DIMENSIONAL INVERTIBILITY

P. H. DOYLE AND J. G. HOCKING

We report here upon another aspect of our continuing investigation of invertibility (see [5, 6]) and its applications in the theory of manifolds.

All spaces considered here are separable and metric.

A separable metric space X will be said to be *k-invertible*, $0 \leqq k \leqq \dim X$, if for each nonempty open set U and each compact proper subset C of dimension $\leqq k$, there is a homeomorphism h of X onto itself such that $h(C)$ lies in U. Then we say that X is *strongly k-invertible* if for each nonempty open set U and each *closed* proper subset C of dimension $\leqq k$, there is a homeomorphism h of X onto itself such that $h(C)$ lies in U.

Clearly, "strongly k-invertible" implies "k-invertible" and the two properties coincide in compact spaces. If $\dim X = n$, then "invertible" and "strongly n-invertible" are equivalent but, for instance, E^n is n-invertible and not invertible. We remark that k-invertibility is a strong form of near-homogeneity and says that compact k-dimensional subsets are "small under homeomorphisms." In the case of an n-manifold, k-invertibility is equivalent to the condition that every compact set of dimension k lie in an open n-cell.

We first collect some results on 0-invertible spaces, most of these results being simple generalizations of theorems to be found in [5]. The first of these requires no proof here.

THEOREM 1. *The orbit of any point in a 0-invertible space is dense in the space.*

THEOREM 2. *Each orbit in a 0-invertible space is itself 0-invertible.*

Proof. Let 0 be the orbit of any point in a 0-invertible space X. Let U be an open subset of 0 and C be a compact 0-dimensional proper subset of 0. Then there is an open set V in X such that $V \cap 0 = U$ and, by 0-invertibility, there is a space homeomorphism h such that $h(C)$ lies in V. But by definition of 0 as an orbit, $h(C)$ also lies in 0, hence $h(C)$ lies in $V \cap 0 = U$.

COROLLARY. *Each 0-invertible space is a union of disjoint, dense homogeneous, 0-invertible subspaces.*

Received February 21, 1962.

THEOREM 3. *If X is* 0-*invertible and contains a nondegenerate connected open set, then X is connected.*

Proof. If U is a nondegenerate open connected set in X, let p be any point in U.

For each point x in X, there is a space homeomorphism h_x such that $h_x(x \cup p) = h_x(x) \cup h_x(p)$ lies in U. Thus X is a union $\bigcup_x h_x^{-1}(U)$ of connected sets, each containing the point p.

COROLLARY. *If X is* 0-*invertible and is locally connected at any point, then X is connected or X is the* 0-*sphere.*

THEOREM 4. *If X is* 0-*invertible and is locally Euclidean at any point, then X is a manifold.*

Proof. If X contains an open cell U as an open set, then X is connected by Theorem 3 and, as in the proof of Theorem 3, $h_x^{-1}(U)$ is an open cell neighborhood of the point x for each point x in X.

THEOREM 5. *If X is strongly* 0-*invertible and contains an open set with compact closure, then X is compact.*

Proof. Let U be an open set in X with compact closure $\bar{U}$. Given any infinite set A in X such that A has no limit point, the set A contains an infinite sequence $\{a_n\}$ having no limit point in X. But then the sequence $\{a_n\}$ can be carried into U by a space homeomorphism h in view of strong 0-invertibility. In $\bar{U}$, the sequence $\{h(a_n)\}$ has a limit point. This contradiction shows that X is compact.

COROLLARY. *A locally compact, strongly* 0-*invertible space is compact.*

Every 2-manifold is 0-invertible and every compact 2-manifold is strongly 0-invertible because any compact 0-dimensional set in a 2-manifold lies in an arc in the manifold. In higher dimensions, however, 0-invertibility has more force. The following result is an interesting characterization of the 3-sphere.

THEOREM 6. *A strongly* 0-*invertible* 3-*manifold is S^3.*

Proof. We employ the characterization of R. H. Bing [1] and show that every polygonal simple closed curve in such a 3-manifold lies in an open 3-cell. Let M^3 be a strongly 0-invertible 3-manifold and let J be a polygonal simple closed curve in M^3. A sufficiently thin tubular neighborhood of J may be chosen to be a polyhedral solid torus T in M^3. Since every longitudinal simple closed curve in T is isotopic to J, if we can show that there is such a curve which lies in an open 3-cell, the proof will be complete.

