CONTINUOUSLY INVERTIBLE SPACES

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In [4] we introduced the concept of an invertible space. (A topological space S is *invertible* if, for each open set U in S, there is an *inverting homeomorphism for* U, $h : S \rightarrow S$, of S onto itself such that $h(S - U)$ lies in U.) This paper is a continuation of the investigation of invertibility in which we concentrate upon that aspect contained in the following definition: A topological space S is *continuously invertible* if it is invertible and if, for each open set in S, there is an inverting homeomorphism that is isotopic to the identity mapping.

It is easy to see that the n-sphere S^n enjoys this property for, given an open set U in S^n, there is an inverting homeomorphism for U in each component of the space of homeomorphisms of S^n onto itself. It was this observation which suggested our present investigation. We acknowledge a debt of gratitude to the unknown referee of our paper [4] who brought to our attention the papers by Dancer [2], L. Whyburn [6] and Wilder [7]. While they were of little assistance, these papers do contain results allied to our own and hence provide us with a link to the past.

Let S be a continuously invertible space and let $\mathscr{G}(S)$ denote the group of homeomorphisms of S onto itself. (Note that each element of $\mathscr{G}(S)$ is an inverting homeomorphism for *some* open set, the identity mapping included.) Let $\mathscr{G}(S)$ be the subgroup of $\mathscr{G}(S)$ consisting of all homeomorphisms that are isotopic to the identity mapping. If x is a point of S, we let

$$O_x = \{y \mid y = g(x)\ ,\quad g \in \mathscr{G}(S)\}$$

denote the *total orbit* of x and we let

$$P_x = \{y \mid y = h(x)\ ,\quad h \in \mathscr{G}(S)\}$$

denote the *continuous orbit* of x.

By Theorem 8 of [4], each total orbit O_x is dense in S and an obvious modification of the same argument proves that each continuous orbit P_x is also dense in S. We note that each continuous orbit is connected. For if y is any point of the continuous orbit P_x, then the isotopy path of x during an isotopy carrying x onto y is a continuum. Therefore P_x is a union of continua having the point x in common.

THEOREM 1. *If S is a continuously invertible space, then every continuous orbit (and every total orbit) is connected and dense in S.*

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Also, S itself is connected.

THEOREM 2. *If S is a continuously invertible T_1 space and if c is a cut point of S, then the continuous orbit P_c of c is S itself.*

Proof. Suppose there were a point x in $S - P_c$. Since $S - c = A \cup B$ set, we may assume that x lies in A. The set B is open under the hypotheses and hence there is an inverting homeomorphism h, isotopic to the identity, which carries x into B. But then the isotopy path of x must pass through the cut point c and hence the continuous orbits P_x and P_c intersect. This is impossible.

COROLLARY. *No continuously invertible Hausdorff continuum has a cut point.*

Proof. If such a continuum is degenerate, then it has no cut points. If it is nondegenerate, then it has at least two noncut points and the existence of a cut point would contradict Theorem 2.

THEOREM 3. *In a* (nondegenerate) *continuously invertible Hausdorff space, each continuous orbit is arcwise connected and each point in a total orbit lies on an arc in the orbit.*

Proof. The isotopy path of a point is a continuous image of the unit interval in a Hausdorff space and hence is a Peano continuum. Then by Theorem 10 of [4], each orbit is homogeneous.

COROLLARY. *No orbit in a (nondegenerate) continuously invertible Hausdorff space is degenerate.*

COROLLARY. *Every continuously invertible (nondegenerate) Hausdorff space is a union of nondegenerate, dense, disjoint, homogeneous, arcwise connected, continuously invertible subspaces.*

Proof. Each such space is the union of its continuous orbits.

If a continuously invertible Hausdorff space S contains no simple closed curve, then every Peano continuum in S is a dendrite and if S itself is a Peano continuum, then S is a dendrite. But by virtue of the corollary to Theorem 2, a dendrite is not continuously invertible.

THEOREM 4. *Each nondegenerate continuously invertible Peano continuum contains a simple closed curve.*

THEOREM 5. *If the invertible space S contains a separating proper subcontinuum C, then each open set in S contains a separating continuum imbedded in S as is C.*

Proof. This is an application of Theorem 6 of [4].

THEOREM 6. *If an invertible space S is separated by a proper closed set C which is irreducible with respect to separating S, then C contains no open set of S.*

Proof. This is an application of Theorem 5 above.

Next we have an extrinsic characterization of the n-sphere which may be compared with the intrinsic characterization given in [3].

