EQUALITY IN CERTAIN INEQUALITIES

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1. Introduction. Let $\sigma = (\sigma_1, \cdots, \sigma_n)$ be a point on the unit $(n-1)$-simplex S^{n-1}: $\sum_{i=1}^n \sigma_i = 1$, $\sigma_i \geqq 0$. Let $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ and $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n > 0$ be positive numbers and form the function on S^{n-1}

$$F(\sigma) = \sum_{i=1}^n \sigma_i \lambda_i \sum_{i=1}^n \sigma_i \mu_i \, . \tag{1.1}$$

The main purpose of this paper is to examine the structure of the set of points $\sigma \in S^{n-1}$ for which $F(\sigma)$ takes on its maximum value. In case a convex monotone decreasing function f is fitted to the points (λ_i, μ_i)(i.e. $f(\lambda_i) = \mu_i$), $i = 1, \cdots, n$, then it is not difficult to show that the maximum for $F(\sigma)$ on S^{n-1} is the upper bound given by M. Newman [4] in a recent interesting paper. In the case of the Kantorovich inequality [1] the function f is $f(t) = t^{-1}$, $\mu_i = \lambda_i^{-1}$, $i = 1, \cdots, n$. In this case a maximizing σ is $\sigma_1 = 1/2$, $\sigma_n = 1/2$, $\sigma_i = 0$, $i = 2, \cdots, n-1$, and if $\lambda_1 < \lambda_k < \lambda_n$, $k = 2, \cdots, n-1$, it is a corollary of our main result (Theorem 2) that this is the only choice possible for $\sigma \in S^{n-1}$ in order to achieve the maximum value.

We shall assume henceforth in this paper that $\mu_i = f(\lambda_i)$, $i = 1, \cdots, n$, where f is a monotone decreasing convex function defined on the closed interval $[\lambda_1, \lambda_n]$. In 2 we determine the structure of the set of $\sigma \in S^{n-1}$ for which $F(\sigma)$ is a maximum in the case in which f is assumed to be strictly convex. In 3 we investigate the structure of the set of unit vectors x for which the function

$$\varphi(x) = (Ax, x)(f(A)x, x) \tag{1.2}$$

assumes its maximum value on the unit sphere $\| x \| = 1$. Throughout, A is a positive definite hermitian transformation on an n-dimensional unitary space U with inner product (x, y). The eigenvalues of A are λ_i, $0 < \lambda_1 \leqq \cdots \leqq \lambda_n$, with corresponding orthonormal eigenvectors u_i, $Au_i = \lambda_i u_i$, $i = 1, \cdots, n$. Of particular interest in (1.2) is the choice $f(t) = t^{-p}$, $p > 0$.

Finally, in 4, we discuss the applications of the previous results to Grassmann compounds and induced power transformations associated with A. In two recent papers [2, 5] the Kantorovich inequality was applied to the compound to obtain inequalities involving principal subdeterminants of a positive definite hermitian matrix. We shall prove (Theorem 5) that these inequalities are in fact strict except in

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trivial cases. Similar inequalities are obtained for the permanent function together with a discussion of the cases of equality. These inequalities are believed to be new.

2. Maximum values for F. In the rest of the paper M will systematically denote the maximum value of $F(\sigma), \sigma \in S^{n-1}$, and m will denote the largest of $\lambda_1\mu_1$ and $\lambda_n\mu_n$. Also, γ will denote the number $(\lambda_1\mu_n + \lambda_n\mu_1)/2$. The main result of this section is Theorem 2 which describes the structure of those σ for which $F(\sigma) = M$ when f is strictly convex. We first prove

THEOREM 1. *For any $\sigma \in S^{n-1}$ there exists a $\beta \in [0, 1]$ such that*

$$F(\sigma) \leqq (\beta\lambda_1 + (1-\beta)\lambda_n)(\beta\mu_1 + (1-\beta)\mu_n) . \tag{2.1}$$

If f is strictly convex and for some $k, 1 \leqq k \leqq n, \lambda_1 < \lambda_k < \lambda_n$ and $\sigma_k > 0$ then there exists a $\beta \in [0, 1]$ for which (2.1) is a strict inequality.

To prove Theorem 1 we use the following elementary fact.

LEMMA. *If $0 \leqq a_1 \leqq a_2 \leqq a_3$, and $b_1 \geqq b_2 \geqq b_3 \geqq 0$ and*

$$(a_1 - a_3)(b_2 - b_3) \geqq (a_2 - a_3)(b_1 - b_3) \tag{2.2}$$

then for any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in S^2$ there exists a $\beta \in [0, 1]$ such that

$$\sum_{i=1}^{3}\alpha_i a_i \sum_{i=1}^{3}\alpha_i b_i \leqq (\beta a_1 + (1-\beta)a_2)(\beta b_1 + (1-\beta)b_3) . \tag{2.3}$$

If the inequality (2.2) is strict and $\alpha_2 > 0$ then there exists a $\beta \in [0, 1]$ such that (2.3) is strict.

