ON COMPLEX APPROXIMATION

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1. Let C denote the set of complex numbers and G the set of Gaussian integers. In this note we prove the following theorem which is a two-dimensional analogue of Theorem 2 in [3].

THEOREM 1. *If* $\beta, \gamma \in C$, *then there exists* $u \in G$ *such that* $|\beta - u| < 2$ *and*

$$|\beta - u||\gamma - u| < \begin{cases} 27/32 & \text{if } |\beta - r| < \sqrt{11/8} \\ \sqrt{2}\,|\beta - \gamma|/2 & \text{if } |\beta - \gamma| \geq \sqrt{11/8}\,. \end{cases}$$

As an illustration of the application of Theorem 1 to complex approximation, we use it to prove the following result.

THEOREM 2. *If* $\theta \in C$ *is irrational and* $a \in C$, $a \neq m\theta + n$ *where* $m, n \in G$, *then there exist infinitely many pairs of relatively prime integers* $x, y \in G$ *such that*

$$|x(x\theta - y - a)| < 1/2\,.$$

The method of proof of Theorem 2 is due to Niven [6]. Also in [7], Niven uses Theorem 1 to obtain a more general result concerning complex approximation by nonhomogeneous linear forms.

Alternatively, Theorem 2 may be obtained as a consequence of a theorem of Hlawka [5]. This was done by Eggan [2] using Chalk's statement [1] of Hlawka's Theorem.

2. Theorem 1 may be restated in an equivalent form. For $u, b, c \in C$, define

$$g(u, b, c) = |u - (b + c)||u - (b - c)|\,.$$

Then Theorem 1 may be stated as follows.

THEOREM 1′. *If* $b, c \in C$, *then there exist* $u_1, u_2 \in G$ *such that*

(i) $|u_1 - (b + c)| < 2$, $|u_2 - (b - c)| < 2$

and for $i = 1,2$,

(ii) $$g(u_i, b, c) < \begin{cases} 27/32 & \text{if } |c| < \sqrt{11/32} \\ \sqrt{2}\,|c| & \text{if } |c| \geqq \sqrt{11/32}\,. \end{cases}$$

It is clear that Theorem 1′ implies Theorem 1 by taking

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$$b = (\beta + \gamma)/2,\ c = (\beta - \gamma)/2 .$$

To see that Theorem 1 implies Theorem 1′, first apply Theorem 1 with $\beta = b + c$, $\gamma = b - c$ and then apply Theorem 1 with $\beta = b - c$, $\gamma = b + c$.

3. We precede the proof of Theorem 1′ with a few remarks concerning the nature of the proof.

Given $b, c \in C$, introduce a rectangular coordinate system for the complex plane such that b has coordinates $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y)

$$\begin{aligned} g^2(u,b,c) &= |u - b - c|^2|u - b + c|^2 \\ &= ((x - k)^2 + y^2)((x + k)^2 + y^2) \\ &= (x^2 + y^2 + k^2)^2 - 4k^2x^2 . \end{aligned}$$

Now for k a positive real number let $R(k)$ be the set of all points (x, y) such that

$$(x^2 + y^2 + k^2)^2 - 4k^2x^2 < \begin{cases} (27/32)^2 & \text{if } k < \sqrt{11/32} \\ 2k^2 & \text{if } k \geqq \sqrt{11/32} . \end{cases}$$

Theorem 1′ depends upon showing that $R(k)$ under any rigid motion always contains two lattice points, not necessarily distinct. These lattice points correspond to the integers u_1 and u_2 of the theorem.

For $k > 1/\sqrt{2}$, $R(k)$ contains two circles with centers at

$$(\pm\sqrt{k^2 - 1/2},\ 0)$$

and each of radius $1/\sqrt{2}$. Each of these circles contains a lattice point no matter how $R(k)$ is displaced in the plane. In this case, u_1 and u_2 correspond to these lattice points.

For $k < \sqrt{11/32}$, $R(k)$ contains the circle with center at $(0, 0)$ and radius $1/\sqrt{2}$. In this case, $u_1 = u_2$ corresponds to a lattice point in this circle. Finally if $\sqrt{11/32} \leqq k \leqq 1/\sqrt{2}$ $R(k)$ contains a region described by Sawyer [8] which always contains a lattice point no matter how it is displaced and $u_1 = u_2$ corresponds to a lattice point in this region.

