ON THE SPECTRUM OF A TOEPLITZ OPERATOR

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1. Introduction. A Toeplitz operator T is one which transforms a sequence $x = (x_0, x_1, x_2 \cdots)$ into a sequence y according to the formal law

$$\sum_{k=0}^{\infty} c_{n-k} x_k = y_n \,, \qquad n = 0, 1, 2, \cdots . \tag{1}$$

If the complex coefficients c_n satisfy the condition

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty \,, \tag{2}$$

then T carries each l_p space $(1 \leqq p \leqq \infty)$ into itself. Here l_p is the Banach space of all complex sequences $x = (x_0, x_1, \cdots)$ for which the norm

$$\|x\|_p = \left\{\sum_{n=0}^{\infty} |x_n|^p\right\}^{1/p}$$

is finite. As usual, $\|x\|_\infty = \sup |x_n|$.

Under the assumption (2), M. G. Krein [7] has described the spectrum of T as an operator in l_p. His method uses some rather deep theorems on the factorization of absolutely convergent Fourier series. Actually, Krein's emphasis is on the Wiener-Hopf integral operator, which is the continuous analogue of T. Without knowledge of Krein's work, Calderón, Spitzer, and Widom [4] used similar methods to obtain most of the same results on the spectrum of T.

The key to the spectrum of T, in any l_p space, is the continuous closed curve Γ defined by

$$\lambda = F(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \,, \qquad 0 \leqq \theta < 2\pi . \tag{3}$$

For any point $\lambda \notin \Gamma$, it is the winding number of Γ about λ which alone determines the exact spectral character of λ. The precise results, which are due to Krein, will be stated below. Since the spectrum is always a closed set, it follows from these results that the entire curve Γ belongs to the spectrum of T. There remains, however, the finer question: for what reason is a point $\lambda \in \Gamma$ in the spectrum? That is, to which part of the spectrum does λ belong? For operators T satisfying

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the general condition (2), only fragmentary answers to this question are known.

We shall be able to supply a complete answer for the special class of Toeplitz operators T such that

$$(4) \qquad c_n = 0\,, \quad n > l\,, \quad n < -m\,; \quad c_l \neq 0\,, \quad c_{-m} \neq 0\,.$$

Here l and m are certain non-negative integers. Such an operator will be called a *multidiagonal Toeplitz operator*; its associated Toeplitz matrix has at most $l + m + 1$ non-vanishing diagonals. The pair of numbers (l, m) will be called the *diagonal index* of T.

In this paper, Krein's results will be derived, for multidiagonal Toeplitz operators, by an entirely different method, relying upon the elementary theory of difference equations with constant coefficients. This approach gives new insight into the theory, and it has the additional advantage of providing a complete spectral classification of the points on the curve Γ. Here the results go beyond those of Krein. Essentially the same methods will also be applied to yield partial results on the spectra of more general operators. I did most of this work, and presented it in my thesis [5], before I became aware of Krein's paper, where the major features of the spectrum were already described. However, continued interest in the part of the spectrum which lies on the curve Γ; in particular, recent work of A. Brown and P. R. Halmos [3], has seemed to recommend publication.

2. Statement of results. Before the theorems can be stated, we must explain some terminology. For any bounded linear operator T on a Banach space X, the resolvent set $\rho(T)$ is the set of all complex numbers λ for which $(T - \lambda I)$ is one-to-one onto X. The *spectrum* $\sigma(T)$ is the complement of $\rho(T)$, and is composed of three not necessarily disjoint parts. The *point spectrum* $\sigma_p(T)$ is the set of all λ such that $(T - \lambda I)$ is not one-to-one. The dimension of the null-space of $(T - \lambda I)$ is called the *multiplicity* of λ. The *compression spectrum* $\sigma_c(T)$ consists of those λ for which $(T - \lambda I)X$ is not dense, and the deficiency of λ is the number of linearly independent functionals in the dual space X^* which annihilate $(T - \lambda I)X$. Finally, the *essential spectrum* $\sigma_e(T)$ is the set of all λ for which $(T - \lambda I)X$ is not closed.

Now let T be a Toeplitz operator whose coefficients satisfy (2), and let Γ be the associated curve defined in (3) by the continuous function $F(\theta)$. For any point $\lambda \notin \Gamma$, the finite integer

$$(5) \qquad n = n(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} d \arg \{F(\theta) - \lambda\}$$

is called the *winding number* of Γ about λ. Krein's main results may be expressed as follows.

