SOME REPRODUCING KERNELS FOR THE UNIT DISK

G. S. INNIS, JR.

Introduction. Let $S(t)$ denote the class of functions φ analytic in the unit disk U with center 0 and satisfying

$$\int_U\int |\varphi(z)|\,(1-|z|^2)^t\,dxdy < \infty \quad (z = x + iy) \tag{1}$$

for t real. In this paper we shall prove that for λ and ν properly restricted, $|\zeta| < 1$ and $\varphi \in S(t)$, the following formulas are valid:

$$\varphi(\zeta) = \frac{(\lambda+1)^\nu}{\Gamma(\nu)\,\pi}\int_U\int \frac{\varphi(z)\,(1-|z|^2)^\lambda}{(1-\bar{z}\zeta)^{\lambda+2}}\, ln^{\nu-1}\Big(\frac{1-\bar{z}\zeta}{1-|z|^2}\Big)dx\,dy\ , \tag{2}$$

and

$$\varphi^{(m)}(\zeta) = \frac{\lambda+1}{\pi}\iint \bar{z}^m \frac{\varphi(z)\,(1-|z|^2)^\lambda}{(1-\bar{z}\zeta)^{\lambda+2+m}}\sum_{i=0}^{m} a_i ln^{\nu-1-i}\Big(\frac{1-\bar{z}\zeta}{1-|z|^2}\Big)dx\,dy\ , \tag{3}$$

where the a_i are suitably chosen constants (with respect to φ and the variables z and ζ). Finally, if

$$\begin{aligned} F_n(\zeta,\nu,\lambda) = \frac{(-1)^{n+1}}{\pi}\iint & \frac{\varphi(z)\,(1-|z|^2)^\lambda}{\bar{z}^n(1-\bar{z}\zeta)^{\lambda+2-n}} \\ &\cdot\Big[\frac{(\lambda+1)^{\nu-1}}{\Gamma(\nu+n-1)}\, ln^{\nu+n-2}\Big(\frac{1-\bar{z}\zeta}{1-|z|^2}\Big) \\ &+ \frac{1}{\Gamma(n)}\, ln^{n-1}\Big(\frac{1-\bar{z}\zeta}{1-|z|^2}\Big)\Big]dxdy\ , \end{aligned} \tag{4}$$

then $F_n(\zeta, \nu, \lambda)$ has the property that

$$\frac{d^n}{d\zeta^n} F_n(\zeta,\nu,\lambda) = \varphi(\zeta)\ . \tag{5}$$

Formula (2) reduces to the well known results of Ahlfors [1] and Bergman [2] for particular choices of the parameters t, λ, and ν. The author is indebted to Professor Ahlfors for suggesting this problem.

Notation. Define

$$N(z,\lambda) = (1-|z|^2)^\lambda\ ,$$
$$D(z,\zeta,\lambda) = (1-\bar{z}\zeta)^\lambda\ ,$$
$$L(z,\zeta,\nu) = ln^{\nu-1}\Big(\frac{1-\bar{z}\zeta}{1-|z|^2}\Big)$$

Received May 15, 1963. This work was done while the author was a NAS-NRC Postdoctoral Fellow.

where the principal values of the functions on the right are used.

Reproducing Kernels. In this section we shall prove

THEOREM 1. *If $\varphi \in S(t)$ for some t, then*
(a) *for $Re\,\nu \geqq 1$ and $Re\,\lambda > t$, (2) is satisfied and*
(b) *for $Re\,\nu = 1$ and $Re\,\lambda \geqq t$, (2) is satisfied.*

REMARKS. If

$$K_1(z, \zeta, \nu, \lambda) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu)\pi} N(z, \lambda)\, D(z, \zeta, -\lambda, -2)\, L(z, \zeta, \nu)\,,$$

then because $|z| < 1$, $|\zeta| < 1$ and principal values were used in defining N, D and L, K_1 is unambiguously defined. Thus (2) can be written

$$\varphi(\zeta) = \int_U\!\!\int \varphi(z)\, K_1(z, \zeta, \nu, \lambda) dx\, dy\,. \tag{2'}$$

Also, if $\varphi \in S(t)$ and $\varphi \not\equiv 0$, then $t > -1$ as is easily seen by considering (1) in polar coordinates.