Proof. Let θ and ω in $[0, 1]$ be so chosen that $a_2 = \theta a_1 + (1-\theta)a_3$, $b_2 = \omega b_1 + (1-\omega)b_3$ and set $b_2' = \theta b_1 + (1-\theta)b_3$. Then

$$b_2' - b_2 = (\theta - \omega)(b_1 - b_3) . \tag{2.4}$$

Assume first that $a_3 > a_2$ and $b_2 > b_3$. Then $\theta = (a_2 - a_3)/(a_1 - a_3) > 0$ and $\omega = (b_2 - b_3)/(b_1 - b_3)$. Moreover $\theta \geqq \omega$ by (2.2) and if (2.2) is strict then $\theta > \omega$. From (2.4) $b_2' - b_2 \geqq 0$ and we compute that

$$\begin{aligned} L \leqq\ & ((\alpha_1 + \theta\alpha_2)a_1 + (\alpha_2(1-\theta) + \alpha_3)a_3) \\ & ((\alpha_1 + \theta\alpha_2)b_1 + (\alpha_2(1-\theta) + \alpha_3)b_3) , \end{aligned} \tag{2.5}$$

where L is the left side of (2.3). This is (2.3) with $\beta = \alpha_1 + \theta\alpha_2 \in [0, 1]$. If (2.2) is strict then $\theta > \omega$, $b_2' = b_2$, and $\alpha_2 > 0$ together imply that (2.5) is strict.

Suppose next that $a_2 = a_3$. From (2.2) and $(a_1 - a_3) \leqq 0$ we have

$(a_1 - a_3)(b_2 - b_3) = 0$ and hence $a_1 = a_3$ or $b_2 = b_3$. The first alternative yields $a_1 = a_2 = a_3$ and thus $L = a_1 \sum_{i=1}^{3} \alpha_i b_i \leqq a_1 b_1$ which is (2.3) with $\beta = 1$. If $b_2 = b_3$ then (2.3) holds with $\beta = \alpha_1$. This completes the proof of the lemma.

The proof of Theorem 1 is by induction on n. The first nontrivial case is $n = 3$. In general the convexity of f implies that

$$(\lambda_1 - \lambda_3)(\mu_2 - \mu_3) > (\lambda_2 - \lambda_3)(\mu_1 - \mu_3) \tag{2.6}$$

and (2.6) is strict if $\lambda_1 < \lambda_2 < \lambda_3$ and f is strictly convex. The inequality (2.1) follows from the lemma. If $n > 3$ we distinguish the two possibilities $\sigma_1 + \sigma_2 = 1$ and $\sigma_1 + \sigma_2 < 1$. In the first case

$$F(\sigma) = (\sigma_1\lambda_1 + \sigma_2\lambda_2)(\sigma_1\mu_1 + \sigma_2\mu_2) . \tag{2.7}$$

If $\mu_1 = \mu_n$ and hence $\mu_i = \mu_1 = \mu_n$, $i = 1, \cdots, n$, then $F(\sigma) \leqq \lambda_n\mu_n$ which is (2.1) with $\beta = 0$. If $\mu_1 > \mu_n$, and hence $\lambda_1 < \lambda_n$, obtain θ and ω in $[0, 1]$ so that $\lambda_2 = \theta\lambda_1 + (1 - \theta)\lambda_n$, $\mu_2 = \omega\mu_1 + (1 - \omega)\mu_n$ and set $\mu_2' = \theta\mu_1 + (1 - \theta)\mu_n$ to obtain

$$\mu_2' - \mu_2 = (\theta - \omega)(\mu_1 - \mu_n) \geqq 0 . \tag{2.8}$$

The convexity of f again implies that $\theta \geqq \omega$ with strictness in case f is strictly convex and $\lambda_2 > \lambda_n$. Hence

$$\begin{aligned} F(\sigma) &\leqq (\sigma_1\lambda_1 + (\theta\lambda_1 + (1 - \theta)\lambda_n)\sigma_2)(\sigma_1\mu_1 + \sigma_2\mu_2') \\ &= ((\sigma_1 + \theta\sigma_2)\lambda_1 + (1 - \theta)\sigma_2\lambda_n)((\sigma_1 + \theta\sigma_2)\mu_1 + (1 - \theta)\sigma_2\mu_n) \end{aligned}$$

