GENERAL GROUP EXTENSIONS

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Introduction. The aim of this paper is to show how some of the methods useful in studying normal extensions of groups can be used in a problem of more general extensions. The present approach (which might be compared with that of Szep [5]) is made possible because we consider classes of extensions which are still relatively restricted.

If G is an arbitrary subgroup of a group H then the set of all right cosets of G in H forms a mixed group under a naturally defined operation (Loewy [3]). In particular, when G is normal in H then the corresponding mixed group is the ordinary quotient group H/G. This paper is concerned with examining properties of the class of those extensions H of a given group G for which the corresponding mixed group is isomorphic to a given mixed group Γ. As an example of the results, Theorems 2.2 and 2.3 represent analogues of the corresponding theorems of Schreier on factor sets for normal extensions.

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Mixed groups.

1.1 DEFINITION. A *mixed group* is a set Γ on which a product $\alpha\beta \in \Gamma$ is defined for certain pairs $\alpha, \beta \in \Gamma$ such that

(i) a nonempty subset Δ of Γ forms a group under the given product and is called the *nucleus* of Γ;

(ii) for all $\beta \in \Gamma, \alpha\beta$ is defined if and only if $\alpha \in \Delta$; furthermore, $\alpha\beta = \beta$ if and only if $\alpha = 1$, the identity of Δ;

(iii) if $\alpha, \beta \in \Delta$ and $\gamma \in \Gamma$ then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. (See Loewy [3] and Bruck [2; page 35]. The general properties of mixed groups are derived in Baer [1].)

In particular, if H is a group with a subgroup G then the set of all right cosets of G in H forms a mixed group when the product of two elements is defined by $(Gx)(Gy) = Gxy$ whenever $x \in N(G; H)$, the normaliser of G in H. In this case we denote the mixed group by H/G and note that its nucleus is the quotient group $N(G; H)/G$ (See Baer [1]).

1.2 DEFINITION. Two mixed groups Γ and Γ' with nuclei Δ and Δ' respectively are *isomorphic* under a mapping τ if τ is a one-to-one

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mapping of Γ onto Γ' such that $(\alpha\tau)(\beta\tau)$ is defined in Γ' if and only if $\alpha\beta$ is defined in Γ and that in that case $(\alpha\beta)\tau = (\alpha\tau)(\beta\tau)$.

Because of part (ii) of Definition 1.1, and isomorphism τ of Γ onto Γ' induces on the nucleus Δ of Γ a (group) isomorphism onto the nucleus Δ' of Γ'.

As an example, suppose that H is a group with a subgroup G and let ϕ be a homomorphism of H onto a group H^*. If G^* is the image of G under ϕ and $\ker \phi \subseteqq G$ then it is easy to show that the mixed group H/G is isomorphic to H^*/G^*. The notation $H/G \geqq H^*/G^*$ will be used to imply the existence of a homomorphism of H onto H^* with this property.

1.3 DEFINITION. If H is a group with subgroup G and H/G is isomorphic to a mixed group Γ then H/G is a *representation* of Γ and H is an *extension* of G by Γ.

Baer [1; Theorem 3] proves that, except in the case that the mixed group Γ is of order 2 and has unit nucleus, every mixed group Γ has a representation H/G for some groups H and G. (The exceptional case arises because no subgroup G of index 2 in a group H can be its own normaliser.)

1.4. Contrary to the case of normal extensions, not every group G has an extension H by a given mixed group Γ. From the example of 1.2, when H^*/G^* is chosen to be minimal under the quasi-ordering defined there, G^* contains no nontrivial normal subgroup of H^*. In such a case we call H^*/G^* a *cardinal representation* of Γ. (If $|\Gamma| = n$ is finite, and H^*/G^* is a cardinal representation of Γ, then H^* is isomorphic to a permutation group of degree n and G^* corresponds to a subgroup fixing one letter.) Thus a necessary condition that G should have an extension H by Γ is that, for some cardinal representation H^*/G^* of Γ, G^* should be a homomorphic image of G. Examples show, however, that this condition is not sufficient.

2. Extension functions. As a generalisation of the Schreier factor set used in the theory of normal extensions we consider the extension of a group by a mixed group through the medium of a skew product (cf. Redei [4], Szep [5]).

2.1. Let G be a group and Γ be a mixed group with nucleus Δ. We define the skew product $\langle G, \Gamma\rangle$ to be the set $\{\langle a, \alpha\rangle \mid a \in G, \alpha \in \Gamma\}$ on which a binary operation is defined by

$$\langle a, \alpha\rangle\langle b, \beta\rangle = \langle af(\alpha, b, \beta), \phi(\alpha, b, \beta)\rangle$$

with $a, b, f(\alpha, b, \beta) \in G$ and $\alpha, \beta, \phi(\alpha, b, \beta) \in \Gamma$ for some functions f

and ϕ respectively.

