SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

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Let K be an arbitrary algebraic number field. We denote by n the degree of K, by $\mathfrak{f}$ an arbitrary ideal of K, by $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ prime ideals of K, by $\mu(\mathfrak{a})$ the Moebius function of the ideal $\mathfrak{a}$ of K, by $N\mathfrak{a}$ the norm of $\mathfrak{a}$, by $(\mathfrak{a}, \mathfrak{f})$ the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{f}$, and by $h(\mathfrak{f})$ the number of ideal classes H mod $\mathfrak{f}$. It is known that

$$(1)\qquad \begin{aligned} A(x, \mathfrak{f}) := \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a},\mathfrak{f})=1}} 1 = \gamma(\mathfrak{f})x + R(x, \mathfrak{f}),\ R(x, \mathfrak{f}) = O(x^{1-1/n}) , \\ \gamma(\mathfrak{f}) = \alpha \prod_{\mathfrak{p}|\mathfrak{f}} \left(1 - \frac{1}{N\mathfrak{p}}\right) \qquad (\alpha = \alpha(K) > 0) . \end{aligned}$$

According to [1], the proof of the generalized Selberg formula for ideal classes H mod $\mathfrak{f}$ in K:

$$(2)\qquad \sum_{\substack{N\mathfrak{p} \leqq x \\ \mathfrak{p} \in H \bmod \mathfrak{f}}} \log^2 N\mathfrak{p} + \sum_{\substack{N\mathfrak{p}\mathfrak{q} \leqq x \\ \mathfrak{p}\mathfrak{q} \in H \bmod \mathfrak{f}}} \log N\mathfrak{p} \log N\mathfrak{q} = \frac{2}{h(\mathfrak{f})} x \log x + O(x)$$

can be reduced to

$$(3)\qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^2 \frac{x}{N\mathfrak{a}} = \frac{2}{\gamma(\mathfrak{f})} \log x + O(1) ,$$

and (3) is established directly in [1]. First, we generalize (3):

THEOREM 1. *Let $r > 1$ be a rational integer; then*

$$\sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} = \frac{r}{\gamma(\mathfrak{f})} \log^{r-1} x + \sum_{t=1}^{r-2} c_t(r, \mathfrak{f}) \log^t x + O(1) ;$$

the constants $c_t(r, \mathfrak{f})$ resp. the constant in $O(1)$ depends on $K, r, t, \mathfrak{f}$ resp. $K, r, \mathfrak{f}$ only.

The formula

$$\sum_{\mathfrak{a}|\mathfrak{f}} \mu(\mathfrak{a}) = \begin{cases} 1 & \text{for } \mathfrak{f} = 1 , \\ 0 & \text{for } \mathfrak{f} \neq 1 \end{cases}$$

yields

LEMMA 1. *Let $f(x)$ be a complex valued function ($x \geqq 1$); then*

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$$g(x) := \sum_{\substack{Na \leq x \\ (a,f)=1}} f\left(\frac{x}{Na}\right) \quad \textit{implies} \quad f(x) = \sum_{\substack{Na \leq x \\ (a,f)=1}} \mu(a)\, g\left(\frac{x}{Na}\right).$$

Using the Euler summation formula, we find

$$(4) \quad \sum_{m \leq x} \frac{1}{m} \log^{r-1} m = \frac{1}{r} \log^r x + a_r + O\left(\frac{1}{x} \log^{r-1} x\right) \quad (r \text{ integer}, >1).$$

Because of

$$\sum_{\substack{Na \leq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} Na = \sum_{m \leq x} (A(m, f) - A(m-1, f)) \frac{1}{m} \log^{r-1} m,$$

(1) and (4) imply

$$(5) \quad \sum_{\substack{Na \leq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} Na = \frac{\gamma(f)}{r} \log^r x + b_r(f) + O(x^{-1/n} \log^{r-1} x) \quad (r > 1);$$

the constants $b_r(f)$ depend on K, r, f only. Because of

$$\sum_{\substack{Na \leq x \\ (a,f)=1}} \left(\frac{x}{Na}\right)^{1-1/n} \log^{r-1} \frac{x}{Na} = \sum_{m \leq x} (A(m, f) - A(m-1, f)) \left(\frac{x}{m}\right)^{1-1/n} \log^{r-1} \frac{x}{m},$$