which is (2.1) with $\beta = \sigma_1 + \theta\sigma_2$. We proceed to the case $\sigma_1 + \sigma_2 < 1$. Let $\lambda_3' = \sum_{i=3}^{n} \sigma_i\lambda_i/(1 - \sigma_1 - \sigma_2)$, $\mu_3'' = \sum_{i=3}^{n} \sigma_i\mu_i/(1 - \sigma_1 - \sigma_2)$ and observe that $\lambda_1 \leqq \lambda_2 \leqq \lambda_3'$, $\mu_1 \geqq \mu_2 \geqq \mu_3''$ and $F(\sigma) = (\sigma_1\lambda_1 + \sigma_2\lambda_2 + (1 - \sigma_1 - \sigma_2)\lambda_3')$ $(\sigma_1\mu_1 + \sigma_2\mu_2 + (1 - \sigma_1 - \sigma_2)\mu_3'')$. We next verify that (2.2) holds for the choices $\lambda_3' = a_3$, $\lambda_2 = a_2$, $\lambda_1 = a_1$, $\mu_1 = b_1$, $\mu_2 = b_2$, $\mu_3'' = b_3$:

$$\begin{aligned} &(\lambda_1 - \lambda_3)(\mu_2 - \mu_3'') - (\mu_1 - \mu_3'')(\lambda_2 - \lambda_3') \\ &\quad = \mu_2(\lambda_1 - \lambda_3') - \mu_1(\lambda_2 - \lambda_3') + \mu_3''(\lambda_2 - \lambda_1) ; \end{aligned} \tag{2.9}$$

and

$$\mu_3'' = \sum_{i=3}^{n} f(\lambda_i)\sigma_i/(1 - \sigma_1 - \sigma_2) \geqq f\Big(\sum_{i=3}^{n} \lambda_i\sigma_i/(1 - \sigma_1 - \sigma_2)\Big) = f(\lambda_3') = \mu_3' .$$

Hence the expression in (2.9) is at least

$$\mu_2(\lambda_1 - \lambda_3') - \mu_1(\lambda_2 - \lambda_3') + \mu_3'(\lambda_2 - \lambda_1) . \tag{2.10}$$

If $\lambda_2 = \lambda_3'$ the expression (2.10) reduces to 0 and the expression in (2.9) is nonnegative. If $\lambda_2 < \lambda_3'$ then $\lambda_1 < \lambda_3'$ and (2.10) becomes $(\lambda_1 - \lambda_3')(\lambda_2 - \lambda_3')\{(\mu_2 - \mu_3')/(\lambda_2 - \lambda_3') - (\mu_1 - \mu_3')/(\lambda_1 - \lambda_3')\} \geqq 0$. Apply the lemma to obtain $\beta_1 \in [0, 1]$ for which

$$\begin{aligned}(\sigma_1\lambda_1 + \sigma_2\lambda_2 + (1 - \sigma_1 - \sigma_2)\lambda_3')(\sigma_1\mu_1 + \sigma_2\mu_2 + (1 - \sigma_1 - \sigma_2)\mu_3'') \\ \leqq (\beta_1\lambda_1 + (1 - \beta_1)\lambda_3')(\beta_1\mu_1 + (1 - \beta_1)\mu_3'') \\ = \left(\beta_1\lambda_1 + \sum_{i=3}^{n}(1 - \beta_1)\sigma_1\lambda_i/(1 - \sigma_1 - \sigma_2)\right) \\ \left(\beta_1\mu_1 + \sum_{i=3}^{n}(1 - \beta_1)\sigma_i\mu_i/(1 - \sigma_1 - \sigma_2)\right) .\end{aligned}$$

This last expression is a product of convex combinations of λ's and μ's involving only $n - 1$ terms and satisfying the induction hypothesis. Hence there exists $\beta \in [0, 1]$ such that

$$\begin{aligned}F(\sigma) \leqq \left(\beta_1\lambda_1 + \sum_{i=3}^{n}(1 - \beta_1)\sigma_i\lambda_i/(1 - \sigma_1 - \sigma_2)\right) \\ \left(\beta_1\mu_1 + \sum_{i=3}^{n}(1 - \beta_1)\sigma_i\mu_i/(1 - \sigma_1 - \sigma_2)\right) \leqq (\beta\lambda_1 + (1 - \beta)\lambda_n) \\ (\beta\mu_1 + (1 - \beta)\mu_n) .\end{aligned}$$

This establishes (2.10).