We shall denote the identities of G, Γ and $\langle G, \Gamma\rangle$ by 1,1 and $\langle 1, 1\rangle$ respectively. Furthermore we write $\langle G, \alpha\rangle = \{\langle x, \alpha\rangle \mid x \in G\}$, $\langle a, \Gamma\rangle = \{\langle a, \xi\rangle \mid \xi \in \Gamma\}$ and identify G with $\langle G, 1\rangle$ and Γ with $\langle 1, \Gamma\rangle$ under the natural mappings.

2.2 THEOREM. *The skew product $H = \langle G, \Gamma\rangle$ is a group with identity $\langle 1, 1\rangle$ if and only if the following conditions on f and ϕ hold for all $b, c \in G$ and $\alpha, \beta, \gamma \in \Gamma$:*

(1) $$f(1, b, \beta) = b \text{ and } \phi(1, b, \beta) = \beta \; ;$$

(2) $$f(\alpha, b, \beta) f(\phi(\alpha, b, \beta), c, \gamma) = f(\alpha, bf(\beta, c, \gamma), \phi(\beta, c, \gamma)) \; ;$$

(3) $$\phi(\phi(\alpha, b, \beta), c, \gamma) = \phi(\alpha, bf(\beta, c, \gamma), \phi(\beta, c, \gamma)) \; ;$$

(4) *for all $\alpha \in \Gamma$ there exists $\xi \in \Gamma$ such that* $\phi(\xi, 1, \alpha) = 1$.

Proof. (1) is equivalent to $\langle 1, 1\rangle\langle b, \beta\rangle = \langle b, \beta\rangle$, that is that $\langle 1, 1\rangle$ is a left identity.

(2) and (3) are together equivalent to the associative law

$$\{\langle a, \alpha\rangle \langle b, \beta\rangle\} \langle c, \gamma\rangle = \langle a, \alpha\rangle \{\langle b, \beta\rangle \langle c, \gamma\rangle\} \, .$$

(4) is equivalent to $\langle 1, \alpha\rangle$ having a left inverse $\langle x, \xi\rangle$ (with $x = f(\xi, 1, \alpha)^{-1}$). However, in that case $\langle a, \alpha\rangle = \langle a, 1\rangle\langle 1, \alpha\rangle$ has a left inverse $\langle x, \xi\rangle\langle a^{-1}, 1\rangle$ for all $\langle a, \alpha\rangle \in H$.

Thus the stated conditions are necessary and sufficient.

2.3 THEOREM. *If the skew product $H = \langle G, \Gamma\rangle$ is a group with identity $\langle 1, 1\rangle$ then a necessary and sufficient condition that $\langle G, \Gamma\rangle / \langle G, 1\rangle = H/G$ should be isomorphic to Γ under the natural mapping*

$$\tau \colon \langle G, 1\rangle\langle 1, \alpha\rangle = \langle G, \alpha\rangle \to \alpha \qquad (\alpha \in \Gamma)$$

is that

(5) $$\phi(\alpha, b, 1) = \alpha \text{ for all } b \in G \text{ if and only if } \alpha \in \Delta \; ;$$

(6) $$\phi(\alpha, 1, \beta) = \alpha\beta \text{ when } \alpha \in \Delta \, .$$

Proof. Since it is clear that τ is a one-to-one mapping onto Γ the theorem follows from Definition 1.2 when we note:

(5) is equivalent to $\langle G, \alpha\rangle\tau \in \Delta$ if and only if $\alpha \in \Delta$;

(6) is equivalent to $\langle G, \alpha\rangle\langle G, \beta\rangle = \langle G, \alpha\beta\rangle$ for $\alpha \in \Delta$.

2.4. A skew product $H = \langle G, \Gamma\rangle$ which satisfies the conditions (1)–(6) will be called an extension of G by Γ with functions f and ϕ. For such an extension it is easily shown that f and ϕ have the following properties:

(7) $$f(\alpha, 1, 1) = 1 \text{ and } \phi(\alpha, 1, 1) = \alpha\ ;$$

(8) for all $\langle a, \alpha\rangle \in H$ there exists a unique $\xi \in \Gamma$ such that $\phi(\xi, a, \alpha) = 1$;

(9) the mapping $\gamma \to \phi(\gamma, a, \alpha)$ permutes the elements of Γ;

(10) $$\phi(\alpha, b, \beta) = \alpha\beta \text{ when } \alpha \in \Delta\ .$$

2.5 DEFINITION. Two extensions $H_1 = \langle G, \Gamma\rangle_1$ and $H_2 = \langle G, \Gamma\rangle_2$ with functions f_1, ϕ_1 and f_2, ϕ_2 respectively are called *equivalent* if there is an isomorphism of H_1 onto H_2 which leaves each element of G fixed. If, moreover, the coset $\langle G, \alpha\rangle_1$ is mapped onto the coset $\langle G, \alpha\rangle_2$ for each $\alpha \in \Gamma$ then H_1 is said to be *biequivalent* to H_2.

