REMARKS ON CERTAIN ALMOST PRODUCT SPACES

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An almost product space is a differentiable manifold of class C^∞ which has a nontrivial tensor field F_i^j (also of class C^∞) satisfying the conditions

$$F_i^j F_j^k = \delta_i^k ; \qquad i, j, k = 1, 2, \cdots, n . \tag{0.1}$$

It is known that one can find a positive definite riemannian metric such that

$$g_{ij} F_h^i F_k^j = g_{hk} . \tag{0.2}$$

In the following we denote by M^n an almost product metric space which satisfies (0.1) and (0.2). The covariant derivative with respect to the riemannian connection of g_{ij} is denoted as ∇.

If condition (0.1) is replaced by $F_i^j F_j^k = -\delta_i^k$, the space is known as an almost Hermitian space. In this case many subclasses are considered and their properties are studied by several authors. We examine in § 1 and § 2 conditions which correspond to ones which define the subclasses of almost Hermitian space and will study in particular the following two conditions:

$$\nabla_i F_j^i = 0 , \tag{0.3}$$

and

$$\nabla_{(i} F_{jh)} = 0 , \tag{0.4}$$

where $F_{jh} = F_j^i g_{ih}$ which is symmetric in j and h by (0.2).

In addition to these, there are also some special classes of M^n already known. We studied the relations between all these special cases and the result can be described in the following diagram:

$$\begin{array}{ccccc}
\begin{array}{c}\text{Most general almost}\\ \text{product metric space}\end{array} & \longrightarrow & \begin{array}{c}[0.1]\\ \nabla_i F_j^i = 0\end{array} & \longrightarrow & \begin{array}{c}[0.2]\\ \nabla_{(i} F_{jh)} = 0\end{array} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{array}{c}[1.0]\\ N_{ji}^h \mp N_{ji}^k F_k^h = 0\end{array} & \longrightarrow & [1.1] & \longrightarrow & [1.2] \\
\downarrow & & \downarrow & & \downarrow \\
\begin{array}{c}[2.0]\\ N_{ji}^h = 0\end{array} & \longrightarrow & [2.1] & \longrightarrow & \begin{array}{c}[2.2]\\ \nabla_i F_j^h = 0\end{array}
\end{array}$$

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In the diagram the conditions given are the ones defining the special subclasses denoted as [0.1], [1.0] and so on. The arrows are to be rend "properly contains." N_{ji}^h denotes the Nijenhuis tensor of the tensor F_i^j. The conditions in the left hand column are independent of the choice of almost product metric, while going across the top row relates the metric to the almost product structure more and more strongly[1]. Foliated manifold with bundle-like metric studied by Reinhart [6][2] is contained in [1.0], and the case [2.2] is the loccally product riemannian space studied by Tachibana [9] and others. It is shown that [2.0] is a riemannian space having complementary subspaces and such spaces were studied by Wong [12]. Conformally separable riemannian space studied by Yano [13] is contained in this case. In § 2 conditions for some classes are also given by the use of a same tensor M_{ij}^h for the purpose of comparing them. In § 3 and § 4 examples for the situations [1.1], [1.2] and [2.1] are given by almost contact manifolds and tangent bundle of riemannian space to illustrate the above diagram. Finally, we prove in § 5 some properties for classes [0.1] and [0.2] which are analoguous to those for the corresponding cases of an almost complex manifold.

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1. We begin with following:

PROPOSITION 1.1. In M^n the following conditions are equivalent:

(i) $\nabla_j F_i^h = 0$,

(ii) $\nabla_j F_{ih} + \nabla_i F_{jh} = 0$,

(iii) $(\delta_j^m \delta_i^l - F_j^m F_i^l)\nabla_m F_l^h = 0$,

(iv) $(\delta_j^m \delta_i^l + F_j^m F_i^l)\nabla_m F_l^h = 0$,

(v) $\nabla_r F_{jl} + \nabla_j F_{lr} - \nabla_l F_{rj} = 0$.

REMARK 1.1. An almost Hermitian space satisfying (ii) is called K-space [10] and the one satisfying (iv) is called 0*-space [5]. They are wider classes than the one defined by (i) which is a Kaehlerian space. It is also known that in an almost Hermitian space (iii) is equivalent to the vanishing of N_{ji}^h [5].

Proof. It is evident that (i) → (ii), (iii), (iv), (v). Here arrows are to be read "implies."

(ii) → (i): Let $U_{jih} = \nabla_j F_{ih}$, then $U_{jih} = U_{jhi}$. From (ii) we have also $U_{jih} = -U_{ijh}$. Thus we have $U_{jih} = 0$.

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[1] The author thanks the referee for his comments to improve the presentation in a few place.

[2] Number in bracket refers the reference at the end of the paper.

(iii) → (i): Let $V_{jih} = F_j^t \nabla_t F_{ih}$ then $V_{jih} = V_{jhi}$. On the other hand from (iii) we have $V_{jih} = F_i^l \nabla_j F_{lh} = -F_h^l \nabla_j F_{li} = -V_{jhi}$. Thus we have $V_{jih} = 0$ which implies (i).