Using the solid torus T as the 0th stage, we construct a "necklace of Antoine" N in M^3. By the assumption of 0-invertibility, the compact 0-dimensional set N lies in an open 3-cell in M^3. Hence there is a standard decomposition $M^3 = P^3 \cup C$, where P^3 is an open 3-cell and C is a nonseparating continuum of dimension $\leqq 2$ (see [7]), such that $N \cap C$ is empty. Since N and C are compact, there is a positive distance between N and C. Thus there is some stage, say the kth, in the construction of N such that the residual set C fails to meet each solid torus in the kth stage.

Now we add a 2-disk spanning the hole in each solid torus in the kth stage of the construction of N. This results in a connected set consisting of alternately "orthogonal" disks with disjoint solid toroidal rims in the interior of each solid torus in the $(k-1)$st stage. Call these sets $L_i^{(k-1)}$, $i = 1, 2, \cdots, n^{k-1}$, where $n \geqq 3$. There are two cases to consider: (1) In each of the sets $L_i^{(k-1)}$ we can find a simple closed curve passing longitudinally around the hole in the corresponding solid torus in the $(k-1)$st stage and not meeting the residual set C or (2) for some set $L_j^{(k-1)}$, C meets every longitudinal simple closed curve on $L_j^{(k-1)}$.

In case (2), the residual set C does not meet the solid toridal rims of the disks in $L_j^{(k-1)}$ but C must meet at least one of the spanning disks in such a way that no arc from one solid torus of a linking pair to the other can be drawn in the spanning disk without meeting C. Thus C must separate some spanning disk D into components, one of which meets the solid torus spanned by D and another of which meets one of the solid tori linked with that spanned by D. This is impossible. For, in such a case, any longitudinal simple closed curve in the linking solid torus would be linked with C while lying in the complement of C which contradicts the assumption that $M^3 - C = P^3$ is an open 3-cell.

Case (1) reduces to the following situation: Each solid torus in the $(k-1)$st stage of the construction of the necklace N contains a longitudinal simple closed curve lying in the open 3-cell P^3 and these curves are linked just as are the solid tori in the $(k-1)$st stage. We can now replace the solid tori in the $(k-1)$st stage by thinner ones where necessary so that the entire $(k-1)$st stage lies in the open 3-cell P^3. The spanning disks are now added to these tori to obtain the sets $L_i^{(k-2)}$, $i = 1, 2, \cdots, n^{k-2}$, and the argument above can be repeated. The finite regression is now obvious. The contradiction in case (2) at each step forces us back to the first stage in the construction of the necklace N. But then the same argument produces a longitudinal simple closed curve J' in the original solid torus T such that $J' \cap C$ is empty. By our remark above J and J' are isotopic and since J' lies in an open 3-cell, so does J.

COROLLARY. *Every polygonal simple closed curve in a 0-invertible 3-manifold lies in an open 3-cell.*

Proof. The argument for Theorem 6 goes through in this case, too, because the residual set C is closed and there is still a positive distance between C and a necklace N in the complement of C.

Imposing a natural restriction upon the manifold permits us to generalize, not Theorem 6, but its corollary.

THEOREM 7. *In a 0-invertible, combinatorial n-manifold, every polygonal simple closed curve lies in an open n-cell.* (*Hence such manifords are simply connected.*)

Proof. Let M^n be a 0-invertible, combinatorial n-manifold and let J be a polygonal simple closed curve in M^n. In the combinatorial n-manifold, a sufficiently thin tubular neighborhood of J will be a polyhedral solid n-torus T (a homeomorph of the product of an $(n-1)$ disk and the unit circle). In the interior of T we construct a Cantor set N by the method of Blankenship [2]. Then, with the appropriate changes in dimension, the remainder of the proof is identical to that of Theorem 6.

A natural conjecture at this point concerns k-invertibility and the vanishing of the homotopy group $\pi_{k+1}(M^n)$. Such a conjecture is fruitless, however, in view of the following result.

THEOREM 8. *Let $A^{n+1} = S^n \times E^1$, $n \geqq 2$. Then A^{n+1} is an $(n-1)$-invertible manifold* (*and clearly $\pi_n(A^{n+1})$ is not trivial*).

Proof. Assume that A^{n+1} is imbedded in E^{n+1} as the region between two concentric spheres. Then $\bar{A}^{n+1}$ is a closed annulus and there is a map h from $\bar{A}^{n+1}$ onto S^{n+1} such that $h \mid A^{n+1}$ is a homeomorphism and h carries the two components of $\bar{A}^{n+1} - A^{n+1}$ into a pair of points a and b.

