ABSOLUTE CONTINUITY OF INFINITELY DIVISIBLE DISTRIBUTIONS

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1. Introduction and summary. A probability distribution function F is said to be infinitely divisible if and only if for every integer n there is a distribution function F_n whose n-fold convolution is F. If F is infinitely divisible, its characteristic function f is necessarily of the form

$$(1) \qquad f(u) = \exp\left\{iu\gamma + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x)\right\},$$

where $u \in (-\infty, \infty)$, γ is some constant, and G is a bounded, non-decreasing function. J. R. Blum and M. Rosenblatt [1] have found necessary and sufficient conditions that F be continuous and necessary and sufficient conditions that F be discrete. The purpose of this note is to add to the results of Blum and Rosenblatt by giving sufficient conditions under which an infinitely divisible probability distribution F is absolutely continuous. These conditions are that G be discontinuous at 0 or that $\int_{-\infty}^{\infty}(1/x^2)dG_{ac}(x) = \infty$, where G_{ac} is the absolutely continuous component of G. In § 2 some lemmas will be proved, and in § 3 the proof of the sufficiency of these conditions will be given. All notation used here is standard and may be found, for example, in Loève [2].

2. Some lemmas. In this section three lemmas are proved which will be used in the following section.

LEMMA 1. *If F and H are probability distribution functions, and if F is absolutely continuous, then the convolution of F and H, $F * H$, is absolutely continuous.*

This lemma is well known, and the proof is omitted.

LEMMA 2. *If $\{F_n\}$ is a sequence of absolutely continuous distribution functions, and if $p_n \geqq 1$ and $\sum_{n=1}^{\infty} p_n = 1$, then $\sum_{n=1}^{\infty} p_n F_n$ is an absolutely continuous distribution function.*

Proof. By using the Lebesgue monotone convergence theorem it is easy to verify that $\sum_{n=1}^{\infty} p_n f_n$ is the density of $\sum_{n=1}^{\infty} p_n F_n$, where f_n is the density of F_n.

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LEMMA 3. *Let* $\{Y, X_1, X_2, \cdots\}$ *be independent random variables. Assume that the* X_i*'s have the same absolutely continuous distribution* F, *and assume that the distribution of* Y *is Poisson with expectation* λ. *Then* $Z = X_1 + \cdots + X_Y$ *has a distribution function which has a saltus* $e^{-\lambda}$ *at* 0 *and is absolutely continuous elsewhere, and has as characteristic function*

$$f_Z(u) = \exp \lambda \int_{-\infty}^{\infty} (e^{iux} - 1) dF(x) .$$

Proof. Let $E(x)$ be the distribution function degenerate at 0, and let $F^{*n}(x)$ denote the convolution of F with itself n times. Then it is easy to see that the distribution function of Z, $F_Z(z)$, may be written as $F_Z(z) = e^{-\lambda}E(z) + \sum_{n=1}^{\infty} e^{-\lambda}(\lambda^n/n!)F^{*n}(z)$. By lemma 1, each F^{*n} is absolutely continuous and has a density f^{*n}. We need only show that $F_Z(z) - e^{-\lambda}E(z)$ is absolutely continuous. If we write

$$F_Z(z) - e^{-\lambda}E(z) = \sum_{n=1}^{\infty} e^{-\lambda}(\lambda^n/n!) \int_{-\infty}^{z} f^{*n}(t)dt$$

and apply the Lebesgue monotone convergence theorem we obtain this conclusion.

3. **The theorem.** If G is a bounded nondecreasing function used in (1), then we may write $G(x) = G_s(x) + G_{ac}(x)$, where G_s is a singular nondecreasing function and $G_{ac}(x)$ is an absolutely continuous nondecreasing function.

THEOREM. *Let* F *be an infinitely divisible distribution function with characteristic function* (1). *Then* F *is absolutely continuous if at least one of the following two conditions is satisfied*:

(i) G *is not continuous at* 0, *or*

(ii) $\int_{-\infty}^{\infty} (1/x^2) dG_{ac}(x) = \infty$.

Proof. If condition (i) is satisfied, then by Lemma 1 it easily follows that F is absolutely continuous, since in that case F is a convolution of a normal distribution with another infinitely divisible distribution. We now prove that condition (ii) is sufficient. By Lemma 1 it is sufficient to prove that the distribution function F_0 whose characteristic function is

$$\exp \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1 + x^2} \right) \frac{1 + x^2}{x^2} dG_{ac}(x) \tag{2}$$

is absolutely continuous. Let $\varepsilon_n > \varepsilon_{n+1} > 0$ for each n be such that $\varepsilon_n \to 0$ as $n \to \infty$ and such that

$$\lambda_n = \int_{S_n} ((1+x^2)/x^2) dG_{ac}(x) > 0 ,$$

where

$$S_n = (-\varepsilon_{n-1}, -\varepsilon_n] \cup [\varepsilon_n, \varepsilon_{n-1}) , \qquad n = 1, 2, \cdots ,$$

and where $\varepsilon_0 = \infty$. Let U_n be a random variable with characteristic function

$$(3) \qquad f_{U_n}(u) = \exp \int_{S_n} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG_{ac}(x) ,$$

and let

$$H_n(x) = (1/\lambda_n) \int_{(-\infty, x] \cap S_n} ((1+x^2)/x^2) dG_{ac}(x) .$$