(iv) → (i): Put $W_{jih} = F_j^t \varDelta_t F_{ih}$ then $W_{jih} = W_{jhi}$. We have also $W_{jih} = -F_i^l \nabla_j F_{lh} = F_h^l \nabla_j F_{li} = -W_{jhi}$. Thus $W_{jhi} = 0$ implies (i).

(v) → (i): Permute the indices of (v) cyclically and then add the so obtained, then we have $\nabla_r F_{jl} + \nabla_j F_{lr} + \nabla_l F_{rj} = 0$ from which and (v) it follows (i).

PROPOSITION 1.2. For a riemannian space to have complementary subspaces in the sense of Wong [12] it is necessary and sufficient that it is a M^n with vanishing Nijenhuis tensor N_{ij}^h:

$$N_{ij}^h = F_i^l(\nabla_l F_j^h - \nabla_j F_l^h) - F_j^l(\nabla_l F_i^h - \nabla_i F_l^h) . \tag{1.1}$$

Proof. A riemannian space has complementary subspaces $x^\lambda =$ constant and $x^a =$ constant in the sense of Wong if and only if its line element can be written in a suitable coordinate system as

$$\begin{aligned} &ds^2 = g_{\lambda\mu}(x^\nu, x^c)dx^\lambda dx^\mu + g_{ab}(x^\nu, x^c)dx^a dx^b , \\ &\lambda, \mu, \nu = 1, \cdots, p; \qquad a, b, c = p+1, \cdots, n . \end{aligned} \tag{1.2}$$

In this case the tangent $(n-p)$-spaces and tangent p-spaces of these two families of surfaces give rise to two distributions. In the above coordinate neighborhood the almost product structure tensor F_i^j corresponding to these distributions has the components:

$$(F_i^j) = \begin{pmatrix} \delta_\lambda^\mu & 0 \\ 0 & -\delta_a^b \end{pmatrix} . \tag{1.3}$$

It is easily seen that the tensors (1.2) and (1.3) satisfy (0.1) and (0.2). By some easy computation we have $N_{ij}^h = 0$ and

$$\begin{cases} \nabla_i F_\lambda^\mu = 0 , & \nabla_i F_\lambda^a = 2\left\{ {a \atop \lambda i} \right\} , \\ \nabla_i F_a^\lambda = -2\left\{ {\lambda \atop ai} \right\} , & \nabla_i F_a^b = 0 . \end{cases} \tag{1.4}$$

Conversely, if M^n has vanishing Nijenhuis tensor, then we can find a neighborhood at each point such that the tangent spaces of $x^\lambda =$ constant and $x^a =$ constant constitute the distributions defined by F_i^j. In such coordinate system F_i^j has components given in (1.3) and we have $g_{\lambda a} = 0$ from (0.2).

PROPOSITION 1.3. M^n is a locally product riemannian space if and only if $\nabla_j F_i^h = 0$ (Tachibana [9]).

Proof. A riemannian space is called locally product if at each point we can find a coordinate neighborhood such that the line element can be witten as

$$ds^2 = g_{\lambda\mu}(x^\nu)dx^\lambda dx^\mu + g_{ab}(x^c)dx^a dx^b \,. \tag{1.5}$$

So it is a special case of the space in Proposition 1.2. If we note the fact that $\nabla_i F_j^h = 0$ implies $N_{ij}^h = 0$ and that

$$\begin{cases} \left\{ {a \atop \lambda\mu} \right\} = -\frac{1}{2} g^{ab}\partial_b g_{\lambda\mu} \,, & \left\{ {a \atop \lambda b} \right\} = \frac{1}{2} g^{ac}\partial_\lambda g_{bc} \,, \\ \left\{ {\lambda \atop a\mu} \right\} = \frac{1}{2} g^{\lambda\gamma}\partial_a g_{\mu\gamma} \,, & \left\{ {\lambda \atop ab} \right\} = -\frac{1}{2} g^{\lambda\gamma}\partial_\gamma g_{ab} \,, \end{cases} \tag{1.6}$$

then Proposition 1.3 follows from Proposition 1.2 immediately.

REMARK 1.2. From Proposition 1.2 and Proposition 1.3 we see that $N_{ij}^h = 0$ does not imply $\nabla_i F_j^h = 0$.

PROPOSITION 1.4. If M^n satisfies $N_{ij}^h = 0$ and (0.4) that is

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \,, \tag{1.7}$$

then $\nabla_i F_j^h = 0$.

Proof. Since $F_i^l \nabla_k F_{jl} + F_j^l \nabla_k F_{li} = 0$ follows from $F_i^l F_{jl} = g_{ij}$, we have

$$\begin{aligned} N_{ijk} &= N_{ij}^h g_{hk} \\ &= F_i^l(\nabla_l F_{jk} - \nabla_j F_{lk} - \nabla_k F_{jl}) - F_j^l(\nabla_l F_{ik} - \nabla_i F_{lk} + \nabla_k F_{li}) \,. \end{aligned}$$

Thus, if (1.7) holds we have $N_{ijk} = 2\{F_i^l \nabla_l F_{jk} + F_j^l \nabla_i F_{lk}\}$. From $N_{ijk} = 0$ it follows that $(\delta_i^m \delta_j^l + F_i^m F_j^l)\nabla_m F_{lk} = 0$. Then by Proposition 1.1 we get $\nabla_m F_{lk} = 0$.

