QUANTIFIERS AND ORTHOMODULAR LATTICES

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1. Introduction. The "logic" of (non-relativistic) quantum mechanics is currently thought of as being the lattice of closed subspaces of a separable infinite dimensional Hilbert space [7, p. 49]. It has been speculated by P. Jordan [5] that this "logic" ought not to be a lattice at all, but rather what he calls a skew lattice. Given a lattice $L(\cap, \cup)$, Jordan observes that if one has functions $f, F: L \to L$ satisfying the conditions

$$(af \cup b)f = af \cup bf$$
$$af \leqq a$$
$$(a \cap bF)F = aF \cap bF$$
$$a \leqq aF\,;$$

then $L(\wedge, \vee)$ is a skew lattice, where the operations $\wedge$, $\vee$ are defined by:

$$a \wedge b = a \cap bF$$
$$a \vee b = af \cup b\,.$$

Skew lattices themselves will not concern us here; rather we shall be interested in mappings on lattices having the above properties. Such mappings turn out to be generalizations of universal and existential quantifiers. With this thought in mind it seems of interest to begin an investigation of quantifiers on an orthomodular lattice, and in particular to consider the lattice $L(H)$ of closed subspaces of a Hilbert space H, determining all mappings f, F defined on $L(H)$ and satisfying Jordan's prescription.

The remaining part of this section will be devoted to a brief outline of known definitions and theorems that will prove useful in the sequel. These results can essentially be found in [1], [2], [9] and [10] but are included here for convenience. An *orthomodular lattice* is a lattice L with 0 and 1 equipped with an orthocomplementation $': L \to L$ and which satisfies the *orthomodular identity* $e \leqq f \Rightarrow f = e \vee (f \wedge e')$. *Henceforth L will always represent an orthomodular lattice.* If $e, f \in L$ with $e \leqq f$ it is easily shown that the interval $L(e, f) = \{g \in L: e \leqq g \leqq f\}$ is itself an orthomodular lattice with orthocomplementation

$$g \to g^{\#} = e \vee (f \wedge g') = (e \vee g') \wedge f\,.$$

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The semigroup (under function composition) of all monotone maps $\varphi: L \to L$ is denoted by $M(L)$. Two monotone maps φ and ψ are said to be *mutually adjoint* in case $(e\varphi)'\psi \leqq e'$ and $(e\psi)'\varphi \leqq e'$ for all $e \in L$. If $\varphi \in M(L)$ has an adjoint then this adjoint is unique [1, p. 651], and is denoted by φ^*. The subset of $M(L)$ possessing adjoints in $M(L)$ is denoted by $S(L)$. It is shown in [1] that $S(L)$ is a Baer $*$-semigroup (under function composition) and that every $\varphi \in S(L)$ is a hemimorphism of L; i.e., that $0\varphi = 0$ and $(e \vee f)\varphi = e\varphi \vee f\varphi$ for every $e, f \in L$. In fact every $\varphi \in S(L)$ preserves arbitrary suprema whenever they exist in L.

There corresponds to each $e \in L$ a mapping $\varphi_e: L \to L$ given by $f \to f\varphi_e = (f \vee e') \wedge e$, where $\varphi_e = \varphi_e^2 = \varphi_e^* \in S(L)$, $f\varphi_e = f$ if and only if $f \leqq e$, and $f\varphi_e = 0$ if and only if $f \leqq e'$ (see [1, p. 652] and [10, pp. 300–301]). The element f is said to *commute* with e (written fCe) in case $f\varphi_e = f \wedge e$. For future reference we state the next results in the form of a lemma.

LEMMA 1. *Let e, f, $g \in L$. Then*:

(i) $f \leqq e \Rightarrow fCe$.

(ii) $fCe \Rightarrow eCf$.

(iii) $fCe \Rightarrow fCe'$.

(iv) *If any two of the three conditions eCf, fCg, eCg hold, then $(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$ and $(e \wedge f) \vee g = (e \vee g) \wedge (f \vee g)$.*

(v) *If fCe_α for each $\alpha \in A$, and if $\bigvee_{\alpha \in A} e_\alpha$ exists then $fC\bigvee_{\alpha \in A} e_\alpha$.*

Proof. See [2] and [9].

In [11, p. 240] J. von Neumann defines the center of a continuous geometry. This notion can be carried over to an abitrary orthomodular lattice by decreeing that the *center* of L, in symbols $C(L)$, is the set of all those elements of L which commute with every other element. It then follows [2] that 0, $1 \in C(L)$, and $C(L)$ is a Boolean sublattice closed under the formation of orthocomplements and of arbitrary suprema and infima whenever they exist in L.