The proof of Theorem 1 will be preceded by the statement and proof of three lemmas.

LEMMA 1. *For $\varphi \in S(t)$, and for $Re\lambda \geqq t$, (1) implies*

$$\lim_{r\to 1} (1 - r^2)^{Re\lambda+1} \int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta = 0\,.$$

Proof. If $f(r) = \int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta$, then f is a nondecreasing function of r for $0<r<1$ (the trivial case of $\varphi \equiv 0$ is excluded in the sequel). Suppose now that $\limsup\, (1 - r^2)^{Re\lambda+1} f(r) = a > 0$ (a may be infinite). Let $0 < b < a$. Then there exists a sequence $\{r_i\}$ of real numbers, $0 < r_{i-1} < r_i < 1$, converging to 1 such that $f(r) \geqq b(1 - r_i^2)^{-(Re\lambda+1)}$ for $r > r_i$ and $1 - r_i^2 < (1 - r_{i-1}^2)/2$. Then (1) becomes

$$\begin{aligned}\int_0^1\!\!\int_0^{2\pi} r(1 - r^2)^{Re\lambda} |\varphi(re^{i\theta})|\, dr d\theta &\geqq \sum_{i=2}^{\infty} f(r_{i-1}) \int_{r_{i-1}}^{r_i} r(1 - r^2)^{Re\lambda} dr \\ &= \sum_{i=2}^{\infty} \frac{b}{Re\lambda + 1} \left[1 - \left(\frac{1 - r_i^2}{1 - r_{i-1}^2}\right)^{Re\lambda+1}\right] \\ &\geqq \sum_{i=2}^{\infty} \frac{b}{Re\lambda + 1} 1^i \left[1 - \left(\frac{1}{2}\right)^{Re\lambda+1}\right] = \infty\,.\end{aligned}$$

This contradiction implies

$$\lim_{r\to 1} (1 - r^2)^{Re\lambda+1} \int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta = 0\,.$$

LEMMA 2. *If $\varphi \in S(t)$ for some t, $Re\lambda > t$ and $Re\nu \geqq 1$, then*

$$\int_U\int \varphi(z)K_1(z, \zeta, \nu, \lambda)dx\,dy = \int_U\int \varphi(z)K_1(z, \zeta, \nu + 1, \lambda)dx\,dy\ . \tag{6}$$

Proof. Let $K_1(\nu) = K_1(z, \zeta, \nu, \lambda)$. Then

$$K_1(\nu) = [K_1(\nu) - K_1(\nu + 1)] + K_1(\nu + 1)$$

and if

$$f(z, \zeta, \nu, \lambda) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu + 1)\pi}\frac{\varphi(z)}{z - \zeta} N(z, \lambda + 1)D(z, \zeta, -\lambda-1)L(z, \zeta, \nu + 1)\,,$$

then

$$\frac{\partial f}{\partial \bar{z}} = (K_1(\nu) - K_1(\nu + 1))\varphi(z)\ .$$

We are, therefore, in a position to apply Green's formula since the singularity of f at $z = \zeta$ is only apparent ($\lim_{z\to\zeta}(z - \zeta)^{-1} L(z, \zeta, \nu + 1) = 0$). Thus for $0 < r < 1$,

$$\int_{|z|<r}\int \varphi(z)K_1(\nu)dx\,dy = \frac{1}{2i}\int_{|z|=r} f(z, \zeta, \nu, \lambda)dz + \int_{|z|<r}\int \varphi(z)K_1(\nu + 1)dx\,dy\,, \tag{7}$$

and the lemma will be proved if we establish that the line integral in (7) vanishes as $r \to 1$. To show that this is the case, let $\varepsilon > 0$ and $t + \varepsilon < Re\lambda$. Then

$$\begin{aligned} I_r &= \frac{1}{2i}\int_{|z|=r} f(z, \zeta, \nu, \lambda)dz \\ &= C\int_0^{2\pi} \frac{\varphi(re^{i\theta})}{re^{i\theta} - \zeta} N(r, \lambda + 1)D(re^{i\theta}, \zeta, -\lambda -1)L(re^{i\theta}, \zeta, \nu + 1)re^{i\theta}d\theta\,, \end{aligned} \tag{8}$$

and for r near 1,

$$|\,I_r\,| \leqq C_1(1 - r^2)^{Re\lambda+1-\varepsilon/2} \int_0^{2\pi} |\,\varphi(re^{i\theta})\,|\,d\theta \tag{9}$$

where the factor $(1 - r^2)^{\varepsilon/2}$ was used to suppress the logarithm near $r = 1$. On applying Lemma 1 in (9) we get

$$|\,I_r\,| \leqq C_2(1 - r^2)^{\varepsilon/2}\,,$$

and the result follows.