The discussion of the strictness in (2.1) requires the use of (2.1) itself. Let k be the least integer for which both $\sigma_k > 0$ and $\lambda_1 < \lambda_k < \lambda_n$. Then

$$\begin{aligned}F(\sigma) = (\alpha_1\lambda_1 + \alpha_k\lambda_k + \alpha_{k+p}\lambda_{k+p} + \cdots + \alpha_n\lambda_n) \\ (\alpha_1\mu_1 + \alpha_k\mu_k + \alpha_{k+p}\mu_{k+p} + \cdots + \alpha_n\mu_n)\end{aligned} \tag{2.11}$$

in which $\alpha_1 + \alpha_k + \alpha_{k+p} + \cdots + \alpha_n = 1$, $\alpha_j = \sigma_j$, $j = k + p, \cdots, n$, and $\lambda_k < \lambda_{k+p}$. Assume

$$\begin{aligned}\alpha_1 + \alpha_k < 1, \text{ set } \lambda'_{k+p} = \sum_{i=k+p}^{n}\sigma_i\lambda_i/(1 - \alpha_1 - \alpha_k),\ \mu''_{k+p} \\ = \sum_{i=k+p}^{n}\sigma_i\mu_i/(1 - \alpha_1 - \alpha_k)\end{aligned}$$

and (2.11) becomes

$$\begin{aligned}F(\sigma) = (\alpha_1\lambda_1 + \alpha_k\lambda_k + (1 - \alpha_1 - \alpha_k)\lambda'_{k+p}) \\ (\alpha_1\mu_1 + \alpha_k\mu_k + (1 - \alpha_1 - \alpha_k)\mu''_{k+p}) .\end{aligned} \tag{2.12}$$

Clearly $\lambda_1 < \lambda_k < \lambda'_{k+p}$ and we compute that

$$\begin{aligned}(\lambda_1 - \lambda'_{k+p})(\mu_k - \mu'_{k+p}) - (\mu_1 - \mu'_{k+p})(\lambda_k - \lambda'_{k+p}) \\ = \mu_k(\lambda_1 - \lambda'_{k+p}) - \mu_1(\lambda_k - \lambda'_{k+p}) + \mu''_{k+p}(\lambda_k - \lambda_1) ;\end{aligned} \tag{2.13}$$

$$\mu''_{k+p} \geqq f(\lambda'_{k+p}) = \mu'_{k+p} . \tag{2.14}$$

It follows that the expression in (2.13) is at least

$$(\lambda_1 - \lambda'_{k+p})(\lambda_k - \lambda'_{k+p})\{(\mu_k - \mu'_{k+p})/(\lambda_k - \lambda'_{k+p}) - (\mu_1 - \mu'_{k+p})/(\lambda_1 - \lambda'_{k+p})\}$$

and in case f is strictly convex this whole expression is positive. The inequality (2.2) holds strictly with $\lambda_1 = a_1$, $\lambda_k = a_2$, $\lambda'_{k+p} = a_3$, $\mu_1 = b_1$, $\mu_k = b_k$, $\mu'_{k+p} = b_3$ and the strict form of the lemma together with (2.12) implies that there exists $\beta_1 \in [0, 1]$ such that

$$F(\sigma) < \left(\beta_1\lambda_1 + \sum_{i=k+p}^{n} (1 - \beta_1)\sigma_i\lambda_i/(1 - \alpha_1 - \alpha_k)\right) \left(\beta_1\mu_1 + \sum_{i=k+p}^{n} (1 - \beta_1)\sigma_i\mu_i/(1 - \alpha_1 - \alpha_k)\right) . \tag{2.15}$$

Now apply (2.1) to the right side of (2.15) to obtain a $\beta \in [0, 1]$ for which $F(\sigma) < (\beta\lambda_1 + (1 - \beta)\lambda_n)(\beta\mu_1 + (1 - \beta)\mu_n)$.

Assume now that $\alpha_1 + \alpha_k = 1$ and then $F(\sigma)$ becomes $(\alpha_1\lambda_1 + (1 - \alpha_1)\lambda_k)(\alpha_1\mu_1 + (1 - \alpha_1)\mu_k)$. Choose θ and ω in $[0, 1]$ so that $\lambda_k = \theta\lambda_1 + (1 - \theta)\lambda_n$, $\mu_k = \omega\mu_1 + (1 - \omega)\mu_n$, set $\mu''_k = \theta\mu_1 + (1 - \theta)\mu_n$ and note that $\mu''_k - \mu_k = (\theta - \omega)(\mu_1 - \mu_n)$. Then since f is monotone decreasing and strictly convex, $\theta - \omega$ and $\mu_1 - \mu_n$ are both positive. It follows that

$$(\alpha_1\lambda_1 + (1 - \alpha_1)\lambda_k)(\alpha_1\mu_1 + (1 - \alpha_1)\mu_k) < ((\alpha_1 + \theta(1 - \alpha_1))\lambda_1 + (1 - \theta)(1 - \alpha_1)\lambda_n)((\alpha_1 + \theta(1 - \alpha_1))\mu_1 + (1 - \theta)(1 - \alpha_1)\mu_n) .$$

If the quadratic polynomial in β on the right in (2.1) is maximized in $[0, 1]$ we immediately obtain our main result.