The lattice L is said to be *irreducible* in case 0 and 1 are the only elements with unique complements. It can easily be shown that $e \in C(L)$ if and only if e has a unique complement, from which it follows that L is irreducible if and only if $C(L) = \{0, 1\}$.

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2. Quantifiers. Generalizing a notion of Halmos [3, p. 220] a mapping $\varphi: L \to L$ will be called a *quantifier* on L in case φ satisfies:

(Q1) $0\varphi = 0$.

(Q2) $e \leqq e\varphi$ for all $e \in L$.

(Q3) $(e \wedge f\varphi)\varphi = e\varphi \wedge f\varphi$ for all $e, f \in L$.

It should be noted that there are always two special quantifiers on L:

(i) The *discrete* quantifier = identity map;

(ii) The *indiscrete* (or as in Halmos [3] *simple*) quantifier $e\varphi = 1$ for $e \neq 0$, $0\varphi = 0$. We now proceed to investigate some of the properties of quantifiers and will find all quantifiers on an atomic orthomodular lattice whose center is complete as a sublattice.

THEOREM 2. *Let φ be a quantifier on L. Then*:

(i) $1\varphi = 1$.

(ii) $\varphi = \varphi^2$.

(iii) $e \leqq f \Rightarrow e\varphi \leqq f\varphi$.

(iv) $(e\varphi)'\varphi = (e\varphi)'$.

(v) *φ is a projection in $S(L)$.*

(vi) *$(L)\varphi$ = the set of fixed points of φ is a sublattice of L closed with respect to the formation of orthocomplements and of arbitrary suprema and infima whenever they exist in L.*

Proof. (i) $1 \leqq 1\varphi$, so $1 = 1\varphi$.

(ii) $e\varphi^2 = (e\varphi)\varphi = (1 \wedge e\varphi)\varphi = 1\varphi \wedge e\varphi = e\varphi$.

(iii) If $e \leqq f$, then $e \leqq f\varphi$, $e = e \wedge f\varphi$, $e\varphi = (e \wedge f\varphi)\varphi = e\varphi \wedge f\varphi$, so that $e\varphi \leqq f\varphi$.

(iv) Note that $0 = 0\varphi = [(e\varphi)' \wedge (e\varphi)]\varphi = (e\varphi)'\varphi \wedge e\varphi$. By the orthomodular identity, $(e\varphi)' \leqq (e\varphi)'\varphi$ implies that $(e\varphi)'\varphi = (e\varphi)' \vee [(e\varphi)'\varphi \wedge e\varphi] = (e\varphi)'$.

(v) This follows immediately from (ii), (iii) and the observation that $(e\varphi)'\varphi = (e\varphi)' \leqq e'$.

(vi) If $e\varphi = e$ then $e'\varphi = (e\varphi)'\varphi = (e\varphi)' = e'$. The result is now clear in view of the fact that $\varphi = \varphi^2 \in S(L)$ and the known properties of $S(L)$.

LEMMA 3. *Let φ be a quantifier on L, a an atom in L, $e = a\varphi$. Then*:

(i) *$\varphi|_{L(0\ e)}$ is indiscrete.*

(ii) *e is an atom in $(L)\varphi$.*

(iii) *For any $f \in L$, either $e \wedge f = 0$ or $e \leqq f\varphi$.*

Proof. (i) Let $b \in L(0, e)$, so $b\varphi \leqq e\varphi = e$. Since a is an atom, $a \wedge b\varphi = a$ or 0. But if $a \wedge b\varphi = a$, $e = a\varphi = (a \wedge b\varphi)\varphi = a\varphi \wedge b\varphi = e \wedge b\varphi = b\varphi$; while $a \wedge b\varphi = 0$ implies that $0 = 0\varphi = (a \wedge b\varphi)\varphi = e \wedge b\varphi = b\varphi$, whence $b = 0$.

(ii) If $b \in (L)\varphi$ and $b \leqq e$, then $b = b\varphi = 0$ or e.

(iii) Suppose $e \wedge f \neq 0$. Then $(e \wedge f)\varphi = e \wedge f\varphi \neq 0$, and by (ii), $e \leqq f\varphi$.