(1) implies

$$(6) \quad \sum_{\substack{Na \leq x \\ (a,f)=1}} \left(\frac{x}{Na}\right)^{1-1/n} \log^{r-1} \frac{x}{Na} = O(x).$$

By the binomial theorem and

$$\sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} \frac{1}{s+1} = \frac{1}{r},$$

(5) yields

$$(7) \quad \sum_{\substack{Na \leq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} \frac{x}{Na} = \frac{\gamma(f)}{r} \log^r x + \sum_{s=0}^{r-1} d_s(r, f) \log^s x + O(x^{-1/n} \log^{r-1} x);$$

the constants $d_s(r, f)$ depend on K, s, r, f only.

As shown in [1],

$$(8) \quad \sum_{\substack{Na \leq x \\ (a,f)=1}} \frac{\mu(a)}{Na} = O(1), \qquad \sum_{\substack{Na \leq x \\ (a,f)=1}} \frac{\mu(a)}{Na} \log \frac{x}{Na} = O(1).$$

Proof of Theorem 1. By (3), Theorem 1 is correct for $r = 2$. Suppose $r > 2$ and

$$\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^s \frac{x}{N\mathfrak{a}} = \sum_{t=1}^{s-1} c_t(s, \mathfrak{f}) \log^t x + O(1) \qquad (1 < s < r) . \tag{9}$$

In Lemma 1, let $f(x) := x \log^{r-1} x$; then

$$\begin{aligned} g(x) &= x \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} \frac{x}{N\mathfrak{a}} \\ &= \frac{\gamma(\mathfrak{f})}{r} x \log^r x + x \sum_{s=1}^{r-1} d_s(r, \mathfrak{f}) \log^s x + O(x^{1-1/n} \log^{r-1} x) , \end{aligned} \tag{10}$$

by (7). Lemma 1, (10), (9), (6), and (8) imply

$$\begin{aligned} x \log^{r-1} x &= \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \mu(\mathfrak{a}) \Big(\frac{\gamma(\mathfrak{f})x}{rN\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \frac{x}{N\mathfrak{a}} \sum_{s=1}^{r-1} d_s(r, \mathfrak{f}) \log^s \frac{x}{N\mathfrak{a}} \\ &\quad + O\Big(\Big(\frac{x}{N\mathfrak{a}} \Big)^{1-1/n} \log^{r-1} \frac{x}{N\mathfrak{a}} \Big) \Big) \\ &= \frac{\gamma(\mathfrak{f})x}{r} \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \sum_{s=2}^{r-1} d_s(r, \mathfrak{f}) \sum_{t=1}^{s-1} c_t(s, \mathfrak{f}) \log^t x + O(x) ; \end{aligned}$$

let

$$c_t(r, \mathfrak{f}) := -\frac{r}{\gamma(\mathfrak{f})} \sum_{s=t+1}^{r-1} d_s(r, \mathfrak{f})\, c_t(s, \mathfrak{f}) \qquad (t = 1, 2, \cdots, r-2) .$$

This proves Theorem 1.

The fact that

$$c_{s-1}(s, \mathfrak{f}) = \frac{s}{\gamma(\mathfrak{f})}$$

was not used in the preceding proof.

Now we derive two consequences of (2). The well-known relation (Landau (1903))

$$T(x) := \sum_{N\mathfrak{p}\leq x} \log N\mathfrak{p} = O(x)$$

implies

$$T(x, H \bmod \mathfrak{f}) := \sum_{\substack{N\mathfrak{p}\leq x\\ \mathfrak{p}\in H \bmod \mathfrak{f}}} \log N\mathfrak{p} = O(x) \tag{11}$$

By

$$\sum_{\substack{N\mathfrak{p}\leq x\\ \mathfrak{p}\in H \bmod \mathfrak{f}}} \log^2 N\mathfrak{p} = \sum_{m\leq x} (T(m, H \bmod \mathfrak{f}) - T(m-1, H \bmod \mathfrak{f})) \log m ,$$

(11) gives

$$\sum_{\substack{Np\leq x \\ p\in H \bmod f}} \log^2 Np = T(x, H \bmod f)\log x + O(x) . \tag{12}$$

According to Landau (1903), we have

$$s(x) := \sum_{Np\leq x} \frac{\log Np}{Np} = \log x + O(1) . \tag{13}$$

Using

$$\sum_{N\ \leq x} \frac{\log^2 Np}{Np} = \sum_{m\leq x} (S(m) - S(m-1))\log m ,$$

