INTEGRAL CLOSURE OF RINGS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Let K be an ordiary differential field of characteristic zero with field of constants C. Let R be a differential subring of K containing C and having quotient field K. A differential subring V of an extension differential field M of K is called a fundamental differential ring (over R) if V contains R and if, for each v in V, there exist $v_2, \cdots, v_n$ in V, n depending on v, such that $v, v_2, \cdots, v_n$ form a fundamental system of solutions of a homogeneous linear differential equation with coefficients in K. Throughout this paper, $\{\cdots\}$ denotes differential ring adjunction, $<\cdots>$ differential field adjunction.

THEOREM 1. *Let K, C, R, M, V be as above. Then V is a fundamental differential ring over R if and only if $V = R\{v_{\alpha i}, \alpha \in A, 1 \leqq i \leqq n_\alpha\}$, A an indexing set, where for each α in A, $v_{\alpha 1}, v_{\alpha 2}, \cdots, v_{\alpha n_\alpha}$ form a fundamental system of solutions of a homogeneous linear differential equation over K.*

Proof. If V is a fundamental differential ring over R, we may let $A = V$; the interest attaches to the converse. It amounts to proving that every differential polynomial with coefficients in R in the $v_{\alpha i}$ is one element of a fundamental system of solutions of a homogeneous linear differential equation over K, all the elements of which system of solutions belong to V. By use of induction, we may reduce the problem to consideration of the four differential polynomials s', $s + t$, st, and rs, $r \in K$. We treat the polynomials s' and $s + t$; the polynomials st and rs are treated in a like manner.

Let $s^{(n)} + a_{n-1}s^{(n-1)} + \cdots + a_0 s = 0$, $a_i \in K$, $0 \leqq i \leqq n-1$. (There is no loss of generality in supposing that the leading coefficient of this differential equation is 1.) If $a_0 = 0$, then s' already satisfies a homogeneous linear differential equation (of order $n-1$) over K; if $a_0 \neq 0$, we differentiate the expression

$$\left(\left(\frac{1}{a_0}\right)s^{(n)} + \left(\frac{a_{n-1}}{a_0}\right)s^{(n-1)} + \cdots + \left(\frac{a_1}{a_0}\right)s' + s\right)$$

to obtain a homogeneous linear differential equation of order n in s'

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with coefficients in K.

To prove the result for $s+t$, let s, t be in V with $s+t \neq 0$; let $s, s_2, \cdots, s_n$ be n elements of V forming a fundamental system of solutions of a homogeneous linear differential equation over K, and the same for $t, t_2, \cdots, t_m$. Let $s_1 = s, t_1 = t$. Let $u_1 = s_1 + t_1$ and choose $u_2, u_3, \cdots, u_r$ from among $s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_m$ such that $u_1, u_2, \cdots, u_r$ form a basis over the constants for the vector space spanned over the constants by $s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_m$. Let $W(z_1, z_2, \cdots, z_p)$ denote the wronskian of the p elements $z_1, z_2, \cdots, z_p$. Consider the linear differential operator of order r, $\mathscr{L}(y) = W(y, u_1, \cdots, u_r)/W(u_1, \cdots, u_r)$. (Since $u_1, \cdots, u_r$ are linearly independent over constants, their wronskian is nonzero.) $\mathscr{L}(u_r) = 0$, $1 \leqq \lambda \leqq r$, and $\mathscr{L} \neq 0$ since the coefficient of $y^{(r)}$ is $1 = W(u_1, \cdots, u_r)/W(u_1, \cdots, u_r)$. We shall prove that all the coefficients of $\mathscr{L}$ are in K; $\mathscr{L}(y) = 0$ will then be the sought-after differential equation.