THEOREM 7. *Let M be a set in E^{n+1} which is continuously invertible and which contains an n-sphere, S. Then M is the n-sphere S.*

Proof. Suppose that $M - S$ is not empty. There is no loss of generality in assuming that there are points of M in the bounded component A of $E^{n+1} - S = A \cup B$. By Theorem 5, there is an n-sphere S' in $A \cap M$ and, in particular, there is an isotopy H_t which carries S onto S' where $H_t(S)$ lies in M for each t, $0 \leqq t \leqq 1$.

Now M cannot contain an $(n + 1)$-cell for if it were locally Euclidean of dimension $n + 1$ at any point, then Theorem 1 of [3] would imply that M is an $(n + 1)$-sphere imbedded in E^{n+1} which is impossible. Thus there must be a point p lying in the annular region between S and S' such that p is not in M. Similarly, there is a point q in the unbounded domain B such that q is not in M. Clearly, the continuous cycles S and $p \cup q$ are linked, whence the the isotopic cycle S' must be linked with $p \cup q$. This is contradictory.

THEOREM 8. *The only continuously invertible Peano continua in the plane are the simple closed curves.*

Proof. By Theorem 4, each such continuum contains a simple closed curve and then Theorem 7 applies.

We conclude this report with a few results on continuously invertible plane continua which serve only to indicate a direction in which further study may be fruitful.

THEOREM 9. *Let C be a continuously invertible plane continuum that is not a simple closed curve. If x and y are two points in the same continuous orbit in C, then there is a unique arc in C having*

x and y as endpoints.

Proof. By Theorem 3 there is at least one arc joining x and y. If there were another, then C would contain (and hence be) a simple closed curve.

THEOREM 10. *Let C be a continuously invertible plane continuum that is not a simple closed curve. Then every Peano continuum in C is a simple arc.*

Proof. From a previous remark and Theorem 7 we know that every Peano continuum in C is a dendrite. If there is a dendrite in C other than a simple arc, then there would be a simple triod T in C. Now $C - T$ is not empty and hence there is an isotopy carrying T into its complement. The isotopy path of T is a Peano continuum in C and hence is a dendrite. But then the isotopy path of the branch point b of T contains uncountably many branch points in the isotopy path of T. This is impossible because no dendrite contains uncountably many branch points.

THEOREM 11. *No proper subcontinuum of a continuously invertible plane continuum separates the plane.*

Proof. If some proper subcontinuum separates the plane, then by Theorem 5 there is a separating subcontinuum in every open set of the continuum. A construction as in the proof of Theorem 7 using Theorem 7, Chap. 1 of [5] will then yield a contradiction.

THEOREM 12. *Let C be a continuously invertible plane continuum that is not a simple closed curve. Then every proper subcontinuum of C is arcwise connected.*

Proof. Let C' be a proper subcontinuum of C and suppose there are points x, y in C' which lie on no arc in C'. Let U be an open set of C such that $\bar{U}$ lies in $C - C'$. Then there is an isotopy of C with terminal mapping h such that $h(C') = C''$ lies in U. Letting $h(x) = x'$ and $h(y) = y'$, we note that there are arcs xx', yy' from C' to C'' and that these arcs must pass into $C - C'$, as we go from x to x' and y to y'. It follows that the continuum $D = C' \cup xx' \cup yy' \cup C''$ separates the plane. Thus by Theorem 11, we must have $D = C$. But this, too, is impossible since D obviously contains an open arc as an open set and hence Theorem 1 of [3] concludes that D is a simple closed curve, contrary to hypothesis.

THEOREM 13. *Let C be a continuously invertible plane continuum that is not a simple closed curve. Then every proper subcontinuum of C is an arc in some continuous orbit of C.*

Proof. Suppose there were a continuum C' in C which did not lie entirely in a continuous orbit. Let x and y be points of C' such that y is not in P_x. Then the arc from x to y given by Theorem 12 together with an arc in P_x having x as an interior point will contain a triod T. Then the same argument as in Theorem 10 yields a contradiction. The fact that a proper subcontinuum is an arc easily follows from Theorem 9.

THEOREM 14. *The only decomposable, continuously invertible plane continua are the simple closed curves.*

Proof. Suppose there is a decomposable continuously invertible plane continuum C that is not a simple closed curve. By definition, $C = C_1 \cup C_2$ where C_1 and C_2 are proper subcontinua. Then by Theorem 13, both C_1 and C_2 are simple arcs. Hence the open set $C_1 - C_2$ contains an open arc, whence Theorem 1 of [3] proves that C is a simple closed curve after all.

In [1], R. H. Bing has shown that a homogeneous plane continuum that contains an arc is necessarily a simple closed curve. Since a continuously invertible continuum contains an arc, we have the following result.

COROLLARY. *The only homogeneous, continuously invertible plane continua are the simple closed curves.*

The existence of an indecomposable, continuously invertible plane continuum remains an open question. In §8 of [1], Bing gives an example which may enjoy these properties but this has not been established.

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