AN EFFECTLESS CUTTING OF A VIBRATING MEMBRANE

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Let G be a multiply connected domain bounded by an outer boundary Γ_0, inner boundaries $\Gamma_1, \Gamma_2, \cdots$, and possibly some other inner boundaries $\gamma_1, \gamma_2, \cdots$. Let u be the eigenfunction corresponding to the lowest eigenvalue λ_1 of the membrane problem

$$\Delta u + \lambda_1 u = 0 \quad \text{in } G \tag{1}$$

with

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma_0, \Gamma_1, \cdots \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \gamma_1, \gamma_2, \cdots . \end{aligned} \tag{2}$$

We shall show that there exists a cut $\tilde{\gamma}$ consisting of a finite set of analytic arcs along which $(\partial u/\partial n) = 0$ which separates any given one of the fixed holes, say Γ_1, from the outer boundary Γ_0 and the other holes $\Gamma_2, \Gamma_3, \cdots$. This means that the membrane G may be cut in two along $\tilde{\gamma}$ without lowering its lowest eigenvalue. This fact is used in the preceding paper of J. Hersch to establish an upper bound for λ_1.

We assume that $\Gamma_0, \Gamma_1, \cdots$ have continuous normals and that $\gamma_1, \gamma_2, \cdots$ are analytic. Then it is well-known that u has the following properties:

(3) (a) $u > 0$ in G, and $\dfrac{\partial u}{\partial n} < 0$ on $\Gamma_0, \Gamma_1, \cdots$.

(b) u is analytic in $G + \gamma_1 + \gamma_2 + \cdots$.

(c) u_{xx} and u_{yy} do not vanish simultaneously.

(The last property follows from (3a) and (1)).

We define G_1 to be the set of points of G from which the fall lines, i.e. the trajectories of

$$\begin{aligned} \frac{dx}{dt} &= -u_x \\ \frac{dy}{dt} &= -u_y \end{aligned} \tag{4}$$

reach Γ_1. By property (3a) G_1 contains a neighborhood in G of Γ_1, and its exterior contains neighborhoods in G of $\Gamma_0, \Gamma_2, \cdots$. Since u_x

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and u_y are continuous, G_1 is open.

Let $\tilde{\gamma}$ be the part of the boundary of G_1 that lies in G. Let P be a point of $\tilde{\gamma}$ where the gradient of u does not vanish. Then there is a trajectory γ satisfying (4) through P. Let Q be any other point on γ. Since P is not in G_1, it follows from the definition that Q is not in G_1. On the other hand, if a whole neighborhood of Q were not in G_1, it would follow from the continuity of the trajectories with respect to their initial points that a whole neighborhood of P would be outside G_1. This would contradict the fact that P is a boundary point of G_1.

Thus we have shown that the whole trajectory γ lies in $\tilde{\gamma}$. It cannot go to Γ_1. Since the set of points from which trajectories go to $\Gamma_0, \Gamma_2, \cdots$ is also open, γ cannot go to these boundary components.

We note that u is monotone on γ, and

$$\left|\frac{du}{ds}\right| = |\operatorname{grad} u| . \tag{5}$$

Thus γ is either of finite length, or it must contain a sequence of points $Q_1, Q_2, \cdots$ on which grad u approaches zero. These will have a limit point Q at which $\operatorname{grad} u = 0$. (It may be that Q lies on one of the γ_i. In this case we think of u extended across γ_i as an analytic function by reflection).

There is a neighborhood of Q in which the trajectories can be determined by examining the first few terms of the power series for u. Using property (3c), we find that γ is of finite length. This is, of course, true in both the t and $-t$ directions.

The free boundary curves γ_i are composed of trajectories of (4) and critical points, i.e., points where $\operatorname{grad} u = 0$. Hence it follows from the uniqueness of the initial value problem for (4) that if γ ends on γ_i, the end point must again be a critical point. Thus, each trajectory γ in $\tilde{\gamma}$ connects two critical points.

It follows from properties (3b) and (3c) and the implicit function theorem that a critical point Q is either an isolated critical point or lies on an analytic arc of critical points. These arcs are again isolated.

Thus we have shown that $\tilde{\gamma}$ is composed of a finite number of analytic arcs of finite length along which $(\partial u/\partial n) = 0$, and a finite number of critical points. We delete any isolated points of $\tilde{\gamma}$.

The fact that $\tilde{\gamma}$ separates Γ_1 from $\Gamma_0, \Gamma_2, \cdots$ is clear from the definition of G_1.

The above considerations apply to any function with properties (3).

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