Let σ be a differential isomorphism of $K\langle s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_m\rangle$ over K; then $\sigma(s_\mu) = \sum_{i=1}^n c_{\mu i} s_i$, $1 \leqq \mu \leqq n$ and $\sigma(t_\nu) \sum_{j=1}^m d_{\nu j} t_j$, $1 \leqq \nu \leqq m$, where the $c_{\mu i}$ and $d_{\nu j}$ are constants. This is true because $s_1, s_2, \cdots, s_n$ span over constants the vector space of solutions of the homogeneous linear differential equation over K satisfied by s_1; similarly for $t_1, \cdots, t_m$. These two sets of equations taken together imply $\sigma(u_\lambda) = \sum_{k=1}^r e_{\lambda k} u_k$, $1 \leqq \lambda \leqq r$, $e_{\lambda k}$ constants, for each $\sigma(u_\lambda)$ is in the vector space spanned over the constants by $s_1, \cdots, s_n; t_1, \cdots, t_m$.

This implies that $W(y, \sigma u_1, \cdots, \sigma u_r) = (\det(e_{\lambda k}))W(y, u_1, \cdots, u_r)$, and similarly $W(\sigma u_1, \cdots, \sigma u_r) = (\det(e_{\lambda k}))W(u_1, \cdots, u_r)$. Therefore the coefficients a_p, $0 \leqq p \leqq r$, of $\mathscr{L}(y)$ are invariant under σ, for all differential isomorphisms σ of $K\langle s_1, \cdots, s_n; t_1, \cdots, t_m\rangle$ over K. By Theorem 2.6, pg. 16 of [1], a_p is in K, as required. This proves the theorem.

The above theorem has the following immediate consequence.

COROLLARY. *If M is a universal differential field extension of K* ([2], Sec. 5, esp. pg. 771, Theorem), *the set V of all elements of M satisfying a homogeneous linear differential equation over K forms a fundamental differential ring.*

The following lemma isolates the key property of fundamental differential rings that will be used to prove integral closure. An element w in an extension differential field of K is called a wronskian over K if $w \neq 0$ and w'/w belongs to K.

LEMMA. *Let V be a fundamental differential ring over R. Then any nonzero differential ideal I of V contains a wronskian over K.*

Proof. Let u_1 be a nonzero element of the differential ideal I of V, and let $u_2, u_3, \cdots, u_n$ be $n-1$ elements of V such that $u_1, u_2, \cdots, u_n$ form a fundamental system of solutions of a homogeneous linear differential equation over K. Then $W(u_1, u_2, \cdots, u_n)$ is a nonzero element of I: it is nonzero since $u_1, u_2, \cdots, u_n$ are linearly independent over constants; it belongs to I because each term in the expansion of the determinant defining $W(u_1, \cdots, u_n)$ contains a derivative of u_1 as a factor. Since $W(u_1, \cdots, u_n)$ is a wronskian over K, the proof is complete.

DEFINITION. A differential ring is called differentiably simple if it has no differential ideals other than zero and itself.

THEOREM 2. *Let R be differentiably simple (in particular, $R = K$), and for every wronskian w over K belonging to V, let there exist a nonzero h in R such that h/w is in V. Then V too is differentiably simple. (When $R = K$, the assumption is that V contains the inverse of every wronskian over K which belongs to V.)*

Proof. Let I be a nonzero differential ideal of V. To prove that $I = V$, let w be a wronskian over K in I; such exist by the lemma. Now by hypothesis, there is a nonzero h in R with h/w in V. Thus $w \cdot h/w = h$ is in I, so that $I \cap R$ is not the zero ideal of R. Since $I \cap R$ is a differential ideal of R and R is differentiably simple, $I \cap R = R$, so that $1 \in I \cap R$, and $1 \in I$. Thus $I = V$ as required.

The next theorem is a sort of converse to the previous theorem. (Here V need not be a fundamental differential ring over R; V can be any differential subring of M containing R.)

