EVALUATION OF AN E-FUNCTION WHEN THREE OF ITS UPPER PARAMETERS DIFFER BY INTEGRAL VALUES

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1. Introduction. If $p \geqq q+1$, [1, p. 353]

$$E(p; \alpha_r : q; \rho_s : z) = \sum_{r=1}^{p} z^{\alpha_r} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_r + n) \prod_{t=1}^{p}{}' \Gamma(\alpha_t - \alpha_r - n)}{n! \prod_{s=1}^{q} \Gamma(\rho_s - \alpha_r - n)} (-z)^n , \tag{1}$$

where, if $p = q + 1$, $|z| < 1$. The dash in the product sign indicates that the factor for which $t = r$ is omitted.

Now, if two or more of the α's are equal or differ by integral values, some of the series on the right cease to exist. For instance, if $\alpha_1 = \alpha + l$, $\alpha_2 = \alpha$, where l is zero or a positive integer, it has been shown [2, p. 30] that the first two series can be replaced by the expression

$$\begin{aligned} &(-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + n) \prod_{t=3}^{p} \Gamma(\alpha_t - \alpha - l - n)}{n!(l+n)! \prod_{s=1}^{q} \Gamma(\rho_s - \alpha - l - n)} \Delta_n z^n \\ &+ z^{\alpha} \sum_{n=0}^{l-1} \frac{\Gamma(\alpha + n)(l - n - 1)! \prod_{t=3}^{p} \Gamma(\alpha_t - \alpha - n)}{n! \prod_{s=1}^{q} \Gamma(\rho_s - \alpha - n)} (-z)^n , \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Delta_n = {} & \psi(l+n) + \psi(n) - \psi(\alpha + l + n - 1) - \log z \\ & + \sum_{t=3}^{p} \psi(\alpha_t - \alpha - l - n - 1) - \sum_{s=1}^{q} \psi(\rho_s - \alpha - l - n - 1) . \end{aligned}$$

Here

$$\psi(z) = \frac{d}{dz} \log \Gamma(z+1) , \tag{3}$$

so that

$$\frac{d}{dz} \Gamma(z+1) = \Gamma(z+1) \psi(z) . \tag{4}$$

It will now be shown that, in the case in which

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$$\alpha_1 = \alpha,\ \alpha_2 = \alpha + l,\ \alpha_3 = \alpha + l + m\ ,$$

where l and m are zero or positive integers, the first three series can be replaced by the expression

$$
\begin{aligned}
&\tfrac{1}{2}(-1)^l z^{\alpha+l+m}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{n!(m+n)!(l+m+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-m-n)}\Delta_n \\
&-(-1)^l z^{\alpha+l}\sum_{n=0}^{m-1}\frac{\Gamma(\alpha+l+n)(m-n-1)!\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-l-n)z^n}{n!(l+n)!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-l-n)}\Theta_n \qquad (5)\\
&+ z^{\alpha}\sum_{n=0}^{l-1}\frac{\Gamma(\alpha+n)(l-n-1)!(l+m-n-1)!\prod_{t=4}^{p}\Gamma(\alpha_t-\alpha-n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s-\alpha-n)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times(-z)^n\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_n = \pi^2 &+ \left\{\begin{aligned}&\log z + \psi(\alpha+l+m+n-1) - \psi(l+m+n) - \psi(m+n)\\ &-\psi(n) - \sum_{t=4}^{p}\psi(\alpha_t-\alpha-l-m-n-1)\\ &+\sum_{s=1}^{q}\psi(\rho_s-\alpha-l-m-n-1)\end{aligned}\right\}^2 \\
&+\chi(\alpha+l+m+n-1) - \chi(l+m+n) - \chi(m+n) - \chi(n) \\
&+\sum_{t=4}^{p}\chi(\alpha_t-\alpha-l-m-n-1) - \sum_{s=1}^{q}\chi(\rho_s-\alpha-l-m-n-1)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta_n &= \log z + \psi(\alpha+l+n-1) - \psi(l+n) - \psi(m-n-1) - \psi(n) \\
&- \sum_{t=4}^{p}\psi(\alpha_t-\alpha-l-n-1) + \sum_{s=1}^{q}\psi(\rho_s-\alpha-l-n-1)\ .
\end{aligned}
$$

Here

$$\chi(z) = \psi'(z) = \sum_{n=1}^{\infty}\frac{1}{(z+n)^2}\ , \qquad (6)$$

where $|\operatorname{amp} z| < \pi$.

2. Proof of the formula. If $\alpha_1 = \alpha, \alpha_2 = \alpha + l, \alpha_3 = \alpha + l + m + \varepsilon$, where l and m are zero or positive integers and ε is small, it follows from (2) and (1) that the first 3 terms of (1) are equal to $A + B + C$, where

$$A = (-1)^l z^{\alpha+l} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + n)\Gamma(m - n + \varepsilon)\prod_{t=4}^{p}\Gamma(\alpha_t - \alpha - l - n)}{n!(l + n)!\prod_{s=4}^{q}\Gamma(\rho_s - \alpha - l - n)} K_n z^n ,$$

where

$$K_n = \psi(l + n) + \psi(n) + \psi(m - n - 1 + \varepsilon) - \psi(\alpha + l + n - 1)$$
$$- \log z + \sum_{t=4}^{p}\psi(\alpha_t - \alpha - l - n - 1) - \sum_{s=1}^{q}\psi(\rho_s - \alpha - l - n - 1) ,$$