LEMMA 2'. *Lemma 2 is valid for $Re\lambda \geqq t$ if $Re\nu = 1$.*

Proof. The proof of this lemma is similar to that of Lemma 2 except that the factor of $(1-r^2)^{\varepsilon/2}$ is not needed to suppress the logarithm and, therefore, the range of λ can be extended.

LEMMA 3. *If $Re\nu \geqq k$, $Re\lambda > -1$ and p is a positive integer, then*

$$\begin{aligned}(10)\qquad &\int_0^1 r^{2p-1} N(r,\lambda)\, L(r, 0, \nu-k+1)dr \\ &= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \frac{\Gamma(\nu-k+1)}{2(\lambda+i+1)^{\nu-k+1}} .\end{aligned}$$

Proof. Induction on p will be used. If $p=1$, (10) reads

$$\int_0^1 rN(r,\lambda)\, L(r, 0, \nu-k+1)dr = \frac{\Gamma(\nu-k+1)}{2(\lambda+1)^{\nu-k+1}} .$$

Substituting

$$t = (\lambda+1)L(r, 0, 2) , \quad dt = (\lambda+1)\frac{2r}{1-r^2}dr$$

in the left hand side, we get

$$\int_0^1 rN(r,\lambda)L(r, 0, \nu-k+1)dr = \frac{1}{2(\lambda+1)^{\nu-k+1}}\int_0^\infty e^{-t}t^{\nu-k}dt$$

where the path of integration in the right hand member is the half line through the origin inclined at the angle arg $(\lambda+1)$. That integral is $\Gamma(\nu-k+1)$, and the result is established for $p=1$. Suppose that (10) has been proved for $p-1$. The left hand side of (10) can be written in the form

$$\begin{aligned}&\int_0^1 r^{2p-3}N(r,\lambda)L(r, 0, \nu-k+1)dr - \int_0^1 r^{2p-3}N(r,\lambda+1)L(r, 0, \nu-k+1)dr \\ &= \frac{\Gamma(\nu-k+1)}{2(\lambda+1)^{\nu-k+1}} + \sum_{i=1}^{p-2}(-1)^i\left[\binom{p-2}{i+1}+\binom{p-2}{i}\right]\frac{\Gamma(\nu-k+1)}{2(\lambda+1+i)^{\nu-k+1}} \\ &\quad + (-1)^{p-1}\frac{\Gamma(\nu-k+1)}{2(\lambda+p)^{\nu-k+1}} = \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\frac{\Gamma(\nu-k+1)}{2(\lambda+i+1)^{\nu-k+1}} .\end{aligned}$$

Proof of Theorem 1. This proof will be accomplished by showing that the mth derivative of φ evaluated at 0 is given by the mth derivative of (2) evaluated at 0. Induction will be used.

It is clear that (1) implies the absolute convergence of (2), and that if $Re\,\lambda$ is large enough, differentiation with respect to ζ, λ, and ν will commute with integration. Differentiating (2) m times with respect to ζ, one gets

$$\varphi^{(m)}(\zeta) = \frac{\lambda+1}{\pi}\int_U\!\!\int \bar{z}^m \varphi(z) N(z,\lambda) D(z,\zeta,-\lambda-2-m) \sum_{i=0}^{m} a_i L(z,\zeta,\nu-i)\,dx\,dy \tag{11}$$

if $Re\,\nu \geqq m+1$ and the a_i are properly chosen constants.