We borrow another idea of Halmos [3] and call a mapping $\varphi: L \to L$ a *symmetric closure operator* in case φ is a hemimorphism, $\varphi = \varphi^2$, and for each $e \in L$, $e \leqq e\varphi$ and $(e\varphi)'\varphi \leqq e'$. It is shown by Halmos [3, Th. 3] that in a Boolean lattice the notions of quantifier and symmetric closure operator coincide. It is interesting to observe that in an orthomodular lattice this result does not quite carry over. By Theorem 2, every quantifier is a symmetric closure operator. One can show without difficulty that every center valued symmetric closure operator is a quantifier; however, if $a \notin C(L)$, the mapping $\alpha_a: L \to L$ defined by $e\alpha_a = (e \vee a) \wedge (e \vee a')$ is an example of a symmetric closure operator that is not a quantifier. It follows from this that L is a Boolean lattice if and only if every symmetric closure operator is a quantifier.

One can define the *central cover* of e, in symbols $e\nu$, to be the infimum of the set of central elements z such that $e \leqq z$, provided of course that such an infimum exists. Note that if $e\nu$ exists for all $e \in L$, then ν is a non-trivial example of a quantifier. Clearly $0\nu = 0$, $e \leqq e\nu$ for every $e \in L$, and $e = e\nu$ if and only if $e \in C(L)$. We also have that if $e \leqq f$, then $e \leqq f\nu \in C(L)$, whence $e\nu \leqq f\nu$. It then follows that $e\nu^2 = (e\nu)\nu = e\nu$, and $(e\nu)'\nu = (e\nu)' \leqq e'$ so that $\nu = \nu^2 = \nu^* \in S(L)$. But then ν is a center valued symmetric closure operator, hence a quantifier.

LEMMA 4. *Suppose that $e\nu$ exists for each $e \in L$. Then φ is a center valued quantifier on L if and only if $\varphi = \nu \circ \alpha$, where α is a quantifier on $C(L)$. The decomposition is unique in that $\alpha = \varphi|_{C(L)}$.*

Proof. Evidently $\nu \circ \alpha$ is always a center valued quantifier. If conversely φ is a center valued quantifier, set $\alpha = \varphi|_{C(L)}$. Then $e \leqq e\nu \leqq e\varphi$, whence $e\varphi \leqq e\nu\varphi = e\nu\alpha \leqq e\varphi^2 = e\varphi$ for all $e \in L$, from which it follows that $\varphi = \nu \circ \alpha$. The uniqueness of the decomposition is obvious.

LEMMA 5. *Let φ be a quantifier on L, a any atom of L. Then $a\varphi \wedge f \neq 0 \Rightarrow a\varphi C f$.*

Proof. If $a = a\varphi$ the result is obvious, so we can suppose $a < a\varphi$. Suppose that $a\varphi \wedge f \neq 0$ and $a\varphi C f$ fails. Routine computation shows that $k = a\varphi\varphi_{f'} \vee (a\varphi \wedge f)$ is a complement of $(a\varphi)'$, $k \neq a\varphi$, and $k \wedge a\varphi \neq 0$. By Lemma 3, $a\varphi \leqq k\varphi$, and since $k \wedge (a\varphi)' = 0$, $(k \wedge (a\varphi)')\varphi = k\varphi \wedge (a\varphi)' = 0$. Applying the orthomodular identity to $a\varphi \leqq k\varphi$, we have $k\varphi = a\varphi \vee (k\varphi \wedge (a\varphi)') = a\varphi$, whence $k \leqq a\varphi$. A second application of the orthomodular identity now produces $a\varphi = k \vee (a\varphi \wedge k') = k$, a contradiction. Hence $a\varphi \wedge f \neq 0 \Rightarrow a\varphi C f$.

LEMMA 6. *If φ is a quantifier on an atomic lattice L, then $a < a\varphi$ for an atom a implies that $a\varphi$ is central.*

Proof. If b is any atom such that $b < b\varphi$, then $b' < b'\varphi = 1$, so that if $a\varphi \wedge b' = 0$, we would have $(a\varphi \wedge b')\varphi = a\varphi \wedge b'\varphi = a\varphi = 0$, a contradiction. Hence $a\varphi \wedge b' \neq 0$, and by Lemma 5, $a\varphi C b$.

