MARKOV PROCESSES WITH STATIONARY MEASURE

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In [1] we studied Markov processes with a finite positive stationary measure. Here the process is assumed to have a positive stationary measure which might be infinite. Most of the results proved in [1] remain true also in this case. Some proofs that remain valid in this case will be replaced here by simpler proofs.

The main problem studied here, and in [1], is the behaviour at ∞ of $\mu(x_n \in A \cap x_0 \in B)$ where μ is the stationary measure and x_n is the Markov process.

In addition we study the quantities

$$\mu(x_n \in A \text{ for some } n \cap x_0 \in B)\ , \quad \mu((x_n \in A \text{ infinitely often}) \cap x_0 \in B)\ .$$

For Markov chains the results given here are well known even without the assumption of the existence of a stationary measure.

DEFINITIONS AND NOTATION. The notation here will be the same as in [1]. Let (Ω, Σ, μ) be a measure space where $\mu \geqq 0$ but is not necessarily finite.

Let $x_n(\omega)$ be a sequence of measurable real functions defined on Ω.

Let the measure $\mu(x_0^{-1}(\ \ ))$, on the real line, be σ finite.

ASSUMPTION 1. *The process is stationary*:

$$\mu(x_{n+k} \in A \cap x_{m+k} \in B) = \mu(x_n \in A \cap x_m \in B)\ .$$

ASSUMPTION 2. *If $i < j < k$ let A be a Borel set on the line such that $\mu(x_k \in A) < \infty$ then*:

The conditional probability that $x_k \in A$, given x_j and x_i, is equal to the conditional probability that $x_k \in A$ given x_j.

$L_2 = L_2(\Omega, \Sigma, \mu)$ will be the space of real square integrable function. Let B_n be the subspace of L_2 generated by functions of the form

$$I(x_n^{-1}(A)) \text{ where } \mu(x_n^{-1}(A)) < \infty\ .$$

By $I(\sigma)$ we denote the characteristic function of σ.

Let E_n be the self adjoint projection on B_n.

It was shown in [1] that Assumption 2 implies

1. $$E_i E_j E_k = E_i E_k \qquad i < j < k\ .$$

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Finally let $T(n)$ be the transformation from B_0 to B_n defined by

$$T(n)I(x_0 \in A) = I(x_n \in A) .$$

It is easily seen that $x \in B_n$ if and only if $x(\omega) = f(x_n(\omega))$ a.e. and $f(x_n(\omega))$ is square integrable.

Thus

$$T(n)f(x_0(\omega)) = f(x_n(\omega))$$

and

2. a. $\| T(n)x \| = \| x \|$
   b. $T(n)B_0 = B_n$
   c. $(T(n+k)x, \ T(m+k)y) = (T(n)x, \ T(m)y)$.

See [1] Lemma 2.4.

1. Behaviour at ∞. Following [1] let us define

$$C_m = \bigcap_{n=m}^{\infty} B_n \qquad C_{-m} = T(m)^{-1}C_0 \cap C_0$$

$$H = \bigcap_{m=1}^{\infty} C_{-m} .$$

Theorems 3.6 and 3.7 of [1] hold here thus:

$$\textit{If } x \perp H \textit{ then } \text{weak} \lim_{n \to \infty} T(n)x = 0 .$$

Also by Theorem 3.9 of [1] H is invariant under $T(n)$, and $T(n) = T^n$ is a unitary operator on H.

LEMMA 1. *The subspace H is generated by characteristic functions of a Boolean ring.*

Proof. It is enough to show that if $x \in H$ then $I(x^{-1}(A)) \in H$ and if $I(\sigma_1), I(\sigma_2) \in H$ then $I(\sigma_1 \cap \sigma_2) \in H$.

If $x \in H$ then $x \in B_n$ so $I(x^{-1}(A)) \in B_n$. Also $x = T(n)y_n$ where $y_n \in C_0$.

Now

$$y_n(\omega) = f_n(x_0(\omega)) \quad \text{for } y_n \in B_0 .$$

Also $I(y_n^{-1}(A)) \in B_m$ for all m and n. Thus

$$x(\omega) = T(n)y_n(\omega) = f_n(x_n(\omega))$$
$$x^{-1}(A) = x_n^{-1}(f_n^{-1}(A))$$

and

$$I(x^{-1}(A)) = T(n)I(x_0^{-1}(f_n^{-1}(A)))$$

where

$$I(x_0^{-1}(f_n^{-1}(A)) = I(y_n^{-1}(A)) \in B_m \quad \text{for all } m .$$

Thus

$$I(x^{-1}(A)) \in H .$$

Finally if $I(\sigma_1) \in H$ and $I(\sigma_2) \in H$ then $I(\sigma_1 \cap \sigma_2) \in B_n$ for all n. Also

$$\sigma_1 = x_n^{-1}(A_n) \qquad \sigma_2 = x_n^{-1}(B_n)$$

where

$$I(x_0^{-1}(A_n)) \in C_0 \qquad I(x_0^{-1}(B_n)) \in C_n .$$

Thus

$$I(\sigma_1 \cap \sigma_2) = I(x_n^{-1}(A_n \cap B_n))$$

where

$$I(x_0^{-1}(A_n \cap B_n)) \in C_0 .$$

In the rest of the paper it is assumed that if $I(\sigma) \in H$ then $I(\sigma)$ contains an atom in H. This is equivalent to assuming that H is generated by $I(\sigma_i)$ where σ_i are disjoint measurable sets.