If N is any compact $(n-1)$-dimensional set in A^{n+1}, then $h(N)$ is a compact $(n-1)$-dimensional set in $S^{n+1} - (a \cup b)$. Since $h(N)$ does not separate S^{n+1}, there is a polygonal arc J in $S^{n+1} - h(N)$ from a to b and $S^{n+1} - J$ is an $(n+1)$-cell. Whence $h^{-1}(S^{n+1} - J)$

is an $(n+1)$-cell in A^{n+1} containing N and therefore A^{n+1} is $(n-1)$-invertible.

The next result is a slight generalization of our characterization theorem [4].

THEOREM 9. *The only strongly $(n-1)$-invertible n-manifold is S^n.*

Proof. If M^n is strongly $(n-1)$-invertible, then M^n is compact. Choose any standard decomposition $M^n = P^n \cup C$. Since C is a continuum of dimension $\leqq n-1$ and P^n is an open n-cell, there is a space homeomorphism carrying C into P^n. Then Corollary 1 of Theorem 2 in [7] applies to show that M^n is an n-sphere.

THEOREM 10. *The only $(n-1)$-invertible, noncompact n-manifold is E^n.*

Proof. Let M^n be an $(n-1)$-invertible, noncompact n-manifold. Since M^n is locally compact, it is a union $\bigcup_{j=1}^{\infty} A_j$ where we may choose A_1 to be a closed n-cell and where A_j is compact and lies in the interior of A_{j+1} for each j (Theorem 2.60 of [8]). Let U be an open n-cell in A_1 with bi-collored boundary. Each set BdA_j has dimension $\leqq n-1$ and hence there is a homeomorphism h_j of M^n onto itself such that $h_j(BdA_j)$ lies in U.

We claim that $h_j(A_j)$ also lies in U. For BdA_j separates M^n and if $h_n(A_n)$ does not lie in U, then $h_n(M^n - A_n)$ must lie in U. But then $\overline{h_j(M^n - A_j)} = h_j(\overline{M^n - A_j})$ is compact whence $M^n = (M^n - A_j) \cup A_j$ is the union of two compact sets and is compact. This contradiction proves that $h_j(A_j)$ lies in U.

From here we see that $\{h_j^{-1}(U)\}$ is a sequence of open n-cells. We may select a monotone increasing subsequence inductively (or else all A_j lie in some $h_j^{-1}(U)$ which completes the proof). Therefore M^n is the union of a monotone increasing sequence of n-cells and, in view of [3], $M^n = E^n$.

To finish this report, we collect some immediate consequences of the Poincare duality and the Hurewicz theorem.

THEOREM 11. *Let M^n be a compact, triangulated, orientable, k-invertible n-mainfold. Then the homotopy groups $\pi_p(M^n)$ are trivial for $1 \leqq p \leqq k$.*

COROLLARY 1. *If M^n is as in Theorem 11, then M^n has trivial integral homology groups in dimensions $1, 2, \cdots, k$ and $n-k, \cdots,$*

$n-1$.

COROLLARY 2. *If M^n is as in Theorem 11, and if $k \geqq [n/2]$ (the largest integer in $n/2$), then M^n is a homotopy sphere.*

Recent results of Stallings [9] and Zeeman [10] provide immediate proofs of the following result.

THEOREM 12. *A strongly $[n/2]$-invertible polyhedral n-manifold, $n \geqq 5$, is an n-sphere.*

REFERENCES

1. R. H. Bing, *Necessary and sufficient conditions that a 3-manifold be S^3*, Annals of Math., **68** (1958), 17-37.
2. W. A. Blankenship, *Generalization of a construction of Antoine,* Annals of Math., **53** No. 2 (1951), 276-297.
3. M. Brown, *The monotone union of open n-cells is an open n-cell,* Proc. Amer. Math Soc., **12** No. 5 (1961), 812-814.
4. P. H. Doyle and J. G. Hocking, *A characterization of euclidean n-space,* Mich. Math. J., **7** (1960), 199-200.
5. ———, *Invertible spaces,* Amer. Math. Monthly (to appear)
6. ———, *Continuously invertible spaces,* Pacific J. Math. (to appear).
7. ———, *A decomposition theorem for manifolds,* Proc. Amer. Math. Soc. (to appear).
8. J. G. Hocking and G. S. Young, *Topology,* Addison-Wesley, Reading Mass (1961).
9. J. Stallings, *Polyhedral homotopy spheres*, Bull. Amer. Math. Soc., **66** (1960), 485-488.
10. E. C. Zeeman, *The generalized Poincare conjecture,* Bull. Amer. Math. Soc., **67** (1961), 270.

MICHIGAN STATE UNIVERSITY