Consider next an atom $b = b\varphi$. Since $a < a\varphi$, we can write $a\varphi = a \vee (a\varphi \wedge a')$. Now $a < a\varphi \Rightarrow a'\varphi = 1 \Rightarrow b \wedge a' \neq 0 \Rightarrow bCa$. Similarly, $(a\varphi \wedge a')'\varphi = [(a\varphi)' \vee a]\varphi = (a\varphi)' \vee a\varphi = 1$, so that $bC(a\varphi \wedge a')$. By Lemma 1, $a\varphi C b$. This shows that $a\varphi$ commutes with every atom. In an atomic orthomodular lattice each element is the supremum of the atoms it dominates, so by Lemma 1 (v), $a\varphi \in C(L)$.

COROLLARY. *Let L be atomic and irreducible. Then L admits only the discrete and the indiscrete quantifiers.*

If $L = L_1 \times L_2$, and φ_1, φ_2 are quantifiers on L_1, L_2, respectively, it is easily shown that $\varphi = \varphi_1 \times \varphi_2$ is a quantifier on L. It can also be shown that an element a is central in L_1 if and only if $(a, 0)$ is central in L. Let $h \in C(L)$, and let us adopt the notation $\nu_h = \nu|_{L(0,h)}$, $\alpha_h =$ a quantifier on $C(L(0, h)) = C(L) \cap L(0, h)$, $I_h =$ the identity map restricted to $L(0, h)$. We then have:

THEOREM 7. *Let L be atomic with $C(L)$ complete as a sublattice. Then φ is a quantifier on L if and only if $\varphi = (\nu_h \circ \alpha_h) \times I_{h'}$ for some $h \in C(L)$.*

Proof. Any mapping of the form $(\nu_h \circ \alpha_h) \times I_{h'}$ is evidently a quantifier on L. If conversely φ is a quantifier on L, set $h = \sup\{a_\nu\varphi : a_\nu$ is an atom, $a_\nu\varphi \in C(L)\}$, and write $L = L(0, h) \times L(0, h')$. If $b \leqq h$ is an atom, then since each $a_\nu\varphi$ is central, $b = b \wedge h = h\varphi_b = (\bigvee_\nu a_\nu\varphi)\varphi_b = \bigvee_\nu (a_\nu\varphi\varphi_b) = \bigvee_\nu (a_\nu\varphi \wedge b)$. Thua for some index μ, $b \leqq a_\mu\varphi$; by Lemma 3, $b\varphi = a_\mu\varphi \in C(L)$. This shows that $\varphi|_{L(0\ h)}$ is center valued, and by Lemma 4 we may write $\varphi|_{L(0\ h)} = \nu_h \circ \alpha_h$. By Lemma 6, $\varphi|_{L(0\ h')} = I_{h'}$. Hence $\varphi = (\nu_h \circ \alpha_h) \times I_{h'}$.

It should be observed that since Halmos [3] has essentially determined all quantifiers on a Boolean lattice, the above theorem enables us to find all quantifiers on an atomic orthomodular lattice with complete center.

3. Atomic bisection and irreducibility. Generalizing a property of the lattice of closed subspaces of a Hilbert space, let us say that a pair (b, c) of distinct atoms can be *bisected* in case there exists an atom $a \leqq b \vee c$ such that $a \neq b$ and $a \neq c$. The lattice L has the *atomic bisection property* if L is atomic and every pair of distinct atoms can be bisected. The reader will no doubt notice that the results of this section will go through under the weakened hypothesis that pairs of orthogonal atoms can be bisected. This, however, is only an apparent weakening since for non-orthogonal atoms (b, c), we can write $b \vee c = b \vee [(b \vee c) \wedge b']$; any atom $a \leqq (b \vee c) \wedge b'$ will then bisect (b, c).

LEMMA 8. *If L has the atomic bisection property, then so does any interval $L(e, f)$.*

Proof. The lemma follows immediately from the fact that the mapping $a \rightarrow a \wedge e'$ is an orthocomplement preserving lattice isomorphism of $L(e, f)$ onto $L(0, f \wedge e')$.

LEMMA 9. *If L has the atomic bisection property, then L is irreducible.*

Proof. Suppose that L is atomic, and $h \in C(L)$ with $h \neq 0$, $h \neq 1$. Choose atoms b, c with $b \leqq h$, $c \leqq h'$. Then for any atom $a \leqq b \vee c$, $a = a \wedge 1 = a \wedge (h \vee h') = (a \wedge h) \vee (a \wedge h')$, so that $a \leqq h$ or $a \leqq h'$. But $a \leqq h$ implies that $a \leqq (b \vee c) \wedge h = (b \wedge h) \vee (c \wedge h) = b$, and since b is an atom, $a = b$. Similarly $a \leqq h'$ implies $a = c$, so atomic bisection fails.