REMARK 1.3. Proposition 1.4 is an analogue of a theorem of Kaehlerian space.

REMARK 1.4. From Remark 1.2 and Proposition 1.4 we see that $N_{ij}^h = 0$ does not imply (1.7). An example in § 4 shows that (1.7) does not imply $N_{ij}^h = 0$.

If a differentiable manifold has a system of completely integrable distribution, it is called foliated manifold. It is well known that one can introduce a positive definite riemannian metric g_{ij}, another system of distribution and consequently an almost product structure tensor F_i^j such that (0.1) and (0.2) hold. We call a space obtained in this way as a foliated metric space for convenience. Yano [14] has proved the following:

PROPOSITION 1.5. For a M^n to be a foliated metric space it is necessary and sufficient that

$$N_{ij}^h - N_{ij}^l F_l^h = 0 \qquad \text{or} \qquad N_{ij}^h + N_{ij}^l F_l^h = 0 \tag{1.8}$$

holds.

REMARK 1.5. If we put

$$M_{ij}^h = \nabla_i F_j^h + F_i^l F_j^m \nabla_l F_m^h + F_i^l (\nabla_l F_j^h - \nabla_j F_l^h) , \tag{1.9}$$

then $N_{ij}^h - N_{ij}^l F_l^h = 0$ if and only if $M_{ij}^h = M_{ji}^h$.

2. Put $T_{jih} = \nabla_j F_{ih} + F_j^m F_i^l \nabla_m F_{lh}$. Then we have

PROPOSITION 2.1. In M^n the condition (1.7) is equivalent to the following:

$$T_{jih} = -T_{ijh} . \tag{2.1}$$

Proof. Substitute (1.7) in the expression of T_{jih}, we have

$$\begin{aligned} T_{jih} = &-(\nabla_i F_{hj} + \nabla_h F_{ji}) - F_j^m F_i^l (\nabla_l F_{hm} + \nabla_h F_{ml}) \\ &- (\nabla_i F_{hj} + F_j^l F_i^m \nabla_m F_{hl}) = -T_{ijh} , \end{aligned}$$

because of

$$\nabla_h F_{ji} + F_j^m F_i^l \nabla_h F_{ml} = 0 , \tag{2.2}$$

which follows form $F_i^l F_l^h = \delta_i^h$ by covariant differentiation ∇_j, and then contracting with F_t^i.

Conversely,

$$T_{jih} = \nabla_j F_{ih} - F_j^m F_h^l \nabla_m F_{il} = 2\nabla_j F_{hi} - T_{jhi} ,$$

that is

$$2\nabla_j F_{hi} = T_{jih} + T_{jhi} . \tag{2.3}$$

From which we have

$$\begin{aligned} 2(\nabla_j F_{hi} + \nabla_h F_{ij} + \nabla_i F_{jh}) \\ = T_{jih} + T_{jhi} + T_{hji} + T_{hij} + T_{ihj} + T_{ijh} . \end{aligned} \tag{2.4}$$

Substitute (2.1) in (2.4) we have (1.7).

REMARK 2.1. An almost Hermitian space satisfying a condition corresponding to (1.7) is an almost Kaehlerian space.

PROPOSITION 2.2. In M^n (1.7) implies (0.3), that is

$$\nabla_l F_j^l = 0 . \tag{2.5}$$

Proof.
$$g^{ji}T_{jih} = g^{ji}\nabla_j F_{ih} + g^{ji}F_j^m F_{i\,m}^{\,l} F_{lh}$$
$$= 2g^{ji}\nabla_j F_{ih} = 2\nabla_j F_h^j .$$

Thus from (2.1) we get $\nabla_j F_h^j = -\nabla_j F_h^j$, that is $\nabla_j F_h^j = 0$.

REMARK 2.2. An almost Hermitian space satisfying a condition corresponding to (2.5) was first considered by Apte and is called A-space [1].

PROPOSITION 2.3. In M^n (2.5) is equivalent to $M_{ij}^j = 0$.

Proof. From $\nabla_k(F_j^l F_l^i) = 0$ we have $F_l^m \nabla_k F_m^l = 0$. On the other hand, as $F_i^i =$ constant, we have $\nabla_j F_i^i = 0$. Thus $M_{ji}^i = -F_j^l \nabla_m F_l^m$. Then, Proposition 2.3 follows from the fact that F_j^l is non-singular.

PROPOSITION 2.4. In M^n, (1.7) is equivalent to $F_j^l M_{li}^h + F_i^l M_{lj}^h = 0$.