THEOREM 3. *Let V, but not necessarily R, be differentiably simple, and let w be a wronskian over K belonging to V. Then there is a nonzero h in R such that h/w is in V. (Thus if $R = K$, $1/w$ is in V.)*

Proof. Since K is the quotient field of R, there exist b, c in R, with $c \neq 0$, such that $w' = (b/c)w$. Let I denote the set of elements of V of the form $vc^{-p}w$, p a nonnegative integer, v an element of V. I can readily be shown to be an ideal of V; we shall prove that I is closed under differentiation. If $vc^{-p}w \in I$, then $(vc^{-p}w)' = v'c^{-p}w - pvc^{-p-1}c'w + vc^{-p}w' = (v'c)c^{-p-1}w - (pvc')c^{-p-1}w + (bv)c^{-p-1}w = (v'c - pvc' + bv)c^{-p-1}w$ is an element of V and hence of I. Thus I is a differential ideal of V, and is nonzero since w is in I. Since V is differentiably simple, $I = V$, and $1 \in I$. Thus $1 = vc^{-p}w$ for some $v \in V$, $p \geqq 0$. Then, if $c^p = h$, we have $h/w = v \in V$, with h an element of R. This proves the theorem.

The following theorem with $K = C$ generalizes a consequence of a result of Ritt ([4], Sec. 1, pg. 681) to the effect that if C is the field of complex numbers, the ring $C\,[e^{\lambda x}$, all complex $\lambda]$ is integrally closed in its quotient field. In fact, Theorem 4 also implies that $C\,[x, e^{\lambda x}]$ is is integrally closed in its quotient field.

THEOREM 4. *Let K be a differential field of characteristic zero with field of constants C. Let K be differential algebraic over C. Let V be a fundamental differential ring over K which contains the inverse of every wronskian over K in it. Then V is integrally closed in its quotient field* (*it is differentiably simple by Theorem* 2).

Proof. Let u be an element of the quotient field M of V integral over V: that is, there exist elements v_i in V, $1 \leqq i \leqq n$, such that $u^n + \sum_{i=1}^{n} v_i u^{n-i} = 0$, and there exist v_{n+1}, v_{n+2} in V with $u = v_{n+1}/v_{n+2}$. Let v_i be a solution of a homogeneous linear differential equation $\mathscr{L}_i(y) = 0, 1 \leqq i \leqq n+2$, where $\mathscr{L}_i(y) = \sum_{j=0}^{n_i} b_{ij} y^{(j)}, 1 \leqq i \leqq n+2, 0 \leqq j \leqq n_i$; $b_{in_i} = 1, 1 \leqq i \leqq n+2$. Furthermore let $v_{ik}, 1 \leqq k \leqq n_i$, be for each i a fundamntal system of solutions of $\mathscr{L}_i(y) = 0$, with $v_{i1} = v_i$. Let Y be a differential indeterminate, and, for each i, j, let $P_{ij}(Y) \in C\{Y\}$ be a differential polynomial of lowest order r_{ij} say satisfied by b_{ij} over C and such that the degree of P_{ij} in $Y^{(r_{ij})}$ is as small as possible among these differential polynomials of order r_{ij}. Define the separant S_{ij} of P_{ij} as the (partial) derivative of P_{ij} with respect to $Y^{(r_{ij})}$. One verifies, using the minimal property of the P_{ij}, that $S_{ij}(b_{ij})$ is nonzero. Then $b_{ij}^{(r_{ij}+1)}$ is $S_{ij}^{-1}(b_{ij})$ multiplied by a differential polynomial over C in b_{ij} of order at most r_{ij}. This implies that $C\{b_{ij}\} = C[b_{ij}^{(p)}, 0 \leqq p \leqq r_{ij}]$, all i, j. (This argument is well known.)