LEMMA 10. *If L is atomic complete and modular then irreducibility of L is both a necessary and sufficient condition for L to have the atomic bisection property.*

Proof. By a theorem of I. Kaplansky [6] every complete modular orthocomplemented lattice is a continuous geometry. The result now follows immediately from [8, p. 80, Th. 2.4].

Combining the above results with the observation that if the pair (b, c) of distinct atoms cannot be bisected, then $L(0, b \vee c)$ is the Boolean lattice $\{0, b, c, b \vee c\}$, we have the following theorem:

THEOREM 11. *Let L be atomic. Then the following conditions are mutually equivalent:*

(i) *L has the atomic bisection property.*

(ii) *Every interval $L(e, f)$ is irreducible.*

(iii) *$L(0, b \vee c)$ is irreducible for all atoms b, c in L. If in addition L is complete and modular, we can add*:

(iv) *L is irreducible.*

4. Determination of the weak quantifiers on $L(H)$. By a *weak quantifier* on L we will mean a mapping $\varphi\colon L \rightarrow L$ satisfying axioms (Q2) and (Q3) of § 2. A glance at the proof of Theorem 2 shows that parts (i)–(iii) will carry over for weak quantifiers. It should be noted that the reason for our interest in weak quantifiers is that these (together with their duals) are precisely the functions needed for constructing a skew lattice by the method of Jordan.

THEOREM 12. *Let φ be a weak quantifier on L. Then:*

(i) *$\varphi|_{L(0\varphi, 1)}$ is a quantifier.*

(ii) *φ preserves arbitrary suprema whenever they exist in L.*

Proof. (i) By hypothesis $\varphi|_{L(0\varphi, 1)}$ satisfies axioms (Q2) and (Q3), so we need only note that $(0\varphi)\varphi = 0\varphi$.

(ii) Let $e, f \in L$. Then e, $f \leqq e \vee f$, so that $e\varphi$, $f\varphi \leqq (e \vee f)\varphi$; hence $e\varphi \vee f\varphi \leqq (e \vee f)\varphi$. The reverse inequality follows from the fact that $e \vee f \leqq e\varphi \vee f\varphi$, so $(e \vee f)\varphi \leqq (e\varphi \vee f\varphi)\varphi = e\varphi \vee f\varphi$, since $\varphi|_{L(0\varphi, 1)}$ is a quantifier.

Suppose now that $e = \bigvee_\alpha e_\alpha$ exists. Then $e \vee 0\varphi = (\bigvee_\alpha e_\alpha) \vee 0\varphi = \bigvee_\alpha (e_\alpha \vee 0\varphi)$, so

$$\begin{aligned} e\varphi &= e\varphi \vee 0\varphi = (e \vee 0\varphi)\varphi \\ &= \Big[\bigvee_\alpha (e_\alpha \vee 0\varphi)\Big]\varphi = \bigvee_\alpha (e_\alpha \vee 0\varphi)\varphi \\ &= \bigvee_\alpha (e_\alpha\varphi \vee 0\varphi) = \Big(\bigvee_\alpha e_\alpha\varphi\Big) \vee 0\varphi = \bigvee_\alpha e_\alpha\varphi \ . \end{aligned}$$

It now seems natural to call φ a *discrete* or an *indiscrete* weak quantifier according to whether $\varphi|_{L(0\varphi, 1)}$ is the discrete or the indiscrete quantifier on $L(0\varphi, 1)$. Our immediate goal is to determine all weak quantifiers on a lattice having the atomic bisection property. From Theorem 11 and the Corollary to Lemma 6, we see that every weak quantifier on such a lattice is discrete or indiscrete. It suffices then to find all discrete and indiscrete weak quantifiers on an arbitrary orthomodular lattice L.

Our labor is greatly diminished by the observation that for any weak quantifier φ on L, $e\varphi = (e \vee (0\varphi))\hat{\varphi}$ for each $e \in L$, where $\hat{\varphi} = \varphi|_{L(0\varphi, 1)}$. It follows that if φ is indiscrete, $e\varphi = 0\varphi$ for all $e \leqq 0\varphi$, and $e\varphi = 1$ otherwise. Conversely for any $a \in L$ the prescription $e\varphi = a$ for $e \leqq a$, $e\varphi = 1$ for $e \nleqq a$, defines an indiscrete weak quantifier, so this completely determines all indiscrete weak quantifiers on L.