$$B = z^{\alpha} \sum_{n=0}^{l-1} \frac{\Gamma(\alpha + n)(l - n - 1)!\Gamma(l + m - n + \varepsilon)\prod_{t=4}^{p}\Gamma(\alpha_t - \alpha - n)}{n!\prod_{s=1}^{q}\Gamma(\rho_s - \alpha - n)} (-z)^n ;$$

$$C = z^{\alpha+l+m+\varepsilon}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n + \varepsilon)\Gamma(- l - m - n - \varepsilon)\Gamma(- m - n - \varepsilon)}{n!\prod_{s=1}^{q}\Gamma(\rho_s - \alpha - l - m - n - \varepsilon)}$$
$$\times \prod_{t=4}^{p}\Gamma(\alpha_t - \alpha - l - m - n - \varepsilon) \times (-z)^n .$$

Then $A = D + E$, where

$$D = (-1)^l z^{\alpha+l} \sum_{n=0}^{m-1} \frac{\Gamma(\alpha + l + n)\Gamma(m - n + \varepsilon)\prod\Gamma(\alpha_t - \alpha - l - n)}{n!(l + n)!\prod\Gamma(\rho_s - \alpha - l - n)} K_n z^n ,$$

$$E = (-1)^l z^{\alpha+l+m} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + l + m + n)\Gamma(-n + \varepsilon)\prod\Gamma(\alpha_t - \alpha - l - m - n)}{(m + n)!(l + m + n)!\prod\Gamma(\rho_s - \alpha - l - m - n)} L_n z^n ,$$

where

$$L_n = \psi(l + m + n) + \psi(m + n) + \psi(-n - 1 + \varepsilon)$$
$$- \psi(\alpha + l + m + n - 1) - \log z + \sum\psi(\alpha_t - \alpha - l - m - n - 1)$$
$$-\sum\psi(\rho_s - \alpha - l - m - n - 1).$$

Note. In these formulae t takes the values from 4 to p.
Here, since [2, p. 31]

(7) $$\psi(-z - 1) = \psi(z) + \pi \cot \pi z ,$$

(8) $$\psi(-n - 1 + \varepsilon) = \psi(n - \varepsilon) - \pi \cot \pi\varepsilon .$$

Hence

$$C + E = \frac{\pi^2}{\sin^2 \pi\varepsilon}(-1)^l z^{\alpha+l+m}$$

$$\times\left[\begin{array}{l} z^{\varepsilon}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n+\varepsilon)}{n!\Gamma(m+n+1+\varepsilon)} \\ \qquad\qquad \frac{\prod\Gamma(\alpha_t-\alpha-l-m-n-\varepsilon)(-z)^n}{\Gamma(l+m+n+1+\varepsilon)\times\prod\Gamma(\rho_s-\alpha-l-m-n-\varepsilon)} \\ +\frac{\sin\pi\varepsilon}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{\Gamma(n+1-\varepsilon)(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)} \\ \quad\times\left[\begin{array}{l}\psi(l+m+n)+\psi(m+n)+\psi(n-\varepsilon)-\psi(\alpha+l+m+n-1) \\ -\log z+\sum\psi(\alpha_t-\alpha-l-m-n-1) \\ \qquad\qquad -\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right] \\ -\cos\pi\varepsilon\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{\Gamma(n+1-\varepsilon)(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)}\end{array}\right].$$

The limit of this function when $\varepsilon\to 0$ is obtained by replacing $\pi^2/\sin^2\pi\varepsilon$ by $\frac{1}{2}$, finding the second derivative with regard to ε of the expression in the large bracket, and then making $\varepsilon\to 0$. It is

$$\tfrac{1}{2}(-1)^l z^{\alpha+l+m}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+l+m+n)\prod\Gamma(\alpha_t-\alpha-l-m-n)(-z)^n}{n!(m+n)!(l+m+n)!\prod\Gamma(\rho_s-\alpha-l-m-n)}$$

$$\times\left[\begin{array}{l}\left\{\begin{array}{l}\log z+\psi(\alpha+l+m+n-1)-\psi(m+n)-\psi(l+m+n) \\ -\sum\psi(\alpha_t-\alpha-l-m-n-1)+\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right\}^2 \\ +\chi(\alpha+l+m+n-1)-\chi(m+n)-\chi(l+m+n) \\ +\sum\chi(\alpha_t-\alpha-l-m-n-1)-\sum\chi(\rho_s-\alpha-l-m-n-1) \\ +2\psi(n)\left\{\begin{array}{l}\psi(l+m+n)+\psi(m+n)+\psi(n) \\ \qquad\qquad -\psi(\alpha+l+m+n-1) \\ -\log z+\sum\psi(\alpha_t-\alpha-l-m-n-1) \\ \qquad\qquad -\sum\psi(\rho_s-\alpha-l-m-n-1)\end{array}\right\} \\ -2\chi(n)+\pi^2-\{\psi(n)\}^2+\chi(n)\end{array}\right].$$

From this, with B and D, formula (5) is obtained.

References

1. T. M. MacRobert, *Functions of a complex variable*, London, (1954).
2. T. M. MacRobert, *Evaluation of an E-function when two of the upper parameters differ by an integer*, Proc. Glasgow Math. Assoc., **5** (1961).

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