Proof. We first note that from $F_j^l F_{li} = g_{ji}$ we have

$$F_j^l \nabla_k F_{li} + F_i^l \nabla_k F_{jl} = 0 . \tag{2.6}$$

Now, from (1.9) we have

$$g_{hk}(F_j^l M_{li}^h + F_i^l M_{lj}^h) = F_j^l(\nabla_l F_{ik} + \nabla_i F_{lk}) + F_i^l(\nabla_j F_{lk} + \nabla_l F_{jk}). \tag{2.7}$$

If $F_j^l M_{li}^h + F_i^l M_{lj}^h = 0$, we have from (2.6) and (2.7) the following

$$F_j^l(\nabla_l F_{ik} + \nabla_i F_{lk} + \nabla_k F_{li}) = -F_i^l(\nabla_j F_{lk} + \nabla_l F_{jk} + \nabla_k F_{jl}) . \tag{2.8}$$

Denote the left hand side of (2.8) as P_{jik}, then we have $P_{jik} = -P_{ijk}$. But it is evident that $P_{jik} = P_{jki}$. Thus we have $P_{jik} = 0$ from which (1.7) follows as F_i^j is nonsingular. The converse is evident.

3. Now we give some examples from contact manifolds. Consider a $(2m+1)$-dimensional $(n = 2m+1)$ differentiable manifold $\bar{M}^n$ with contact structure η, that is a structure defined by a 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$, then it is known [7] that there exists a (ϕ, ξ, η, g)-structure with

$$\eta = \eta_i dx^i , \qquad d\eta = \phi = \frac{1}{2}\phi_{ij} dx^i \wedge dx^j ;$$

$$\left(\phi_{ij} = \partial_i \eta_j - \partial_j \eta_i,\ \partial_i \eta_j = \frac{\partial \eta_j}{\partial x^i}\right) .$$

Stating more precisely, there exist a vector field ξ^i, a tensor field ϕ_i^j and a positive definite metric tensor g_{ij} such that

$$\begin{gathered}\text{rank}\,|\phi_i^j| = 2m\,, \qquad \phi_j^i\xi^j = 0\,, \qquad \phi_j^i\eta_i = 0\,,\\ \xi^i\eta_i = 1\,, \qquad \phi_j^i\phi_k^j = -\delta_k^i + \xi^i\eta_k\,,\\ g_{ij}\xi^j = \eta_i\,, \qquad g_{ij}\phi_h^i\phi_k^j = g_{hk} - \eta_h\eta_k\,,\\ \phi_{ij} = g_{jh}\phi_i^h = \nabla_i\eta_j - \nabla_j\eta_i\,,\end{gathered}$$

where ∇ denotes the covariant derivative with respect to the riemannian connection of g_{ij}. It is evident that

$$(\nabla_j\eta_k)\xi^k = 0\,, \tag{3.1}$$

and it is also known that

$$\nabla_i\xi^i = 0 \tag{3.2}$$

and

$$(\nabla_k\eta_j)\xi^k = 0\,. \tag{3.3}$$

In such space we can define the following almost product structure [3]:

$$F_i^j = (2\xi^j\eta_i - \delta_i^j). \tag{3.4}$$

It is easily seen that the pair F_i^j and g_{ij} defined in this way satisfy (0.1) and (0.2). Thus we have an almost product metric space M^n from the given contact manifold $\bar{M}^n$. In this section we restrict our consideration to the M^n thus obtained.

PROPOSITION 3.1. An almost product metric space M^n obtained from a contact manifold (as above) satisfies (2.5).

Proof. From (3.4) we have

$$\nabla_h F_i^j = 2(\nabla_h\xi^j)\eta_i + 2\xi^j(\nabla_h\eta_i)\,.$$

So, by (3.3) and (3.2) we have

$$\nabla_j F_i^j = 2(\nabla_j\xi^j)\eta_i + 2\xi^j(\nabla_j\eta_i) = 0\,.$$

PROPOSITION 3.2. For an almost product metric space M^n obtained from a contact manifold (as above) to satisfy (1.7), it is necessary and sufficient that ξ^i is a Killing vector field.

Proof. As $F_{ij} = 2\eta_i\eta_j - g_{ij}$, we have $(1/2)\nabla_h F_{ij} = (\nabla_h\eta_i)\eta_j + \eta_i(\nabla_h\eta_j)$, from which it follows

$$(3.5)\qquad \frac{1}{2}(\nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi})$$
$$= \eta_h(\nabla_i \eta_j + \nabla_j \eta_i) + \eta_i(\nabla_h \eta_j + \nabla_j \eta_h) + \eta_j(\nabla_i \eta_h + \nabla_h \eta_i) .$$

Thus, if (1.7) holds, we have the following by contracting ξ^h with the right hand side of (3.5):

$$(3.6)\qquad \nabla_i \eta_j + \nabla_j \eta_i + \eta_i(\xi^h \nabla_h \eta_j + \xi^h \nabla_j \eta_h) + \eta_j(\xi^h \nabla_i \eta_h + \xi^h \nabla_h \eta_i) = 0 .$$

Taking account of (3.1) and (3.3) we have from the above relation the following:

$$(3.7)\qquad \nabla_i \eta_j + \nabla_j \eta_i = 0 ,$$

that is, ξ^i is a Killing vector field. The converse is evident from (3.5).

REMARK 3.1. From Proposition 3.2 we see that (2.5) does not imply (1.7).