Let $F(\zeta) = \int_U\!\!\int \varphi(z) K_1(\nu)\,dx\,dy$. Then $F(0) = \int_U\!\!\int \varphi(z) K_1(z,0,\nu,\lambda)\,dx\,dy$ which by (1) can be written

$$\begin{aligned} F(0) &= \frac{(\lambda+1)^\nu}{\Gamma(\nu)\pi}\int_0^1 rN(r,\lambda)L(r,0,\nu)\,dr \int_0^{2\pi}\varphi(re^{i\theta})\,d\theta \\ &= \frac{2(\lambda+1)^\nu}{\Gamma(\nu)}\varphi(0)\int_0^1 rN(r,\lambda)L(r,0,\nu)\,dr\ . \end{aligned}$$

By Lemma 3 this last integral is $\Gamma(\nu)/2(\lambda+1)^\nu$, and the desired result follows.

Suppose now that $Re\,\nu > 1$. Because of a complication in the inductive hypothesis, it will also be necessary to show that $F'(0) = \varphi'(0)$. Notice, however, that if we differentiate F with respect to ζ two terms arise, and in one of these the exponent of ln is $\nu - 2$. If $Re\,\nu < 2$, this would cause trouble. This difficulty is avoided if we first apply Lemma 2 to F to write it in a form for which $Re\,\nu \geqq 2$. Then

$$F'(0) = \frac{(\lambda+1)^\nu}{\Gamma(\nu)\pi}\int\!\!\int \bar{z}\varphi(z)N(z,\lambda) [(\lambda+2)\,L(z,0,\nu) - (\nu-1)\,L(z,0,\nu-1)]\,dx\,dy\ .$$

By splitting this into two integrals and proceeding just as above, we derive

$$F'(0) = \varphi'(0)\ .$$

Suppose now that it has been established that $F^{(p-1)}(0) = \varphi^{(p-1)}(0)$. Use Lemma 2 to write F in a form for which $Re\,\nu \geqq p+1$.

Let the following be taken as the inductive hypothesis:

$$F^{(p-1)}(0) = \varphi^{(p-1)}(0)\ , \tag{12a}$$

$$a_0 + \sum_{i=1}^{p-1} a_i \frac{(\lambda+1)^i}{(\nu-1)(\nu-2)\cdots(\nu-i)} = (p-1)!\ , \tag{12b}$$

and

$$a_0 + \sum_{i=1}^{p-1} a_i \frac{(\lambda+k)^i}{(\nu-1)(\nu-2)\cdots(\nu-i)} = 0 \tag{12c}$$

for $k = 2, 3, \cdots, p$. When $p = 2$, (12a) was proved above. In this case $a_0 = \lambda + 2$ and $a_1 = -(\nu - 1)$ so that both (12b) and (12c) are satisfied. Consider now $F^{(p)}(0)$ when $F^{(p-1)}(\zeta)$ is given by the right hand side of (11) with $m = p - 1$.

$$
\begin{aligned}
F^{(p)}(0) = \frac{(\lambda+1)^\nu}{\Gamma(\nu)\pi} \int_U\!\!\int \bar{z}^p \varphi(z)\, N(z, \lambda) \\
\left[(\lambda + 1 + p)\sum_{i=0}^{p-1} a_i L(z, 0, \nu - i) - \sum_{i=0}^{p-1} a_i(\nu - i) L(z, 0, \nu - i - 1)\right] dx\, dy\,.
\end{aligned} \tag{13}
$$

After some algebra (13) becomes

$$
\begin{aligned}
F^{(p)}(0) = \frac{2(\lambda+1)^\nu}{p!\,\Gamma(\nu)} \varphi^{(p)}(0) \Bigg[b_0 \frac{\Gamma(\nu)}{2(\lambda+1)^\nu} - b_1 \binom{p}{1} \frac{\Gamma(\nu)}{2(\lambda+2)^\nu} \\
+ \cdots (-1)^p b_p \frac{\Gamma(\nu)}{2(\lambda + p + 1)^\nu} \Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
b_0 &= a_0(\lambda + 1 + p) + \frac{\lambda + 1}{\nu - 1}[a_1(\lambda + 1 + p) - a_0(\nu - 1)] \\
&\quad + \frac{(\lambda+1)^2}{(\nu-1)(\nu-2)}[a_2(\lambda + 1 + p) - a_1(\nu - 2)] \\
&\quad + \cdots - a_{p-1}(\nu - p) \frac{(\lambda+1)^p}{(\nu-1)(\nu-2)\cdots(\nu-p)} \\
&= (\lambda + 1 + p)(p-1)! - (\lambda + 1)(p - 1)! \\
&= p! \qquad \text{by (12b)}
\end{aligned}
$$