(13) implies

$$\sum_{Np\leq x} \frac{\log^2 Np}{Np} = \frac{1}{2}\log^2 x + O(\log x) . \tag{14}$$

LEMMA 2. *We have*

$$\sum_{\substack{Npq\leq x \\ pq\in H \bmod f}} \log^2 Np \log Nq = \frac{\log x}{2} \sum_{\substack{Npq\leq x \\ pq\in H \bmod f}} \log Np \log Nq + O(x\log x) .$$

Proof. Denote by $H(q) \bmod f$ the class of all ideals a of K with $aq \in H \bmod f$; then (12), (13) and the definition of $T(x, H \bmod f)$ in (11) give

$$\begin{aligned}\sum_{\substack{Npq\leq x \\ pq\in H \bmod f}} \log^2 Np \log Nq &= \sum_{\substack{Nq\leq x \\ (q,f)=1}} \log Nq \left(T\left(\frac{x}{Nq}, H(q) \bmod f\right)\log\frac{x}{Nq} + O\left(\frac{x}{Nq}\right)\right) \\ &= \sum_{\substack{Npq\leq x \\ pq\in H \bmod f}} \log Nq \log Np (\log x - \log Nq) \\ &\quad + O(x\log x) .\end{aligned}$$

This proves Lemma 2.

THEOREM 2. *We have*

$$\log x \sum_{\substack{Npq\leq x \\ pq\in H \bmod f}} \log Np \log Nq + 2 \sum_{\substack{Npqr\leq x \\ pq\in r H \bmod f}} \log Np \log Nq \log Nr$$
$$= \frac{2x}{h(f)}\log^2 x + O(x\log x)$$

where the constant in the remainder term depends on K and f only.

Proof. We write (2) for x/Nr and $H(r) \bmod f$ instead of x and

$H \bmod \boldsymbol{f}$, multiply by $\log N\boldsymbol{r}$, and take summation over all prime ideals $\boldsymbol{r}$ with $(\boldsymbol{r}, \boldsymbol{f}) = 1$ and $N\boldsymbol{r} \leqq x$. By (13) and (14), we find

$$\sum_{\substack{N\boldsymbol{pr} \leqq x \\ \boldsymbol{pr} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \log N\boldsymbol{r} + \sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r}$$
$$= \frac{x}{h(\boldsymbol{f})} \log^2 x + O(x \log x) .$$

The application of Lemma 2 completes the proof.

THEOREM 3. *If*

$$\sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for the principal class $H_0 \bmod \boldsymbol{f}$, *then*

$$\frac{1}{x} \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for all $h(\boldsymbol{f})$ *classes* $H \bmod \boldsymbol{f}$.

Proof. Suppose

$$\sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_1 \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} = O(x)$$

for a certain ideal class $H_1 \bmod \boldsymbol{f}$. Then (2) implies

$$\sum_{\substack{N\boldsymbol{pq} \leqq x \\ \boldsymbol{pq} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x) , \tag{15}$$

and Theorem 2 gives

$$\sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} = O(x \log x) . \tag{16}$$

By (15) and (13), we get

$$\sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} \geqq \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \sum_{\substack{N\boldsymbol{qr} \leqq x/N\boldsymbol{p} \\ \boldsymbol{qr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{q} \log N\boldsymbol{r}$$

$$\begin{aligned} &= \sum_{\substack{N\boldsymbol{p} \leq x \\ \boldsymbol{p} \in H \bmod_0 \boldsymbol{f}}} \log N\boldsymbol{p} \left(\frac{2}{h(\boldsymbol{f})} \frac{x}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O\left(\frac{x}{N\boldsymbol{p}}\right)\right) \\ &= \frac{2x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O(x \log x) \\ &= \frac{x \log x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p} \leqq \sqrt{x} \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} + O(x \log x) . \end{aligned} \tag{17}$$

(17) and (16) imply the contradiction

$$\sum_{\substack{Np \leqq \sqrt{x} \\ p \in H_0 \bmod f}} \frac{\log Np}{Np} = O(1) ,$$

and Theorem 3 is proved.

The special case of Theorem 3 for the rational number field was treated in [2].

BIBLIOGRAPHY

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