Let $\mathscr{L}_\xi$ denotes the Lie derivative with respect to an infinitesimal transformation ξ^i. S. Sasaki and Y. Hatakeyama have proved that $N^i_j = \mathscr{L}_\xi \phi^i_j = 0$ is equivalent to the condition that ξ^i is a Killing vector field, and also proved that the so-called normal contact metric manifold [7] satisfies $N^i_j = 0$. Thus we have:

PROPOSITION 3.3. If $\bar{M}^{2m+1}$ is a normal contact metric manifold, then the almost product metric space M^n obtained from $\bar{M}^{2m+1}$ satisfies (1.7).

PROPOSITION 3.4. If M^n is an almost product metric space obtained from a normal contact metric manifold, then F^j_i does not satisfy $\nabla_h F^j_i = 0$.

Proof. Suppose $\nabla_h F^j_i = 2\{(\nabla_h \xi^j)\eta_i + \xi^i(\nabla_h \eta_i)\} = 0$, we have by contracting with ξ^i the following:

$$\nabla_h \xi^j + \xi^j \xi^i(\nabla_h \eta_i) = 0,$$

from which we have by (3.1), $\nabla_h \xi^j = 0$ which contradict with the fact that $\nabla_h \xi^j = (1/2)\phi^j_h$.

REMARK 3.2. From Proposition 3.3 and Proposition 3.4, we see that (1.7) does not imply $\nabla_i F^h_j = 0$.

REMARK 3.3. All examples in § 3 satisfy also (1.8). Proposition 3.1. gives examples of [1.1], and Proposition 3.3 gives examples of [1.2].

REMARK 3.4. The almost product metric space M^n obtained from a normal contact metric space evidently does not satisfy $N_{ij}^h = 0$. But if ξ^i, η_j be any vector fields satisfying $\xi^i\eta_i = 1$, $g_{ij}\xi^i = \eta_j$, then in the almost product metric space M^n defined by g_{ij} and F_i^j of (3.4) with such ξ^i, η_j, it is easily seen that $N_{ij}^h = 0$ if and only if $\nabla_i\eta_j = \nabla_j\eta_i$ and that $\nabla_i F_j^i = 0$ if and only if $\nabla_i\xi^i = 0$. Such M^n gives example of [2.1].

4. We give in this section another example from tangent bundle of a riemannian space. Let R^m be an m-dimensional riemannian space and $M^n = T(R^m)$, $n = 2m$ be the tangent bundle of R^m, then the local coordinates of an element (a pair consisting a point x^α in R^m and a vector v^α at x^α) are $x^i = (x^\alpha, x^{\alpha*}) = (x^\alpha, v^\alpha)$. We assume in this section that $\alpha^* = m + \alpha$ etc. $\alpha, \beta, \gamma = 1, \cdots, m$; $\alpha^*, \beta^*, \gamma^* = m+1, \cdots, 2m$; $i, j, k = 1, \cdots, 2m = n$. Local coordinate transformation of M^n is given by

$$(4.1) \qquad \begin{pmatrix} \dfrac{\partial x'^\alpha}{\partial x^\beta} & 0 \\ \dfrac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\gamma} v^\gamma & \dfrac{\partial x'^\alpha}{\partial x^\beta} \end{pmatrix}.$$

From this transformation law it is easily seen that if ξ^α is a contravariant vector of R^m then $(\xi^\alpha, (\partial\xi^\alpha/\partial x^\beta)v^\beta)$ is a contravariant vector of $M^n = T(R^m)$ which is called the extended vector of ξ^α.

The riemannian connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ of the metric tensor $g_{\alpha\beta}$ of R^m gives rise a horizontal distribution in $M^n = T(R^m)$. It is known that if ξ^α is a contravariant vector field, then $(0, \xi^\alpha)$ and $\left(\xi^\alpha, -\left\{ {\alpha \atop \beta\gamma} \right\}\xi^\beta v^\gamma\right)$ are respectively a fundamental vector field and a horizontal vector field in $M^n = T(R^m)$ [2]. The fibres and horizontal distributions give rise to an almost product structure in $M^n = T(R^m)$. Its corresponding structure tensor F_i^j is given as follows [4]:

$$(4.2) \qquad \begin{cases} F_\beta^\alpha = -\delta_\beta^\alpha\,, & F_{\beta^*}^\alpha = 0\,, \\ F_\beta^{\alpha^*} = 2\left\{ {\alpha \atop \beta\rho} \right\} v^\rho\,, & F_{\beta^*}^{\alpha^*} = \delta_\beta^\alpha\,. \end{cases}$$