and

$$
\begin{aligned}
b_k &= a_0(\lambda + 1 + p) + \frac{\lambda + k + 1}{\nu - 1}[a_1(\lambda + 1 + p) - a_0(\nu - 1)] \\
&\quad + \frac{(\lambda + k + 1)^2}{(\nu-1)(\nu-2)}[a_2(\lambda + 1 + p) - a_1(\nu - 2)] \\
&\quad + \cdots - a_{p-1}(\nu - p) \frac{(\lambda + k + 1)^p}{(\nu-1)(\nu-2)\cdots(\nu-p)} \\
&= (\lambda + 1 + p)0 + (\lambda + k + 1)0 = 0 \text{ by (12c) for}
\end{aligned}
$$

$k = 2, 3, \cdots, p$. It follows immediately that

$$
F^{(p)}(0) = \varphi^{(p)}(0)
$$

as was to be shown.

The case $Re\,\nu = 1$, $Re\,\lambda \geqq t$ is treated as above except that Lemma 2′ is used in place of Lemma 2. The proof is omitted.

REMARKS. Notice that in proving Theorem 1 we have also established that (11) is a correct formula for the mth derivative of φ.

As mentioned above we are also at liberty to differentiate (2) with respect to ν and λ. It is readily verified that differentiating (2) with respect to λ and using the results of Theorem 1 yields

$$\varphi(\zeta) = \int_U\int \varphi(z)\, K_1(\nu + 1) dx\, dy$$

which is nothing new. However, differentiating (2) with respect to ν and using Theorem 1 we derive the new formula,

$$\varphi(\zeta) = \frac{(\lambda + 1)^\nu}{\Gamma'(\nu)\pi - ln(\lambda + 1)\Gamma(\nu)\pi} \int_U\int \varphi(z) N(z, \lambda) D(z, \zeta, -\lambda - 2) L(z, \zeta, \nu)\, ln(L(z, \zeta, 2)) dx\, dy \,. \tag{14}$$

The integral in (14) is absolutely convergent in spite of the apparent difficulties with $ln(L)$. Further derivations with respect to ζ, ν, and λ are, of course, possible.

An interesting formula results from (11) for the case in which λ is an integer and $\nu = 1$. Here, $a_0 = \Gamma(n + m + 1)/\Gamma(n + 1)$ and the rest of the a's are zero. The θ integral is

$$\int_0^{2\pi} (re^{-i\theta})^m \frac{\varphi(re^{i\theta})}{(1 - re^{-i\theta}\zeta)^{m+n+2}} d\theta = 2\pi \frac{r^{2m}}{(m + n + 1)!} [z^{n+2}\varphi(z)]_{z=r^2\zeta}^{(m+n+1)} \,,$$

and (11) becomes

$$\varphi^{(m)}(\zeta) = \frac{2}{n!} \int_0^1 r^{2m+1} (1 - r^2)^n [z^{n+2}\varphi(z)]_{z=r^2\zeta}^{(m+n+1)} dr \,.$$

This expression is readily checked for $\varphi(z) = z^k$ and, thereby, for any $\varphi \in S(n)$.

Primative Kernels. In this section we shall prove

THEOREM 2. *If $\varphi \in S(t)$ and*

$$K_2^n(z, \zeta, \nu, \lambda) = \frac{(-1)^{n+1}}{\bar{z}^n \pi} N(z, \lambda) D(z, \zeta, -\lambda - 2 + n) \left[\frac{(\lambda + 1)^{\nu - 1}}{\Gamma(\nu + n - 1)} L(z, \zeta, \nu + n - 1) + \frac{1}{\Gamma(n)} L(z, \zeta, n)\right] ,$$

then for $Re\,\nu = 2$ and $Re\,\lambda \geqq t$ or $Re\,\nu \geqq 2$ and $Re\,\lambda > t$,

$$F_n(\zeta, \nu, \lambda) = \int_U\int \varphi(z)\, K_2^n(z, \zeta, \nu, \lambda) dx\, dy \tag{15}$$

has the property that $F_n^{(n)}(\zeta, \nu, \lambda) = \varphi(\zeta)$ (differentiation is with respect to ζ). If $Re\,\lambda \geqq t$ and $\nu = 1$, then

$$H_n(\zeta, \lambda) = \iint \varphi(z)\, K_2^n(z, \zeta, 1, \lambda) dx\, dy \tag{16}$$

has the property that $H_n^{(n)}(\zeta, \lambda) = 2\varphi(\zeta)$.