Now define $\bar{V} = C\{b_{ij}, S_{ij}^{-1}(v_{ij}), v_{ik}$, all $1 \leqq i \leqq n+2, 0 \leqq j \leqq n_i$, $1 \leqq k \leqq n_i\}$; observe $\bar{V} \subset V$. Since $\mathscr{L}_i$ has leading coefficient 1 and $\mathscr{L}_i(v_{ik}) = 0, 1 \leqq i \leqq n+2, 1 \leqq k \leqq n_i$, and because of the above property of each $C\{b_{ij}\}$, one concludes that $\bar{V} = C[b_{ij}^{(p)}, S_{ij}^{-1}(b_{ij}), v_{ik}^{(q)}$, all $1 \leqq i \leqq n+2, 0 \leqq j \leqq n_i, 1 \leqq k \leqq n_i\, 0 \leqq p \leqq r_{ij}, 0 \leqq q \leqq n_i - 1]$. This is what we were after: we have proved that $\bar{V}$ is finitely generated as an ordinary ring over C. We can now apply Theorem 2 of [3] to conclude that the integral closure $\bar{O}$ of $\bar{V}$ in its quotient field $\bar{M}$ is in fact a differential subring of $\bar{M}$. But u is in $\bar{O}$; if we can prove that $\bar{O}$ is contained in $\bar{V}$, the proof will be completed.

So consider the ideal $\bar{I}$ of $\bar{V}$ consisting of all h in $\bar{V}$ such that $h\bar{O} \subset \bar{V}$. By [5], pg. 267, Theorem 9, $\bar{I}$ is nonzero; a fortiori, the ideal I of V consisting of those h in V with $h\bar{O} \subset V$ is also nonzero, since it contains $\bar{I}$. We assert that I is a differential ideal of V: let $\omega \in \bar{O}$; then $h\omega \in V, (h\omega)' = h'\omega + h\omega' \in V$. Since $\bar{O}$ is closed under differentiation by [3], pg. 1393, lemma, $\omega' \in \bar{O}$, so that, since $h \in I$,

$h\omega' \in V$. Thus $h'\omega$ is in V if ω is in $\bar{O}$ and h is in I. In other words, I is a differential ideal of V. Since V is differentiably simple by Theorem 2, and I is nonzero, we conclude that $I = V$. Therefore $1 \in I$. This implies that $\bar{O} = 1 \cdot \bar{O}$ is contained in V, as promised. This completes the proof of Theorem 4.

(The above theorem could be strengthened by use of the following unproved result: a differentiably simple ring of characteristic zero is integrally closed in its quotient field. This result would generalize Theorem 1 of [3].)

Theorem 4 has the following corollary.

COROLLARY. *Let K be a differential field of characterististic zero with field of constants C. Let K be differential algebraic over C. Let M be a universal differential field extension of K. Let V be the subset of M comprising those elements of M satisfying a homogeneous linear differential equation over K. Then V is integrally closed in its quotient field.*

Proof. That V is a fundamental differential ring over K follows from the corollary to Theorem 1. To prove V integrally closed in its quotient field, we shall prove that V contains the inverse of every wronskian over K in it, and then apply Theorem 4.

Now if w is a wronskian over K in V, then $w \neq 0$ and $w' = kw$, $k \in K$. Then $(1/w)' = (-1/w^2)\cdot w' = (-1/w^2)\cdot kw = -k\cdot(1/w)$. So $1/w$ satisfies a (first order) homogeneous linear differential equation over K; by the definition of V, $(1/w)$ belongs to V, as required for the application of Theorem 4.

REMARK. Let $V_1 = V$ and V_{n+1}, $n \geqq 1$, be the differential subring of M consisting of those elements of M satisfying a homegeneous linear differential equation with coefficients in V_n. Then V_{n+1} contains L_n (thus $\bigcup_{n=1}^{\infty} V_n = V_\infty$ is a field), for if $f(\neq 0)$ is in V_n, then $(1/f)' = -f'/f\cdot 1/f$. Thus $1/f$ satisfies a first order homogeneous linear differential equation with coefficients in L_n and so is in V_{n+1}. Since V_{n+1} contains V_n, and now the inverse of every nonzero element in V_n, V_{n+1} contains L_n. But each L_n is differential algebraic over C, and M is still a universal differential extension of L_n. The above corollary thus implies that each V_n is integrally closed in its quotient field L_n, $n \geqq 1$.

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