4. We turn now to the proof of Theorem 1′. As above, for given $b, c \in C$, introduce a coordinate system so that b has coordinate $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y),

$$g^2(u, b, c) = (x^2 + y^2 + k^2)^2 - 4k^2x^2 . \tag{1}$$

Suppose that $|c| = k > 1/\sqrt{2}$. For $i = 1, 2$ let

$$d_i = (\delta_i\sqrt{k^2 - 1/2}, 0)$$

where $\delta_i = (-1)^{i+1}$ and let $u_i \in G$ be a closest Gaussian integer to d_i (i.e. $|d_i - u_i| \leqq |d_i - t|$, $t \in G$). Then, omitting the subscripts,

$$|d - (b + \delta c)| = |\delta\sqrt{k^2 - 1/2} - \delta k| = k - \sqrt{k^2 - 1/2} < 1/\sqrt{2}\,.$$

Hence

$$|u - (b + \delta c)| \leqq |u - d| + |d - (b + \delta c)| < 2(1/\sqrt{2}) < 2$$

and condition (i) is satisfied.

Now let u_i have coordinates (x_i, y_i). Then, again omitting subscripts, since $|d - u| \leqq 1/\sqrt{2}$, we have

$$(2) \qquad (x - \delta\sqrt{k^2 - 1/2})^2 + y^2 \leqq 1/2\,,$$

equality holding if and only if d is the center of a unit square with Gaussian integers as vertices. Also, since for any two real numbers a and b, $2ab \leqq a^2 + b^2$, equality holding if and only if $a = b$, we have

$$(3) \qquad 2\delta x\sqrt{k^2 - 1/2} \leqq x^2 + k^2 - 1/2\,,$$

equality holding if and only if $x = \sqrt{k^2 - 1/2}/\delta$. Thus

$$\begin{aligned}(1 + 2\delta x\sqrt{k^2 - 1/2})^2 &= 4\delta x\sqrt{k^2 - 1/2} + 4x^2(k^2 - 1/2) + 1\\ &\leqq 2x^2 + 2k^2 - 1 + 4x^2(k^2 - 1/2) + 1\\ &= k^2(2 + 4x^2)\end{aligned}$$

and since k and $k^2(2 + 4x^2)$ are positive,

$$1 + 2\delta x\sqrt{k^2 - 1/2} \leqq k\sqrt{2 + 4x^2}\,.$$

Hence

$$\begin{aligned}(4) \qquad 1/2 - (x - \delta\sqrt{k^2 - 1/2})^2 &= 1 + 2\delta x\sqrt{k^2 - 1/2} - x^2 - k^2\\ &\leqq k\sqrt{2 + 4x^2} - x^2 - k^2\,.\end{aligned}$$

Using (4) and (2), we have

$$\begin{aligned}x^2 + k^2 + y^2 &\leqq k\sqrt{2 + 4x^2} + (x - \delta\sqrt{k^2 - 1/2})^2 - 1/2 + y^2\\ &\leqq k\sqrt{2 + 4x^2}\,,\end{aligned}$$

$$(5) \qquad (x^2 + k^2 + y^2)^2 \leqq 2k^2 + 4k^2x^2\,.$$

Thus, from (1) and (5), $g^2(u, b, c) \leqq 2k^2$, the equality holding if and only if equality holds in both (2) and (3). If equality holds in (2), then there exist four possible choices for u, at least two of these choices having unequal first coordinates. Now equality holds in (3) if and only if, for fixed k, x is unique. Thus if equality holds in (2), u may be chosen so that equality does not hold in (3). For this choice of u, $g^2(u,b,c) < 2k^2$ which establishes condition (ii).

Next suppose $|c| = k < \sqrt{11/32}$. Now there exists $u \in G$ such that $|u - b| \leqq 1/\sqrt{2}$. Thus

$$|u - (b \pm c)| \leqq |u - b| + |c| < 2(1/\sqrt{2}) < 2 .$$

Also, if u has coordinates (x, y), $x^2 + y^2 \leqq 1/2$ and thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &< \frac{1}{4} + 2\left(\frac{11}{32}\right)\frac{1}{2} + \left(\frac{11}{32}\right)^2 = \left(\frac{27}{32}\right)^2 \end{aligned}$$

which establishes the theorem for $|c| < \sqrt{11/32}$.

Finally, for $\sqrt{11/32} \leqq |c| = k \leqq 1/\sqrt{2}$, we use a result due to Sawyer [8] which states that the region defined by $|x| \leqq 3/4 - y^2$, $|y| \leqq 1/2$ always contains a lattice point no matter how it is displaced in the plane. Thus there exists $u \in G$ with coordinates (x, y) such that $|x| \leqq 3/4 - y^2$, $|y| \leqq 1/2$.