Proof. The proof will be by induction. Consider $F_1(\zeta)$. To differentiate under the integral sign in (15) it is sufficient to show that the given and resulting integrals are absolutely convergent. However,

$$\int_U\!\!\int |\varphi(z)\, K_2^1(z, \zeta, \nu, \lambda)|\, dx\, dy = \int_{|z|\leqq r}\!\!\int + \int_{r<|z|<1}\!\!\int .$$

The integral over the annulus offers no difficulty and for small r,

$$|\varphi(z)\, K_2^1(z, \zeta, \nu, \lambda)| \leqq C\frac{1}{r}$$

where C is constant. Thus

$$\int_{|z|\leqq r}\!\!\int |\varphi(z)\, K_2^1(z, \zeta, \nu, \lambda)|\, dx\, dy \leqq 2\pi r C .$$

Because $Re\,\nu \geqq 2$, all of the integrals occurring after differentiation are absolutely convergent and, hence,

$$\begin{aligned} F_1'(\zeta, \nu, \lambda) &= \int_U\!\!\int \varphi(z) \frac{\partial}{\partial\zeta} K_2^1(z, \zeta, \nu, \lambda)\, dx\, dy \\ &= \int_U\!\!\int \varphi(z)\, [K_1(\nu) + K_1(1) - K_1(\nu - 1)]\, dx\, dy \\ &= \varphi(\zeta) . \end{aligned}$$

Similarly $H_1'(\zeta, \lambda) = 2\varphi(\zeta)$ and thus

$$H_1(\zeta, \lambda) = 2F_1(\zeta, \nu, \lambda) + C .$$

Suppose now that it has been established that for some $n \geqq 2$,

(a) $F_{n-1}(\zeta, \nu, \lambda)$ is an $(n-1)$st primative and

(b) $H_{n-1}(\zeta, \lambda) = 2F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$ where

P is a polynomial of degree $n - 2$ in ζ. The absolute convergence of the needed integrals can be established as above. Therefore, from (15) we get

$$\begin{aligned} F_n'(\zeta, \nu, \lambda) = \frac{(-1)^{n+1}}{\pi} \int_U\!\!\int & \frac{\varphi(z)}{\bar{z}^{n-1}} N(z, \lambda)\, D(z, \zeta, -\lambda - 1 + n) \\ & \Big[(\lambda + 2 - n) \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 1)} L(z, \zeta, \nu + n - 1) \\ & + (\lambda + 2 - n) \frac{1}{\Gamma(n)} L(z, \zeta, n) \\ & - \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 2)} L(z, \zeta, \nu + n - 2) \\ & - \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1)\Big] dx\, dy . \end{aligned} \tag{17}$$

The last two terms in this square bracket yield $F_{n-1}(\zeta, \nu, \lambda)$. Now let us add and subtract $2(\lambda + 2 - n)L(z, \zeta, n-1)/[(\lambda+1)\Gamma(n-1)]$ to the first two terms to write them as

$$\frac{\lambda + 2 - n}{\lambda + 1}\Big[\frac{(\lambda+1)^{\nu-1}}{\Gamma(\nu+n-2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n-1)} L(z, \zeta, n-1)$$
$$+ \frac{(\lambda+1)^{\nu-1}}{\Gamma(\nu+n-2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n-1)} L(z, \zeta, n-1)$$
$$- \frac{2}{\Gamma(n-1)} L(z, \zeta, n-1)\Big]$$

where the first term comes from the first term of (17) with ν replaced by $\nu + 1$ and the third term comes from the second term of (17) with $\nu = 2$. Thus (17) yields

$$\begin{aligned} F'_n(\zeta, \nu, \lambda) &= F_{n-1}(\zeta, \nu, \lambda) - \frac{\lambda+2-n}{\lambda+1}[F_{n-1}(\zeta, \nu+1, \lambda) \\ &\quad + F_{n-1}(\zeta, 2, \lambda) - H_{n-1}(\zeta, \lambda)] \\ &= F_{n-1}(\zeta, \nu, \lambda) + Q(\zeta, \nu, \lambda) \end{aligned}$$

where Q is a polynomial of degree $(n-2)$ in ζ.