In case φ is discrete it is immediate that $e\varphi = [e \vee (0\varphi)]\hat{\varphi} = e \vee (0\varphi)$. Our next task is to find necessary and sufficient conditions on a which will assure us that the mapping $e \to e \vee a$ is a discrete weak quantifier. We will use the notation $M(b, c)$ to indicate that the pair (b, c) is modular; i.e., that $a \leqq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$.

THEOREM 13. *Given $a \in L$ the mapping ψ_a defined by $b\psi_a = b \vee a$ is a discrete weak quantifier if and only if $M(e, a')$ for all $e \in L$. It then follows that L is modular if and only if ψ_a is a discrete weak quantifier for every $a \in L$.*

Proof. We clearly need only concern ourselves with axiom (Q3). Suppose first that ψ_a is a weak quantifier. Then for any $e, f \in L$, $(e \wedge f\psi_a)\psi_a = e\psi_a \wedge f\psi_a$, so $[e \wedge (f \vee a)] \vee a = (e \vee a) \wedge (f \vee a)$. Now if $b \geqq a$ then $b = b \vee a$, whence $(e \wedge b) \vee a = (e \vee a) \wedge b$. With an obvious change in notation, $b \leqq a' \Rightarrow (b \vee e) \wedge a' = b \vee (e \wedge a')$. Hence $M(e, a')$ for all $e \in L$.

Suppose conversely that $M(e, a')$ for all $e \in L$. In particular, if $b' \leqq a'$, then $(b' \vee e') \wedge a' = b' \vee (e' \wedge a')$, so that $b \geqq a \Rightarrow (b \wedge e) \vee a = b \wedge (e \vee a)$. Given $f \in L$, $f\psi_a \geqq a$, so $(e \wedge f\psi_a) \vee a = (e \vee a) \wedge f\psi_a$, $(e \wedge f\psi_a)\psi_a = e\psi_a \wedge f\psi_a$. Thus ψ_a is a weak quantifier. That ψ_a is discrete follows from the observation that if $e \in L(a, 1)$, then $e = e \vee a = e\psi_a$.

The only remaining problem is to consider the lattice of closed subspaces $L(H)$ of a Hilbert space H and determine which elements satisfy the conditions of Theorem 13. If H is finite dimensional, $L(H)$ is modular, so ψ_A is a weak quantifier for every $A \in L(H)$. The answer for the infinite dimensional case is contained in the next theorem.

THEOREM 14. *Let H be an infinite dimensional Hilbert space. Then $M(E, A)$ for all $E \in L(H)$ if and only if A or $A^\perp$ is finite dimensional.*

Proof. If A is finite dimensional and $B \leqq A$, we need only show that for any $E \in L(H)$, $(B + E) \wedge A \leqq B + (E \wedge A)$. In case $A^\perp$ is finite dimensional, we must show that $B \geqq A$ implies that $B \wedge (E + A^\perp) \leqq (B \wedge E) + A^\perp$. Both results follow from the standard pointwise argument.

Suppose that both A and $A^\perp$ are infinite dimensional. Using a minor variation of the construction outlined in Halmos [4, § 15], one can produce a closed subspace N and a vector y such that

$$A^\perp < A^\perp + \langle y \rangle$$
$$A^\perp \vee N^\perp = (A^\perp + \langle y \rangle) \vee N^\perp$$
$$A^\perp \wedge N^\perp = (A^\perp + \langle y \rangle) \wedge N^\perp = 0 .$$

It is then immediate that modularity fails for the pair (N, A).

The main purpose of this paper was to determine all weak quantifiers on $L(H)$. Since $L(H)$ has the atomic bisection property, this has essentially been done, but for convenience we assemble the results into one final theorem.

THEOREM 15. (i) *Every weak quantifier on $L(H)$ is discrete or indiscrete.*

(ii) *Given $A \in L(H)$ the mapping $B\varphi = A$ if $B \leqq A$, $B\varphi = 1$ if $B \nleqq A$ is an indiscrete weak quantifier; every indiscrete weak quantifier takes this form.*

(iii) *The mapping $B\psi_A = B \vee A$ is a discrete weak quantifier if and only if A or $A^{\perp}$ is finite dimensional. If φ is a discrete weak quantifier then $\varphi = \psi_{(0\varphi)}$.*

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