THEOREM 2. *If*

$$\gamma \geqq m \text{ and } \lambda_1 < \lambda_n \text{ and } \mu_1 > \mu_n \tag{2.16}$$

then

$$M = (\lambda_n\mu_1 - \lambda_1\mu_n)/4(\lambda_n - \lambda_1)(\mu_1 - \mu_n) . \tag{2.17}$$

If

$$\gamma \leqq m \text{ or } \lambda_1 = \lambda_n \text{ or } \mu_1 = \mu_n \tag{2.18}$$

then

$$M = m . \tag{2.19}$$

Let f be strictly convex and suppose that

$$\lambda_1 = \cdots = \lambda_p < \lambda_{p+1} \leqq \cdots \leqq \lambda_{n-q} < \lambda_{n-q+1} = \cdots = \lambda_n .$$

Then $F(\sigma) = M$, $\sigma \in S^{n-1}$, if and only if σ has the form

$$\sigma = (\sigma_1, \cdots \sigma_p, 0, \cdots, 0, \sigma_{n-q+1}, \cdots, \sigma_n) ,$$

$\sum_{j=1}^{p}\sigma_j = \beta_0$, $\sum_{j=n-q+1}^{n}\sigma_j = 1 - \beta_0$, *where*

$$\beta_0 = \begin{cases}(\gamma - \lambda_n\mu_n)/(\lambda_n - \lambda_1)(\mu_1 - \mu_n) \text{ if (2.16) holds,} \\ 0 \text{ or } 1 \text{ if (2.18) holds.}\end{cases} \tag{2.20}$$

We remark that if $\gamma = m$ then the expression on the right in (2.17) reduces to m.

3. Applications. As customary $f(A)$ will designate the linear transformation defined for any $x \in U$ by

$$f(A)x = \sum_{i=1}^{n} \mu_i(x, u_i)u_i, \ (\mu_i = f(\lambda_i)) \ . \tag{3.1}$$

On the unit sphere $\|x\| = 1$ define the real valued function

$$\varphi(x) = (Ax, x)(f(A)x, x) \ . \tag{3.2}$$

We compute directly from (3.1) that

$$\varphi(x) = \sum_{i=1}^{n} \lambda_i |(x, u_i)|^2 \sum_{i=1}^{n} \mu_i |(x, u_i)|^2 \tag{3.3}$$

and by setting $\sigma_i = |(x, u_i)|^2$, $i = 1, \cdots, n$, we have $\sigma = (\sigma_1, \cdots, \sigma_n) \in S^{n-1}$ and

$$\varphi(x) = F(\sigma) \ . \tag{3.4}$$

Thus by direct application of Theorem 2 we have

THEOREM 3. *Then maximum value of $\varphi(x)$ for x on the unit sphere $\|x\| = 1$ is the number M in the statement of Theorem 2. Moreover $\varphi(x_0) = M$ can always be achieved with a unit vector x_0 in the subspace spanned by those eigenvectors of A corresponding to λ_1 and λ_n. If f is strictly convex and $\varphi(x_0) = M$ then x_0 must lie in the sum of the null spaces of $A - \lambda_1 I$ and $A - \lambda_n I$. In particular, if λ_1 and λ_n are simple eigenvalues of A, f is strictly convex and $\varphi(x_0) = M$ then x_0 must lie in the two dimensional subspace spanned by u_1 and u_n.*

In Theorem 3 take $f(t) = t^{-p}$, $p > 0$. Let $\theta = \lambda_1/\lambda_n$ denote the condition number of A. Assume that $\theta < 1$ (otherwise $\lambda_1 = \lambda_n$ and A is a multiple of the identity). There are two cases to consider: $p > 1$; $p \leqq 1$. In case $p > 1$, $m = \lambda_1^{1-p}$ and the condition (2.16), $\gamma \geqq m$, becomes

$$g(\theta) = \theta^{p+1} - 2\theta + 1 \geqq 0 \ . \tag{3.5}$$

We note that g is convex, $g(1) = 0$, $g'(\theta) = 0$ for $\theta = (2/(p+1))^{1/p}$, and hence g has precisely one root in $(0, 1)$, call it θ_p. It is easy to see that $\theta_p > 1/2$ for all $p > 1$. In general, if $0 < \theta \leqq \theta_p$ then Theorem 2 yields

$$M = \lambda_1^{1-p}(\theta^{p+1} - 1)^2/4\theta(\theta - 1)(\theta^p - 1) ; \tag{3.6}$$

and if $1 \geqq \theta > \theta_p$ then

$$M = \lambda_1^{1-p} . \tag{3.7}$$

In case $p \leqq 1$, $m = \lambda_n^{1-p}$ and the condition (2.16), $\gamma \geqq m$, becomes $g(\eta) \geqq 0$ where $\eta = \theta^{-1}$. But $g(\eta) \geqq 0$ for $\eta \geqq 1$ and $\eta = \theta^{-1} \geqq 1$ so the upper bound for $F(\sigma)$ is M given in (3.6).