One easily sees that $\lambda_n < \infty$ and that $H_n(x)$ is an absolutely continuous distribution function of a bounded random variable. For each positive integer n we may write, by Lemma 3, that

$$U_n = X_{n,1} + X_{n,2} + \cdots + X_{n,Z_n} - \int_{S_n} (1/x) dG_{ac}(x)$$

where Z_n is a random variable with Poisson distribution with expectation λ_n, where $\{X_{n,1}, X_{n,2}, \cdots\}$ have the common absolutely continuous distribution function $H_n(x)$, and where $\{Z_n, X_{n,1}, X_{n,2} \cdots\}$ are independent. If we assume that

$$\{\{Z_n, X_{n,1}, X_{n,2}, \cdots\}, n = 1, 2, \cdots\}$$

are all independent, then the distribution function of

$$U_0 = \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{Z_n} X_{n,j} - \int_{S_n} (1/x) dG_{ac}(x) \right)$$

is equal to F_0. Now let us define a sequence of events $\{C_n\}$ by

$$C_1 = [Z_1 \neq 0] , \qquad C_2 = [Z_1 = 0][Z_2 \neq 0] ,$$

and, in general,

$$C_n = [Z_n \neq 0] \bigcap_{i=1}^{n-1} [Z_i = 0] .$$

These events are easily seen to be disjoint. If we define

$$(4) \qquad C_0 = \left(\bigcup_{n=1}^{\infty} C_n \right)^c = \bigcap_{n=1}^{\infty} [Z_n = 0] ,$$

then $\Omega = \bigcup_{n=1}^{\infty} C_n$, where Ω is the sure event. The distribution function of U_0 is

$$F_{U_0}(u) = \sum_{n=1}^{\infty} P([U_0 \leqq u] \mid C_n)P(C_n) + P([U_0 \leqq u]C_0) .$$

By (4) and by hypothesis, we obtain

$$P([U_0 \leqq u]C_0) \leqq P(C_0) = \lim_{n\to\infty} \exp\left\{-\int_{-\infty}^{-\varepsilon_n} + \int_{\varepsilon_n}^{\infty} (1/x^2)dG_{ac}(x)\right\} = 0 .$$

Also, $P([U_0 \leqq u] \mid C_n)$ is the distribution function of $X_{n,1} + W_n$, where $X_{n,1}$ and W_n are independent random variables. Since the distribution function of $X_{n,1}$ is absolutely continuous, it follows by Lemma 1 that $P([U_0 \leqq u] \mid C_n)$ is absolutely continuous for each n. Lemma 2 then implies that $F_{U_0}(u)$ is absolutely continuous, which concludes the proof of the theorem.

The condition given in this theorem is not necessary, as is shown in the following example. Let $\gamma = 0$ in (1), and let α, β be real numbers which satisfy $\beta > 1, 1 > \alpha > \beta/2$. For $j = 1, 2, \cdots$, let us denote

$$x_j = j^{-\alpha} \quad \text{and} \quad \rho_j = j^{-\beta} .$$

Let G be a pure jump function with jumps at x_j and $-x_j$ of size ρ_j for every j. (The total variation of G is $2\sum \rho_j < \infty$.) In this case we obtain

$$f(u) = \exp 2 \sum_{n=1}^{\infty} \left(\cos\frac{u}{n^{\alpha}} - 1\right) \frac{n^{2\alpha} - 1}{n^{\beta}} .$$

We shall show that there is a constant K such that for all $|u| \geqq \pi$, the inequality

$$(5) \qquad 0 < f(u) < \exp(-K|u|^{2-\beta/\alpha})$$

is true. This is equivalent to showing that

$$(6) \qquad \sum_{n=1}^{\infty} \frac{n^{2\alpha} + 1}{n^{\beta}} \sin^2 \frac{|u|}{2n^{\alpha}} > K|u|^{2-\beta/\alpha} .$$

Let us consider, for each fixed $|u| \geqq \pi$ the integer N defined by

$$N = \left[\frac{1}{2}\left(\frac{2|u|}{\pi}\right)^{1/\alpha} + 1\right] ,$$

where the square brackets have their usual meaning. It is easy to verify that $0 < |u|/2N^{\alpha} < \pi/2$, and thus we may write

$$\frac{N^{2\alpha} + 1}{N^{\beta}} \sin^2 \frac{|u|}{2N^{\alpha}} > N^{2\alpha-\beta} \sin^2 \frac{|u|}{2\left[\left(\frac{2|u|}{\pi}\right)^{1/\alpha}\right]^{\alpha}}$$

$$> K|u|^{2-\beta/\alpha} ,$$

where K does not depend on u. This inequality implies that inequality (6) is true, thus implying (5). Inequality (5) implies that $f(u) \in L_1(-\infty, +\infty)$, which in turn implies that $f(u)$ is the characteristic function of an absolutely continuous distribution. (See Theorem 3.2.2 on page 40 in [3].)

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REFERENCES

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3. Eugene Lukacs, *Characteristic Functions*, Hafner, New York, 1960.

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