If $|x| < 1/2$, then

$$|u - (b \pm c)| \leqq |u - b| + |c| = \sqrt{x^2 + y^2} + |c| \leqq \sqrt{2} .$$

Also since $|x^2 - k^2| \leqq 1/2$,

$$\begin{aligned} g^2(u, b, c) &= (x^2 - k^2)^2 + 2y^2(x^2 + k^2) + y^4 \\ &< \frac{1}{4} + 2\frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{16} = \frac{11}{16} \leqq 2|c|^2 . \end{aligned}$$

If $1/2 \leqq |x| \leqq 3/4 - y^2$, then

$$x^2 + y^2 \leqq \frac{9}{16} - \frac{1}{2}y^2 + y^4 = \frac{1}{2} + \left(y^2 - \frac{1}{4}\right)^2 \leqq \frac{9}{16} .$$

Hence

$$|u - (b + c)| \leqq \sqrt{x^2 + y^2} + |c| \leqq \frac{3}{4} + \frac{1}{\sqrt{2}} < 2 .$$

Also $-x^2 \leqq -1/4$ so $y^2 - x^2 \leqq 0$. Thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &\leqq \left(\frac{9}{16}\right)^2 + 0 + \frac{1}{4} < \frac{11}{16} \leqq 2|c|^2 . \end{aligned}$$

This completes the proof of Theorem 1′.

5. To prove Theorem 2, we require a well-known result of Ford [4] which states that for any irrational $\theta \in C$, there exist infinitely many pairs of relatively prime $h, k \in G$ such that

$$|k(k\theta - h)| < 1/\sqrt{3}\,. \tag{6}$$

For θ and a is in the statement of Theorem 2, choose h, k satisfying (6) and let $t \in G$ be such that $|t - ka| \leqq 1/\sqrt{2}$. Since h and k are relatively prime, there exist $r, s \in G$ such that $rh - sk = t$ and hence

$$|rh - sk - ka| \leqq 1/\sqrt{2}\,. \tag{7}$$

Now, in Theorem 1, let

$$\beta = \frac{r\theta - s - a}{k\theta - h}\,, \qquad \gamma = \frac{r}{k}$$

and set

$$x = r - ku,\ y = s - hu$$

where u is the Gaussian integer whose existence is guaranteed by the theorem. Then $x, y \in G$ and

$$|x\theta - y - a|\,|x| = |\beta - u|\,|\gamma - u|\,|k|\,|k\theta - h|\,.$$

Hence if $|\beta - \gamma| < \sqrt{11/8}$ we have, using Theorem 1 and (6),

$$|x\theta - y - a|\,|x| < \frac{27}{32}|k(k\theta - h)| < \frac{27}{32}\,\frac{1}{\sqrt{3}} < \frac{1}{2}\,.$$

If $|\beta - \gamma| \geqq \sqrt{11/8}$, using Theorem 1 and (7), we have

$$\begin{aligned} |x\theta - y - a|\,|x| &< \frac{1}{2}\sqrt{2}\,|\gamma - \beta|\,|k(k\theta - h)| \\ &= \frac{1}{2}\sqrt{2}\left|\frac{hr - ks - ka}{k(k\theta - h)}\right|\,|k(k\theta - h)| \leqq \frac{1}{2}\,. \end{aligned}$$

Thus for each pair h, k satisfying (6) we have a solution in G of

$$|x(x\theta - y - a)| < 1/2\,. \tag{8}$$

To show that there are infinitely many solutions to (8), we note that since $|\beta - u| < 2$ and $a \neq m\theta + n$, $m, n \in G$, we have with the use of (6).

$$0 < |x\theta - y - a| = |\beta - u|\,|k\theta - h| < 2/(\sqrt{3}\,|k|)\,. \tag{9}$$

If there are only a finite number of solutions of (8), let M be the minimum of $|x\theta - y - a|$ for these solutions. Then from (9), for every h, k satisfying (6) we have $|k| < 2/(\sqrt{3}M)$ and

$$|h| \leqq |h - k\theta| + |k\theta| < 1/(\sqrt{3}\,|k|) + |k||\theta| < N\,,$$

say. But this is impossible since there are infinitely many pairs $h, k \in G$ which satisfy (6).

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