Thus if θ is a one-to-one mapping of H_1 onto H_2 then in order for this mapping to be a biequivalence we must have

(i) $\langle a, 1\rangle_1\theta = \langle a, 1\rangle_2$;

(ii) $\{\langle a, \alpha\rangle_1\langle b, \beta\rangle_1\}\theta = \langle a, \alpha\rangle_1\theta\langle b, \beta\rangle_1\theta$

(iii) $\langle 1, \alpha\rangle_1\theta = \langle x_\alpha, \alpha\rangle_2$ for some $x_\alpha \in G$.

2.6 THEOREM. *If $H_1 = \langle G, \Gamma\rangle_1$ and $H_2 = \langle G, \Gamma\rangle_2$ are two extensions of G by Γ with functions f_1, ϕ_1 and f_2, ϕ_2 respectively then H_1 is biequivalent to H_2 if and only if there is a function $\alpha \to x_\alpha$ of Γ into G such that*

(11) $$x_\alpha f_2(\alpha, bx_\beta, \beta) = f_1(\alpha, b, \beta)x_{\phi_1(\alpha, b, \beta)}\ ;$$

(12) $$\phi_2(\alpha, bx_\beta, \beta) = \phi_1(\alpha, b, \beta)\ .$$

Proof. If θ is a biequivalence of H_1 onto H_2 then define x_α by 2.5 (iii). Then $\langle a, \alpha\rangle_1\theta = \langle ax_\alpha, \alpha\rangle_2$. Therefore

$$\{\langle a, \alpha\rangle_1\langle b, \beta\rangle_1\}\theta = \langle af_1(\alpha, b, \beta)x_{\phi_1(\alpha, b, \beta)}, \phi_1(\alpha, b, \beta)\rangle_2$$

and $\langle a, \alpha\rangle_1\theta\langle b, \beta\rangle_1\theta = \langle ax_\alpha f_2(\alpha, bx_\beta, \beta), \phi_2(\alpha, bx_\beta, \beta)\rangle_2$ and so (11) and (12) are together implied by 2.5(ii).

Conversely, if we are given (11) and (12) and define θ as the one-to-one mapping of H_1 onto H_2 given by $\langle a, \alpha\rangle_1\theta = \langle ax_\alpha, \alpha\rangle_2$ then 2.5 (i), (ii) and (iii) follow, so θ is the required biequivalence.

2.7. Let $H = \langle G, \Gamma\rangle$ be an extension of G by a mixed group Γ with functions f and ϕ. The kernel Δ of Γ is isomorphic to $N(G; H)/G$ (by 1.1). Therefore $G = \langle G, 1\rangle$ is a normal subgroup (and H is a normal extension of G) if and only if $\Delta = \Gamma$, and Γ is a group. Alternatively, using (5) and (6) we have that $\phi(\alpha, b, 1) = \alpha$ (for all $\alpha \in \Gamma$, $b \in G$) as a necessary and sufficient condition that H be a normal extension of G.

A second important case is when H is a splitting extension (i.e.

$\Gamma = \langle 1, \Gamma \rangle$ is a subgroup of H) so that $H = G\Gamma$ and $G \cap \Gamma = 1$. In terms of the extension functions, H is a splitting extension if and only if $f(\alpha, 1, \beta) = 1$ for all $\alpha, \beta \in \Gamma$. To prove this we note that, since $\langle 1, \alpha \rangle \langle 1, \beta \rangle = \langle f(\alpha, 1, \beta), \phi(\alpha, 1, \beta) \rangle$, the condition is certainly necessary. It is also sufficient because when it holds we also have

$$\langle 1, \alpha \rangle^{-1} = \langle f(\xi, 1, \alpha)^{-1}, \xi \rangle = \langle 1, \xi \rangle \in \Gamma$$

where ξ is defined as in Theorem 2.2.

2.8 THEOREM. *An extension $H = \langle G, \Gamma \rangle$ of G by Γ with functions f and ϕ is a splitting extension if and only if for some function $\alpha \rightarrow x_\alpha$ of Γ into G we have*

$$f(\alpha, x_\beta, \beta) = x_\alpha^{-1} x_{\phi(\alpha, x_\beta, \beta)} \,. \tag{13}$$

Proof. Apply Theorem 2.6 with $f_1(\alpha, 1, \beta) = 1$.

COROLLARY. *If the conditions of the theorem are satisfied then $\Gamma^* = \{\langle x_\alpha, \alpha \rangle \mid \alpha \in \Gamma\}$ is a subgroup of H such that $H = G\Gamma^*$ and $G \cap \Gamma^* = 1$.*

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