Notice that H may be empty.

The above assumption holds if x_0 has a countable range or if a "Doeblin Condition" holds:

There exists a measure η on Borel sets on the line and an $\varepsilon > 0$ suct that:

1. *If $\mu(x_0^{-1}(A)) < \infty$ then $\eta(A) < \infty$.*
2. *If $\eta(A) < \varepsilon$ then $T(n)I(x_0^{-1}(A)) \notin B_0$ for some n.*

This condition is enough for if $I(x_0^{-1}(A)) \in H$ then $\eta(A)$ is finite and by 2 contains only finitely many sets in H.

For every set σ_i $T(n)I(\sigma_i)$ is in H hence is either $I(\sigma_i)$ or is disjoint to $I(\sigma_i)$.

Ler Ω_1 be the union of all the σ_i for which

$$T(n)I(\sigma_i) = I(\sigma_i) \quad \text{for some } n .$$

Let Ω_2 be the union of all the sets σ_i such that

$$(T(n)I(\sigma_i), I(\sigma_i)) = 0 \quad \text{for all } n .$$

In this case

$$(T(n)I(\sigma_i), T(m)I(\sigma_i)) = 0 \qquad \text{if } n \neq m ,$$

by 2.c.

Let Ω_3 be the complement set of $\Omega_1 \cup \Omega_2$.
If μ is finite then $\Omega_1 = \Omega$.

THEOREM 1. *Let A be a Borel set on the line such that $x_0^{-1}(A) \subset \sigma_i$ for some i.*

If $\sigma_i \subset \Omega_1$ and n is the smallest integer such that $T(n)I(\sigma_i) = I(\sigma_i)$ then

$$\text{weak} \lim_{k\to\infty} T(kn + d)I(x_0^{-1}(A)) = \mu(\sigma_i)^{-1}\, \mu(x_0^{-1}(A))T(d)I(\sigma_i)\ .$$

If $\sigma_i \subset \Omega_2$ then

$$\text{weak} \lim_{n\to\infty} T(n)I(x_0^{-1}(A)) = 0\ .$$

Proof. If $T(n)I(\sigma_i) = I(\sigma_i)$ define

$$g(\omega) = I(x_0^{-1}(A)) - \mu(\sigma_i)^{-1}\mu(x_0^{-1}(A))I(\sigma_i)\ .$$

Now $g(\omega) \perp H$ hence

$$T(kn + d)g(\omega) = T(kn + d)I(x_0^{-1}(A)) - \mu(\sigma_i)^{-1}\mu(x_0^{-1}(A))T(d)I(\sigma_i)$$

and this expression tends weakly to zero when $k \to \infty$. If $x_0^{-1}(A) \subset \sigma_i$ where $\sigma_i \subset \Omega_2$ then the functions $T(n)I(x_0^{-1}(A))$ are disjoints.

THEOREM 2. *If $x_0^{-1}(A) \subset \Omega_3$ then*

$$\text{weak} \lim_{n\to\infty} T(n)I(x_0^{-1}(\mathrm{A})) = 0\ .$$

Proof. It is enough to note that $I(x_0^{-1}(A)) \perp H$, for $\Omega_1 \cup \Omega_2$ contains all the sets σ_i.

Let

$$U(n, A) = I(\bigcup_{m=n}^{\infty} x_m \in A)$$
$$U(A) = \lim_{n\to\infty} U(n, A)\ .$$

Thus

$$(U(0, A), I(x_0^{-1}(B))) = \mu((x_n \in A \text{ for some } n) \cap x_0 \in B)$$
$$(U(A), I(x_0^{-1}(B))) = \mu((x_n \in A \text{ infinitely often}) \cap x_0 \in B)\ .$$

THEOREM 3. *Let A be a Borel set such that $x_0^{-1}(A) \subset \sigma_i$ for some i. If $\sigma_i \subset \Omega_1$ and $T(n)I(\sigma_i) = I(\sigma_i)$ then*

$$U(m, A) = U(A) = \sum_{d=0}^{n-1} T(d)I(\sigma_i)\ .$$

If $\sigma_i \subset \Omega_2$ *then* $U(A) = 0$

Proof. If $T(n)I(\sigma_i) = I(\sigma_i)$ then

$$U(A) \leqq U(0, A) \leqq \sum_{d=0}^{n-1} T(d)I(\sigma_i) .$$

On the other hand if $I(\sigma) \leqq T(d)I(\sigma_i)$ then

$$\begin{aligned}(U(A), I(\sigma)) &\geqq \lim_{k\to\infty} (T(kn + d)I(x_0^{-1}(A)), I(\sigma)) \\ &= \mu(\sigma_i)^{-1} \mu(x_0^{-1}(A)) \mu(\sigma) > 0 .\end{aligned}$$

But $U(A)$ is a characteristic function therefore the above equation implies that $U(A) \geqq T(d)I(\sigma_i)$. Thus

$$U(A) = \sum_{d=0}^{n-l} T(d)I(\sigma_i) .$$

If $(T(n)I(\sigma_i), I(\sigma_i)) = 0$ for all n, then $U(n, A)$ is disjoint to $T(m)I(x_0^{-1}(A))$ $m < n$. Thus $U(A)$ is disjoint to $T(m)I(x_0^{-1}(A))$ for all m and therefore $U(A) = 0$.