Let G_{ij} be the riemann metric introduced by S. Sasaki [8] in $T(R^m)$, that is

$$(4.3) \qquad \begin{cases} G_{\alpha\beta} = g_{\alpha\beta} + g_{\rho\sigma}\left\{ {\rho \atop \mu\alpha} \right\}\left\{ {\sigma \atop \nu\beta} \right\} v^\mu v^\nu\,, \\ G_{\alpha\beta^*} = g_{\rho\beta}\left\{ {\rho \atop \lambda\alpha} \right\} v^\lambda\,, \\ G_{\alpha^*\beta^*} = g_{\alpha\beta}\,, \end{cases}$$

then it is easily shown that F_i^j of (4.2) and G_{ij} satisfy (0.1) and (0.2), thus $M^n = T(R^m)$ is an almost product metric space. In this section, let $\bar{\nabla}$ denote the covariant derivative with respect to the riemannian connection $\overline{\left\{ {i \atop jk} \right\}}$ of G_{ij}:

$$(4.4)\quad \begin{cases} \overline{\left\{ {i \atop \beta^*\gamma^*} \right\}} = 0 , \qquad \overline{\left\{ {\alpha \atop \beta^*\gamma} \right\}} = \frac{1}{2} R^{\alpha}_{\lambda\beta\gamma} v^{\lambda} , \\ \overline{\left\{ {\alpha^* \atop \beta^*\gamma} \right\}} = \left\{ {\alpha \atop \beta\gamma} \right\} - \frac{1}{2} \left\{ {\alpha \atop \mu\eta} \right\} R^{\eta}_{\lambda\beta\gamma} v^{\lambda} v^{\mu} , \\ \overline{\left\{ {\alpha \atop \beta\ \gamma} \right\}} = \left\{ {\alpha \atop \beta\gamma} \right\} + \frac{1}{2}\left(R^{\alpha}_{\mu\eta\gamma} \left\{ {\eta \atop \lambda\beta} \right\} + R^{\alpha}_{\mu\eta\beta} \left\{ {\eta \atop \lambda\beta} \right\} \right) v^{\lambda} v^{\mu} , \\ \overline{\left\{ {\alpha^* \atop \beta\ \gamma} \right\}} = \frac{1}{2}\left(R^{\alpha}_{\gamma\lambda\beta} + R^{\alpha}_{\beta\lambda\gamma} + 2\partial_{\lambda} \left\{ {\alpha \atop \beta\lambda} \right\} \right) v^{\lambda} \\ \qquad \cdot \frac{1}{2} \left\{ {\alpha \atop \nu\eta} \right\} \left(R^{\eta}_{\rho\mu\gamma} \left\{ {\rho \atop \lambda\beta} \right\} + R^{\eta}_{\rho\mu\beta} \left\{ {\rho \atop \lambda\beta} \right\} \right) v^{\lambda} v^{\mu} v^{\nu} , \end{cases}$$

where $R^{\alpha}_{\lambda\beta\gamma}$ is the curvature tensor of $g_{\alpha\beta}$ in R^m. Then by straightforward computation we have the following:

$$(4.5)\quad \begin{cases} \bar{\nabla}_{\gamma} F^{\alpha}_{\beta} = R^{\alpha}_{\lambda\tau\gamma} \left\{ {\tau \atop \beta\rho} \right\} v^{\rho} v^{\lambda} , \qquad \bar{\nabla}_{\gamma} F^{\alpha}_{\beta^*} = R^{\alpha}_{\lambda\beta\gamma} v^{\lambda} , \\ \bar{\nabla}_{\gamma} F^{\alpha^*}_{\beta} = (R^{\alpha}_{\gamma\lambda\beta} - R^{\alpha}_{\beta\lambda\gamma}) v^{\lambda} - R^{\eta}_{\lambda\tau\gamma} \left\{ {\alpha \atop \mu\eta} \right\} \left\{ {\tau \atop \beta\rho} \right\} v^{\lambda} v^{\mu} v^{\rho} , \\ \bar{\nabla}_{\gamma} F^{\alpha^*}_{\beta^*} = -R^{\tau}_{\lambda\beta\gamma} \left\{ {\alpha \atop \tau\rho} \right\} v^{\lambda} v^{\rho} , \qquad \bar{\nabla}_{\gamma^*} F^{\alpha}_{\beta} = 0 , \\ \bar{\nabla}_{\gamma^*} F^{\alpha}_{\beta^*} = 0 , \qquad \bar{\nabla}_{\gamma^*} F^{\alpha^*}_{\beta} = 0 , \qquad \bar{\nabla}_{\gamma^*} F^{\alpha^*}_{\beta^*} = 0 . \end{cases}$$

We have also

$$(4.6)\quad \begin{cases} \bar{\nabla}_{\gamma} F_{\alpha\beta} = g_{\alpha\rho} R^{\rho}_{\lambda\nu\gamma} \left\{ {\nu \atop \beta\varepsilon} \right\} v^{\lambda} v^{\varepsilon} + g_{\delta\rho} (R^{\rho}_{\gamma\lambda\beta} - R^{\rho}_{\beta\lambda\gamma}) \left\{ {\delta \atop \varepsilon\alpha} \right\} v^{\lambda} v^{\varepsilon} , \\ \bar{\nabla}_{\gamma} F_{\alpha^*\beta} = g_{\alpha\rho} (R^{\rho}_{\gamma\lambda\beta} - R^{\alpha}_{\beta\lambda\gamma}) v^{\lambda} , \quad \bar{\nabla}_{\gamma} F_{\alpha^*\beta^*} = 0 , \\ \bar{\nabla}_{\gamma^*} F_{\alpha^*\beta} = 0 , \qquad \bar{\nabla}_{\gamma^*} F_{\alpha\beta} = 0 , \qquad \bar{\nabla}_{\gamma^*} F_{\alpha^*\beta^*} = 0 . \end{cases}$$