Assume now that λ_1 and λ_n are both simple eigenvalues of A and we examine the structure of the vector x_0 that maximizes $\varphi(x) = (Ax, x)(A^{-p}x, x)$ on the unit sphere $\|x\| = 1$. By Theorem 3 the maximum value of $\varphi(x) = F(\sigma)$ can only occur for $\sigma_2 = \cdots = \sigma_{n-1} = 0$. Moreover by (2.20) $F(\sigma) = M$ for the unique values

$$\left.\begin{aligned} &\sigma_n = \sigma_n(\theta) = g(\theta)/2(1 - \theta)(1 - \theta^p) &(3.8)\\ &\sigma_1 = \sigma_1(\theta) = \sigma_n(\theta^{-1}) &(3.9)\end{aligned}\right\} \text{if } g(\theta) \geqq 0 \text{ or } p = 1 ;$$

and

$$\sigma_1 = 1, \sigma_n = 0 \text{ if } g(\theta) < 0 \text{ and } p > 1 . \tag{3.10}$$

Summing up these results we have

THEOREM 4. *Let θ designate the condition number of A, $\theta = \lambda_1/\lambda_n$. If either $0 < p \leqq 1$, or $p > 1$ and $0 \leqq \theta \leqq \theta_p$, then for $\|x\| = 1$*

$$(Ax, x)(A^{-p}x, x) \leqq \lambda_1^{1-p}(\theta^{p+1} - 1)^2/4\theta(\theta - 1)(\theta^p - 1) . \tag{3.11}$$

If $p > 1$ and $\theta_p < \theta$ then for $\|x\| = 1$

$$(Ax, x)(A^{-p}x, x) \leqq \lambda_1^{1-p} . \tag{3.12}$$

If λ_1 and λ_n are simple eigenvalues of A then the upper bound in (3.11) is only achieved for unit vectors of the form

$$x_0 = \sqrt{\sigma_n(\theta^{-1})}\, e^{i\omega_1}u_1 + \sqrt{\sigma_n(\theta)}\, e^{i\omega_2}u_n , \tag{3.13}$$

ω_1, ω_2 real. The upper bound in (3.12) is achieved only for unit vectors of the form

$$x_0 = e^{i\omega}u_1 .$$

In case $p = 1$ we have the Kantorovich inequality. In this case (3.11) becomes (for $\|x\| = 1$)

(3.14) $$(Ax, x)(A^{-1}x, x) \leqq (\sqrt{\theta} + \sqrt{\theta^{-1}})^2/4 .$$

If λ_1 *and* λ_n *are simple eigenvalues then the inequality* (3.14) *is strict unless*

(3.15) $$x = x_0 = (e^{i\omega_1}u_1 + e^{i\omega_2}u_n)/\sqrt{2}, \omega_1, \omega_2 \text{ real} .$$

4. Determinants and permanents. In this section we specialize by taking U to be the unitary space of n-tuples with inner product $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$ and A to be an n-square hermitian positive semi-definite matrix. If $1 \leqq k \leqq n$ then $C_k(A)$ will denote the kth compound of A and if $x_1, \cdots, x_k$ are vectors in U then $x_1 \wedge \cdots \wedge x_k$ is the Grassmann product of these vectors, sometimes called a pure vector of grade k [6, p. 16]. The eigenvalues of $C_k(A)$ are all $\binom{n}{k}$ numbers $\lambda_{i_1} \cdots \lambda_{i_k}$, with corresponding eigenvectors $u_{i_1} \wedge \cdots \wedge u_{i_k}$, $1 \leqq i_1 < \cdots < i_k \leqq n$. The smallest and largest of these eigenvalues are $\prod_{j=1}^{k} \lambda_j$ and $\prod_{j=1}^{k} \lambda_{n-j+1}$ respectively. It has been noted in [2] and [5] that the Kantorovich inequality applied to $C_k(A)$ yields

(4.1) $$\det A[i_1, \cdots, i_k] \det A^{-1}[i_1, \cdots, i_k] \leqq (\sqrt{\Delta} + \sqrt{\Delta^{-1}})^2/4$$

where $\Delta = \prod_{j=1}^{k} \lambda_j \lambda_{n-j+1}^{-1}$ *and* $A[i_1, \cdots, i_k]$ *is the principal submatrix of* A *lying in rows and columns numbered* $i_1, \cdots, i_k$.