COROLLARY. *In the first case studied above*

$$\begin{aligned}&\mu((x_n \in A \textit{ for some } n) \cap x_0 \in B) \\ &\qquad = \mu((x_n \in A \textit{ infinitely often}) \cap x_0 \in B) .\end{aligned}$$

In the second case

$$\mu((x_n \in A \textit{ infinitely often}) \cap x_0 \in B) = 0 .$$

REMARKS. Let a Markov chain be defined by the matrix $(P_{i\,j})$ $P_{i,i+1} = 1$ $P_{i,j} = 0$ if $j \neq i + 1$, $-\infty < i, j < \infty$. Then if $\mu(x_n = i) = 1$ Ω can be chosen as the union of countably many atoms. In this case $H = L_2(\Omega)$ and $\Omega = \Omega_2$. Let (P_{ij}) be the matrix of a free random walk (See K. L. Chung Markov Chains p. 23) and again $\mu(x_n = i) = 1$ $-\infty < i < \infty$. In this case for every i and j there is a sufficiently large n such that $\mu(x_n = i \cap x_0 = j) = P_{ij}^{(n)} > 0$. Thus each set $x_0 = i$ is neither in Ω_1 nor in Ω_2 and $\Omega = \Omega_3$.

Let $P(x, A)$ be a transition function of a Markov process with the real numbers as state space. Let μ be a stationary measure that is not finite. One can construct a measure space Ω and the sequence $x_n(\omega)$ with

$$\mu(x_n \in A \cap x_0 \in B) = \int_{x\in B} P^n(x, A) \mu(dx) .$$

Notice that we use alternatively $\mu(B)$ or $\mu(x_0 \in B)$ to mean the same thing. This construction is well known.

Let $\mu(x_0 = 1) > 0$ and let the set $\bigcap_{n=0}^{\infty} \{x \mid P^n(x, 1) = 0\}$ be empty. Then if $\mu(x_0 \in A) > 0$

(*) $\sup_n \mu(x_n = 1 \cap x_0 \in A) > 0$.

Otherwise $P^n(x, 1) = 0$ $x \in A$ except on a set of measure zero.

We will prove that in this case $H = 0$ hence $\Omega = \Omega_2$

and

$$\lim_{n \to \infty} \mu(x_n \in A \cap x_0 \in B) = 0 .$$

If H contained any characteristic function of a set $\{\omega \mid x_0 \in A\}$ (always $H \subset B_0$) then this set intersects the set $\{\omega \mid x_n(\omega) = 1\}$ for some n. But $H \subset B_n$ and this set is an atom in B_n. Therefore $\{\omega \mid x_0 \in A\}$ contains the set $\{\omega \mid x_n(\omega) = 1\}$. There exists an atom in H that contains this set. This proves that H is generated by atoms. Let H be generated by σ_i where $\sigma_1 \supset \{\omega \mid x_n(\omega) = 1\}$. Now

$$\sup_m (I(\sigma_i), T(m)I(\sigma_1)) \geqq \sup_m \mu(\sigma_i \cap x_{n+m} = 1) .$$

But $\sigma_i = \{\omega \mid x_n(\omega) \subset A_i\}$ for $I(\sigma_i) \in B_n$. Hence

$$\sup_m (I(\sigma_i), T(m)I(\sigma_1) \geqq \sup_m \mu(x_n \in A_i \cap x_{n+m} = 1)$$
$$= \sup_m x_0 \in A_i \cap x_m = 1) > 0 .$$

By (*).

Thus for some m $I(\sigma_i) = T(m)I(\sigma_1)$. Now

$$\sup_m \mu(\sigma_1 \cap \sigma_i) = \sup_m (I(\sigma_1), T(m)I(\sigma_i)) \geqq \sup_m \mu(x_n = 1 \cap x_{n+m} = 1) > 0 .$$

They can not be disjoint: for some m,

$$T(m)I(\sigma_1) = I(\sigma_1) .$$

Now

$$\bigcup_{i=1}^{\infty} \sigma_i = \bigcup_{k=0}^{m-1} T(k)I(\sigma_1)$$

and this is a set of finite measure. But Ω had infinite measure. Since $\cup \sigma_i \subset B_0$ there is a set in B_0 disjoint to $\cup \sigma_i$ which contradicts (*).

Reference

1. S. R. Foguel, *Weak and strong convergence for Markov processes*, Pacific J. Math., **10** (1960), 1221–1234.

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