega^{-1}\,v^{-2}\,\sin^2(\omega v)\,dv\,.
\end{aligned} \tag{4.6}$$

The following notations will be used

$$\delta = \eta/(2(2\alpha+1))\,, \quad \varepsilon = \delta\eta/3\,.$$

The number ω will be $t^{-1/2+\varepsilon} = t^{\gamma}$.

Inserting the estimates for h on the right side of (4.6) we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\tilde{g}(t+iv)|\,|t+iv|^{-1}\,\omega^{-1}\,v^{-2}\,\sin^2(\omega v)\,dv \\
&\leqq 2K_1\int_1^{+\infty} v^{-3}\,\omega^{-1}\,dv + 2K_1\int_{t^{1/2+\delta}}^{1} v^{\eta-2\alpha}\,\omega^{-1}\,v^{-2}\,dv \\
&+ K_1\int_{-t^{1/2+\delta}}^{t^{1/2+\delta}} |h(t+iv)|\,t^{\eta-2\alpha}\,dv \\
&= O(t^{1/2-\varepsilon}) + O(t^{1/2-\varepsilon}\,t^{-(2\alpha+1-\eta)(1/2+\delta)}) + K_1 I_t \\
&= O(t^{-\alpha+2\varepsilon}) + K_1 I_t\; t \to 0+\,,
\end{aligned} \tag{4.7}$$

where

$$I_t = \int_{-t^{1/2+\delta}}^{t^{1/2+\delta}} \omega\,|h(t+iv)|\,t^{\eta-2\alpha}\,dv\,.$$

To get a suitable bound for I_t we have to improve the estimate $|h(s)| \leqq 1$ for $|s| \leqq 2$ by aid of (4.1). We apply a theorem of Milloux's ([3] p. 134–137) stating that if $f(\zeta)$ is regular in $|\zeta| < 1$ and $|f(\zeta)| \leqq 1$ there and if $|f(y)| \leqq m < 1$ for real y, $0 \leqq y < 1$, then the estimate

$$|f(\zeta)| < m^{(1/2)(1-|\zeta|)}$$

is valid in the unit circle.

Let us now consider the circle with radius 1 and centre $z = 1$. In this circle $|h(z)| \leqq 1$ and $|h(u)| = K_1 |u|^{2\alpha-1-\eta} |\tilde{g}(u)| \leqq K_2 t^{2\alpha-1-\eta} e^{-c/t}$ for real $u, 0 < u \leqq t$.

We now map this circle onto the unit circle by $w(z) = (z - t)/(z(t - 1) - t)$. Then the line $Im\ z = 0,\ 0 < Re\ z \leqq t$ is mapped on $Im\ w = 0,\ 0 \leqq Re\ w < 1$ and the line $Re\ z = t$ is mapped on a circle with diameter from $w = -(1 - t)^{-1}$ to $w = 0$.

In order to use the theorem of Milloux's mentioned above for $f(w(z)) = h(z)$ we have to estimate

$$\begin{aligned}\min_{|v|<t^{1/2+\delta}} (1 - |w(t + iv)|) &= 1 - |w(t + it^{1/2+\delta})| \\ &= 1 - t^{1/2+\delta}((t^2 - 2t)^2 + ((t - 1)t^{1/2+\delta})^2)^{-1/2} \\ = (((t^{3/2-\delta} - 2t^{1/2-\delta})^2 + (t - 1)^2)^{1/2} - 1)&/((t^{3/2-\delta} - 2t^{1/2-\delta})^2 + (t - 1)^2)^{1/2} \\ &= (2t^{1-2\delta} + O(t))/(1 + O(t^{1-2\delta})) = 2t^{1-2\delta} + O(t)\ .\end{aligned}$$

Thus we are now able to estimate the integral I_t as

$$\begin{aligned}I_t &\leqq \int_{-t^{1/2+\delta}}^{t^{1/2+\delta}} (t^{-1/2+\varepsilon}) [K_2 t^{2\alpha-1-\eta} e^{-c/t}]^{(1/2)(2t^{1-2\delta}+O(t))} t^{\eta-2\alpha} dv \\ &\leqq K_3 t^{\delta+\varepsilon+\eta-2\alpha+(2\alpha-1-\eta)(t^{1-2\delta}+O(t))} e^{-ct^{-2\delta}+O(1)} \\ &= O(1) \quad \text{as} \quad t \to 0^+\ .\end{aligned}$$

Introducing this estimate in (4.7) we get, since $2\varepsilon < \alpha$,

$$\int_{-\infty}^{+\infty} |\tilde{g}(t + iv)|\, |t + iv|^{-1} \omega^{-1} v^{-2} \sin^2(\omega v)\ dv = O(t^{-\alpha+2\varepsilon})\ .$$

Returning to the definition of $\tilde{g}(s)$, we finally apply the lemma with $\alpha, \beta = -\alpha + 2\varepsilon,\ \gamma = (\alpha + \beta - 1)/2 = \varepsilon - 1/2$ and conclude that

$$F(x) = O(x^{\alpha-1/2-\varepsilon}) = O(x^{\alpha-1/2})\ ,$$

and the proof is finished.

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UNIVERSITY OF GOTHENBURG, SWEDEN