We prove

THEOREM 5. *If* $1 \leqq k < n-1$ *and* $\lambda_1, \cdots, \lambda_k$ *together with* $\lambda_n, \cdots, \lambda_{n-k+1}$ *are simple eigenvalues of* A *then the inequality* (4.1) *is always strict.*

Proof. The number $\det A[i_1, \cdots, i_k] \det A^{-1}[i_1, \cdots, i_k]$ is a value of the product of quadratic forms associated with $C_k(A)$ and $C_k(A^{-1})$,

(4.2) $$\begin{gathered}(C_k(A)x_1 \wedge \cdots \wedge x_k, x_1 \wedge \cdots \wedge x_k) \\ (C_k(A^{-1})x_1 \wedge \cdots \wedge x_k, x_1 \wedge \cdots \wedge x_k) ,\end{gathered}$$

and according to (3.15), (4.1) will be strict unless

(4.3) $$x_1 \wedge \cdots \wedge x_k = \frac{1}{\sqrt{2}}(e^{i\omega_1}u_1 \wedge \cdots \wedge u_k + e^{i\omega_2}u_n \wedge \cdots \wedge u_{n-k+1}) .$$

Let $p = \min\{k, n-k\}$, $q = \max\{k+1, n-k+1\}$ and compute successively the Grassmann products of both sides of (4.3) with $u_1, \cdots, u_p$ and $u_n, \cdots, u_q$. We obtain

(4.4) $$x_1 \wedge \cdots \wedge x_k \wedge u_j = \frac{e^{i\omega_2}}{\sqrt{2}}(u_n \wedge \cdots \wedge u_{n-k+1} \wedge u_j), j = 1, \cdots, p ,$$

and

$$(4.5) \quad x_1 \wedge \cdots \wedge x_k \wedge u_j = \frac{e^{i\omega_2}}{\sqrt{2}}(u_1 \wedge \cdots \wedge u_k \wedge u_j), j = q, \cdots, n .$$

Since $u_1, \cdots, u_n$ are linearly independent it follows that the right sides of (4.4) and (4.5) are not 0. Thus

$$(4.6) \quad < x_1, \cdots, x_k, u_j > = < u_1, \cdots, u_k, u_j >, j = 1, \cdots, p ,$$

and

$$(4.7) \quad < x_1, \cdots, x_k, u_j > = < u_1, \cdots, u_k, u_j >, j = q, \cdots, n ,$$

where $< x_1, \cdots, x_k, u_j >$ denotes the subspace spanned by the vectors inside the brackets. Intersect the p subspaces on the left in (4.6) and observe that $< x_1, \cdots, x_k >$ is a subspace of the intersection. Similarly $< x_1, \cdots, x_k >$ is a subspace of the intersection of the $n - q + 1$ spaces on the left in (4.7). On the other hand

$$\bigcap_{j=1}^{p} < u_n, \cdots, u_{n-k+1}, u_j > = < u_n, \cdots, u_{n-k+1} >$$

and

$$\bigcap_{j=q}^{n} < u_1, \cdots, u_k, u_j > = < u_1, \cdots, u_k > .$$

Hence

$$(4.8) \quad \begin{aligned} &\dim \{< u_1, \cdots, u_k > \cap < u_n, \cdots, u_{n-k+1} >\} \\ &= \dim \left\{\bigcap_{j=1}^{p} < x_1, \cdots, x_k, u_j > \cap \bigcap_{j=q}^{n} < x_1, \cdots, x_k, u_j >\right\} > k . \end{aligned}$$

The subspace $< u_1, \cdots, u_k > \cap < u_n, \cdots, u_{n-k+1} >$ is nonempty if and only if $n - k + 1 \leqq k$ in which case its dimension is $2k - n$. But the inequality $2k - n \geqq k$ implies that $k \geqq n$, a contradiction. Thus (4.3) cannot hold and (4.1) is strict.

We remark that in case $k = n - 1$ then $p = 1, q = n, x_1 \wedge \cdots \wedge x_k \wedge u_1 = u_n \wedge \cdots \wedge u_2 \wedge u_1, x_1 \wedge \cdots \wedge x_k \wedge u_n = u_1 \wedge \cdots \wedge u_{n-1} \wedge u_n$ and the above argument fails. In fact, it is not difficult to construct examples for which (4.1) is equality.