As $F_{\alpha\beta}$ is symmetric in α, β it is the same for $\bar{\nabla}_{\gamma} F_{\alpha\beta}$, so we have

$$(4.7)\quad \begin{aligned} \bar{\nabla}_{\gamma} F_{\alpha\beta} = & \frac{1}{2}\left[g_{\alpha\rho} R^{\rho}_{\lambda\nu\gamma} \left\{ {\nu \atop \beta\varepsilon} \right\} + g_{\beta\rho} R^{\rho}_{\lambda\nu\gamma} \left\{ {\nu \atop \alpha\varepsilon} \right\} \right] v^{\lambda} v^{\varepsilon} \\ & \frac{1}{2} g_{\delta\rho} \left[(R^{\rho}_{\gamma\lambda\beta} - R^{\rho}_{\beta\lambda\gamma}) \left\{ {\delta \atop \varepsilon\alpha} \right\} + (R^{\rho}_{\gamma\lambda\alpha} - R^{\rho}_{\alpha\lambda\gamma}) \left\{ {\delta \atop \varepsilon\beta} \right\} \right] v^{\lambda} v^{\varepsilon} . \end{aligned}$$

Using (4.6), the expression (4.7) and then making use of $R_{\lambda\nu\alpha\gamma} = -R_{\lambda\nu\gamma\alpha}$ we have finally:

$$\bar{\nabla}_i F_{jk} + \bar{\nabla}_j F_{ki} + \bar{\nabla}_k F_{ij} = 0 .$$

Thus we have

PROPOSITION 4.1. If M^n is an almost product metric space obtained from the tangent bundle of a riemannian space, then M^n satisfies (1.7).

REMARK 4.1. As the fibres of $T(R^m)$ evidently constitute a system of completely integrable distribution, so $M^n = T(R^m)$ is a foliated metric manifold.

REMARK 4.2. It is known [8] that the horizontal distribution is also completely integrable if and only if R^m is flat. In this case $N_{ij}^h = 0$. Therefore we see that (1.7) does not imply $N_{ij}^h = 0$.

By the way, we give here one more proposition for M^n obtained from the tangent bundle of a riemannian space:

PROPOSITION 4.2. Let $V^i = (u^\alpha, v^\mu \partial_\mu u^\alpha)$ be the extended vector field of u^α. Then $\mathscr{L}_V F_i^j = 0$ if and only if u^α is an affine transformation in R^m.

Proof. By making use of the formula

$$\begin{aligned} t_{\beta\alpha}^\eta &= \nabla_\beta \nabla_\alpha u^\eta + u^\rho R_{\rho\beta\alpha}^\eta \\ &= \partial_{\beta\alpha} u^\eta + u^\rho \partial_\rho \left\{ {\eta \atop \beta\alpha} \right\} + \left\{ {\eta \atop \beta\rho} \right\} \partial_\alpha u^\rho + \left\{ {\eta \atop \alpha\rho} \right\} \partial_\beta u^\rho - \left\{ {\rho \atop \beta\alpha} \right\} \partial_\rho u^\eta \end{aligned} \tag{4.8}$$

we have

$$\begin{cases} \mathscr{L}_V F_\lambda^\kappa = 0 \,, & \mathscr{L}_V F_\lambda^{\kappa^*} = 2 v^\sigma t_{\lambda\delta}^\kappa \,, \\ \mathscr{L}_V F_{\lambda^*}^\kappa = 0 \,, & \mathscr{L}_V F_{\lambda^*}^{\kappa^*} = 0 \,, \end{cases} \tag{4.9}$$

from which the proposition follows.

REMARK 4.3. Proposition 4.2. is an analogue of a theorem of Tachibana and Okumura [11].

5. In this section we assume that M^n satisfies (2.5). Let $\mathscr{L}_v$ denote as above the Lie derivative with respect to an infinitesimal transformation v^i. It is well known that

$$\mathscr{L}_v \nabla_j F_i^h - \nabla_j \mathscr{L}_v F_i^h = F_i^l \mathscr{L}_v \left\{ {h \atop jl} \right\} - F_l^h \mathscr{L}_v \left\{ {l \atop ji} \right\} . \tag{5.1}$$

Contract with j, h and then make use of (2.5) we have

$$-\nabla_l \mathscr{L}_v F_i^l = F_i^m \mathscr{L}_v \left\{ {l \atop lm} \right\} - F_m^l \mathscr{L}_v \left\{ {m \atop li} \right\} . \tag{5.2}$$

PROPOSITION 5.1. In a compact orientable almost product metric space satisfying (2.5), if an infinitesimal projective transformation at the same time leaves F_i^j invariant, then it is an isometry.