To complete the inductive argument, it is necessary to show that $H'_n(\zeta, \lambda) = 2\, F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$.

$$\begin{aligned} H'_n(\zeta, \lambda) &= 2(-1)^{n+1} \int_U\!\!\int \frac{\varphi(z)}{\bar{z}^{n-1}} N(z, \lambda)\, D(z, \zeta, -\lambda - 1 + n) \\ &\quad \Big[\frac{\lambda+2-n}{\Gamma(n)} L(z, \zeta, n) - \frac{1}{\Gamma(n-1)} L(z, \zeta, n-1)\Big] dx\, dy\,. \end{aligned} \tag{18}$$

Using the same techniques as above, the square brackets can be written

$$\frac{\lambda + 2 - n}{\lambda + 1}\Big[\frac{(\lambda+1)^{\nu-1}}{\Gamma(\nu+n-2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n-1)} L(z, \zeta, n-1)\Big]$$
$$- \Big(\frac{\lambda+2-n}{\lambda+1} + 1\Big)\frac{1}{\Gamma(n-1)} L(z, \zeta, n-1)$$

where $\nu = 2$ in the first term. On placing this expression in (18), we get

$$H'_n(\zeta, \lambda) = -2\Big(\frac{\lambda+2-n}{\lambda+1}\Big) F_{n-1}(\zeta, 2, \lambda) + \Big(\frac{\lambda+2-n}{\lambda+1} + 1\Big) H_{n-1}(\zeta, \lambda)\,.$$

By the inductive hypothesis, $H_{n-1}(\zeta, \lambda) = 2\, F_{n-1}(\zeta, \nu, \lambda) + R(\zeta, \nu, \lambda)$ where R is of degree $(n-2)$ in ζ. We have then that

$$H'_n(\zeta, \lambda) = 2\, F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$$

where P is of degree $(n-2)$ in ζ. This proves Theorem 2.

It is interesting to note that F_n and H_n depend analytically on ν and λ and are not necessarily constants (with respect to these two variables).

It is easy to prove

THEOREM 3. *If* (a) $\varphi \in S(Re\,\lambda)$ *and has a zero of order at least* n *at* 0, (b) *either* λ *is not an integer or* λ *is an integer greater than* $n-2$, (c)

$$K_3^n = \frac{\lambda+1}{\pi}\frac{\Gamma(\lambda+3-n)}{\Gamma(\lambda+3)}\bar{z}^{-n} N(z, \lambda)\, D(z, \zeta, -\lambda-2+n)$$

and (d)

$$G_n(\zeta) = \int_U\!\!\int \varphi(z)\, K_3^n(z, \zeta, \lambda)dx\, dy\ , \tag{19}$$

then

$$G_n^{(n)}(\zeta) = \varphi(\zeta)\ .$$

The conditions imposed on λ are sufficient to guarantee that the integral (19) converges absolutely. The proof of the theorem is just a matter of differentiating and is omitted. If, however, $\varphi \in S(Re\,\lambda)$, then for each positive integer n, $z^n\varphi(z)$ is also in $S(Re\,\lambda)$, and, therefore, if we define

$$E_n(\zeta) = \int_U\!\!\int z^n\, \varphi(z) K_3^n(z, \zeta, \lambda)dx\, dy\ , \tag{20}$$

$E_n(\zeta)$ is well defined, absolutely convergent and has the property that

$$E_n^{(n)}(\zeta) = \zeta^n\varphi(\zeta)\ .$$

The simplicity of (20) may make it more useful then either (15) or (16) in some cases.

BIBLIOGRAPHY

1. L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math., **74** (1961), 176.
2. Z. Nehari, *Conformal Mapping*, McGraw-Hill, 1952, p. 252.

HARVARD UNIVERSITY