Once again, if $1 \leqq k \leqq n$ then $P_k(A)$ will denote the kth induced power matrix of A and if $x_1, \cdots, x_k$ are vectors in U then $x_1 \cdots x_k$ will denote the symmetric or dot product of these vectors [3, p. 49]. The eigenvalues of $P_k(A)$ are all $\binom{n+k-1}{k}$ homogeneous products $\lambda_{i_1} \cdots \lambda_{i_k}$ with corresponding eigenvectors $u_{i_1} \cdots u_{i_k}, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n$. Suppose $x_1, \cdots, x_n$ are orthonormal vectors and the multiplicities of the distinct integers in the sequence $i_1 \leqq \cdots \leqq i_k$ are respectively $m_1, \cdots, m_p$. Let $\mu = \mu(i_1, \cdots, i_k) = m_1! \cdots m_p!$. Then the square of the length of the symmetric product $x_{i_1} \cdots x_{i_k}$ is $\mu(i_1, \cdots, i_k)$ [3, p. 50]. Applying the Kantorovich inequality to $P_k(A)$ yields

$$(4.9) \qquad \begin{aligned} &(P_k(A)x_i \cdots x_{i_k}, x_{i_1} \cdots x_{i_k})(P_k(A^{-1})x_{i_1} \cdots x_{i_k}, x_{i_1} \cdots x_{i_k}) \\ &\leqq \mu^2(\sqrt{\delta} + \sqrt{\delta^{-1}})^2/4, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n , \end{aligned}$$

where $\delta = (\lambda_1\lambda_n^{-1})^k$, *and* $x_1, \cdots, x_n$ *is an orthonormal basis of* U. In particular if we let $x_i = e_i$, the unit vector with 1 in the ith position, 0 elsewhere, then (4.9) becomes

$$(4.10) \quad \operatorname{per} A[i_1, \cdots, i_k] \operatorname{per} A^{-1}[i_1, \cdots, i_k] \leqq \mu^2(\sqrt{\delta} + \sqrt{\delta^{-1}})^2/4 ,$$

where $A[i_1, \cdots, i_k]$ *is the* k*-square matrix whose* (s, t) *entry is* $a_{i_s i_t}$, $s, t = 1, \cdots, k$.

THEOREM 6. *If* λ_1 *and* λ_n *are simple eigenvalues of* A *and there are at least three distinct integers in the sequence* $i_1 \leqq \cdots \leqq i_k$ *then the inequality* (4.10) *is strict.*

Proof. According to (3.15), (4.10) will be strict unless

$$(4.11) \qquad e_{i_1} \cdots e_{i_k} = \frac{e^{i\omega_1}}{\sqrt{2k!}} u_1 \cdots u_1 + \frac{e^{i\omega_2}}{\sqrt{2k!}} u_n \cdots u_n .$$

Let y be an arbitrary vector and compute the inner product of both sides of (4.11) with $y \cdots y$ to obtain

$$(4.12) \qquad \prod_{j=1}^{k} (e_{i_j}, y) = \frac{e^{i\omega_1}}{\sqrt{2k!}} (u_1, y)^k + \frac{e^{i\omega_2}}{\sqrt{2k!}} (u_n, y)^k .$$

Set

$$v_1 = \left(\frac{e^{i\omega_1}}{\sqrt{2k!}}\right)^{1/k} u_1,\ v_2 = \left(\frac{e^{i\omega_2}}{\sqrt{2k!}}\right)^{1/k} u_n ,$$

and write $e_{i_j} = \alpha_j v_1 + w_j$, $w_j \in \langle v_1 \rangle^{\perp}$, $j = 1, \cdots, k$. Then for y any vector in $\langle v_1 \rangle^{\perp}$, (4.12) becomes

$$(4.13) \qquad \prod_{j=1}^{k} (e_{i_j}, y) = \prod_{j=1}^{k} (w_j, y) = (v_2, y)^k ,$$

in which w_j, v_2, y are in $\langle v_1 \rangle^{\perp}, j = 1, \cdots, k$. But then from [3, Theorem 3] we conclude that $w_j = \beta_j v_2$, $j = 1, \cdots, k$, for appropriate scalars $\beta_1, \cdots, \beta_k$ and hence $e_{i_j} \in \langle v_1, v_2 \rangle$, $j = 1, \cdots, k$. Since there are at least three linearly independent e_{i_j}, (4.11) must fail and hence (4.10) is strict.

REFERENCES

1. L. V. Kantorovich and V. I. Krylov, *Approximate methods of higher analysis*, New York, Interscience (1958).
2. Marvin Marcus, and N. A. Khan *Some generalizations of Kantorovich's inequality*, Portugaliae Math. **20**, 1, (1961), 33-38.
3. Marvin Marcus and Morris Newman. *Inequalities for the permanent function*, Ann of Math., **75**, 1, (1962), 47-62.
4. Morris Newman. *Kantorovich's inequality*, J. Research Nat. Bur. Standards, **64** (B), (1960), 33-34.
5. Andreas H. Schopf,. *On the Kantorovich inequality*, Numerische Math., **2** (1960), 344-346.
6. J. H. M. Wedderburn,. *Lectures on matrices*, Amer, Math. Soc. Coll. Publ., **17** (1934).

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