Proof. If an infinitesimal projective transformation v^i at the same time leaves F_i^j invariant, then we have

$$\mathscr{L}_v\left\{ {h \atop ji} \right\} = \delta_j^h \psi_i + \delta_i^h \psi_j \tag{5.3}$$

and

$$\mathscr{L}_v F_i^h = 0 \ . \tag{5.4}$$

Substitute these two relations in (5.2) we have

$$(nF_i^m - \delta_i^m F_l^l)\psi_m = 0 \ . \tag{5.5}$$

But we have

$$(nF_i^m - \delta_i^m F_l^l)(nF_m^t + \delta_m^t F_l^l) = \delta_i^t \{n^2 - (F_l^l)^2\} \neq 0 \tag{5.6}$$

as $(F_l^l)^2 = (n - 2p)^2 < n^2$. Thus $nF_i^m - \delta_i^m F_l^l$ is nonsingular and we have $\psi_m = 0$, from which and (5.3) we have

$$\psi_m = \frac{1}{n+1} \nabla_m \nabla_l v^l = 0 \ . \tag{5.7}$$

Thus $\nabla_l v^l =$ constant and $\mathscr{L}_v\left\{ {h \atop ji} \right\} = 0$. Now if the space is compact and orientable, then by use of Green's theorem we have $\nabla_l v^l = 0$ from which and $\mathscr{L}_v\left\{ {h \atop ji} \right\} = 0$ follows the proposition.

PROPOSITION 5.2. In an compact orientable almost product metric space M^n satisfying (2.5), if an infinitesimal conformal transformation leaves at the same time F_i^j invariant, then it is an isometry.

Proof. It is known that

$$\mathscr{L}_v\left\{ {h \atop ji} \right\} = \frac{1}{2} g^{hn}(\nabla_j \mathscr{L}_v g_{ni} + \nabla_i \mathscr{L}_v g_{jn} - \nabla_n \mathscr{L}_v g_{ji}) \ . \tag{5.8}$$

Substitute (5.8) and (5.4) in (5.2) we have

$$F_i^m g^{ln} \nabla_m \mathscr{L}_v g_{ln} - F^{nl} \nabla_i \mathscr{L}_v g_{ln} = 0 \ . \tag{5.9}$$

From $\mathscr{L}_v g_{ln} = \nabla_l v_n + \nabla_n v_l = 2\phi g_{ln}$, we have $\nabla_i \mathscr{L}_v g_{ln} = 2\phi_i g_{ln}$. Substitute this formula into (5.9) we have

$$F_i^m \nabla_m \nabla_l v^l - \phi_i F_l^l = 0 \ . \tag{5.10}$$

But as $\phi = (1/n)\nabla_l v^l$ and $\phi_i = (1/n)\nabla_i\nabla_l v^l$, the above equation is written as

$$\left(F_i^m - \frac{1}{n}\delta_i^m F_t^t\right)\nabla_m\nabla_l v^l = 0 \; . \tag{5.11}$$

From (5.11) we can get the proof of proposition 5.2 just as in the proof of Proposition 5.1.

PROPOSITION 5.3. In an almost product metric space M^n satisfying (2.5) we have

$$(\nabla_h F^{ji})(\nabla_j F_i^h) = \bar{H} - R \,, \tag{5.12}$$

where $R_{jl} = R_{hjl}^h$, $R = g^{lj}R_{jl}$, $\bar{H}_{ji} = R_{hjil}F^{lh}$ and $\bar{H} = F^{ji}\bar{H}_{ji}$.

Proof. Contract with h, k in the following well-known formula

$$\nabla_k\nabla_j F_i^h - \nabla_j\nabla_k F_i^h = R_{kjl}^h F_i^l - R_{kji}^l F_l^h \,, \tag{5.13}$$

and then make use of (2.5) we have

$$\nabla_l\nabla_j F_i^l = R_{jl}F_i^l - \bar{H}_{ji} \; . \tag{5.14}$$

Differentiate $F^{ji}F_{ih} = \delta_h^j$ with ∇_j, then we have

$$F^{ji}\nabla_j F_{ih} = 0 \tag{5.15}$$

by (2.5). From (5.15) we have

$$\begin{aligned}(\nabla_h F^{ji})(\nabla_j F_i^h) &= -F^{ji}(\nabla_h\nabla_j F_i^h) \\ &= -F^{ji}(R_{jl}F_i^l - \bar{H}_{ji}) \qquad \text{by (5.14)} \\ &= \bar{H} - R\end{aligned}$$

as $F_i^l F^{ji} = g^{lj}$.

PROPOSITION 5.4. In an almost product metric space M^n satisfying (1.7) the relation $\bar{H} \leqq R$ holds. The equality holds if and only if the space satisfies $\nabla_i F_j^h = 0$.

Proof. By (1.7) we have

$$(\nabla^h F^{ji})(\nabla_j F_{ih}) = -(\nabla^h F^{ji})(\nabla_i F_{hj} + \nabla_h F_{ji}) \; . \tag{5.16}$$

From which we have

$$2(\nabla^h F^{ji})(\nabla_j F_{ih}) = -(\nabla^h F^{ji})(\nabla_h F_{ji}) \leqq 0 \; .$$

Then Proposition 5.4 follows.

REMARK 5.1. Propositions in this section are analogues of some theorems of Koto [5].

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