THEOREM (Krein). *Let T be a Toeplitz operator satisfying* (2), *defined in any l_p space, $1 \leqq p \leqq \infty$. Then*

1. $\lambda \in \rho(T)$ *if and only if* $\lambda \notin \Gamma$ *and* $n(\lambda) = 0$.
2. *If* $\lambda \notin \Gamma$ *and* $n(\lambda) < 0$, *then* $\lambda \in \sigma_p(T)$ *with multiplicity* $|n|$, *while* $\lambda \notin \sigma_c(T)$ *and* $\lambda \notin \sigma_e(T)$.
3. *If* $\lambda \notin \Gamma$ *and* $n(\lambda) > 0$, *then* $\lambda \in \sigma_c(T)$ *with deficiency* n, *while* $\lambda \notin \sigma_p(T)$ *and* $\lambda \notin \sigma_e(T)$.
4. $\sigma_e(T) \subset \Gamma$. *If* $\lambda \in \Gamma$ *and if either the multiplicity or the deficiency of* λ *is finite, then* $\lambda \in \sigma_e(T)$.

As to whether or not a point on the curve Γ can belong to the point or compression spectrum, the theorem gives no information.

Now let T be a *multidiagonal* Toeplitz operator with diagonal index (l, m). In terms of the coefficients of T, let us construct the polynomial

$$Q(z; \lambda) = \sum_{k=-l}^{m} c_{-k} z^{l+k} - \lambda z^l \tag{6}$$

depending upon the complex parameter λ. Because of our assumptions $c_{-m} \neq 0$, $c_l \neq 0$, Q is of exact degree $l + m$, and $Q(0; \lambda) \neq 0$. For fixed λ, let $s_1 = s_1(\lambda)$ denote the number of zeros of Q, multiplicities counted, which lie in $|z| < 1$. Let $s_2 = s_2(\lambda)$ be the number of zeros in $|z| > 1$. Then $s_1(\lambda) + s_2(\lambda) \leqq l + m$, with equality if and only if $Q(z; \lambda)$ vanishes nowhere on $|z| = 1$. Finally, let $u = u(\lambda)$ denote the number of distinct zeros of Q (multiplicities *not* counted) on $|z| = 1$.

THEOREM 1. *Let T be a multidiagonal Toeplitz operator with diagonal index (l, m), acting on the space l_p $(1 \leqq p \leqq \infty)$. Then*

1. $\lambda \in \rho(T)$ *if and only if* $s_1(\lambda) = l$ *and* $s_2(\lambda) = m$.
2. *For* $1 \leqq p < \infty$, $\lambda \in \sigma_p(T)$ *if and only if* $s_1(\lambda) > l$, *and the multiplicity is* $s_1 - l$. *In* l_∞, $\lambda \in \sigma_p(T)$ *if and only if* $s_1(\lambda) + u(\lambda) > l$, *and the multiplicity is* $s_1 + u - l$.
3. *For* $1 < p < \infty$, $\lambda \in \sigma_c(T)$ *if and only if* $s_2(\lambda) > m$, *and the deficiency is* $s_2 - m$. *In* l_1, $\lambda \in \sigma_c(T)$ *if and only if* $s_2(\lambda) + u(\lambda) > m$, *and the deficiency is* $s_2 + u - m$. *In* l_∞, $s_2(\lambda) > m$ *implies* $\lambda \in \sigma_c(T)$ *with deficiency* $\geqq s_2 - m$; *a point* $\lambda \notin \Gamma$ *belongs to* $\sigma_c(T)$ *if and only if* $s_2(\lambda) > m$, *and the deficiency is* $s_2 - m$.
4. $\sigma_e(T) = \Gamma$.

It is not difficult to show that Krein's results follow from Theorem 1 in case T is a multidiagonal operator. The curve Γ is exactly the set of points λ for which $Q(z; \lambda)$ has one or more zeros on the circle $|z| = 1$. For $\lambda \notin \Gamma$, it follows from the argument principle that

$$s_1(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} d \arg \{Q(e^{i\theta}; \lambda)\} = l - n(\lambda) ,$$

where $n(\lambda)$ is the winding number defined in (5). Also, for $\lambda \notin \Gamma$, $s_1(\lambda) + s_2(\lambda) = l + m$, so $s_2(\lambda) - m = n(\lambda)$. These relations make it clear that the two formulations of the results are equivalent for points $\lambda \notin \Gamma$. However, the formulation in terms of the zeros of $Q(z; \lambda)$ remains meaningful for λ on the curve Γ, where the winding number $n(\lambda)$ is no longer defined.

3. **Proof of Theorem 1.** Let us first record a few fundamentals for reference. The deficiency of the range $(T - \lambda I)l_p$ is the dimension of its annihilator in l_p^*. However, each functional in l_p^* is uniquely representable, for $1 \leqq p < \infty$, in terms of an element of l_q, where $1/p + 1/q = 1$. It follows that (for $1 \leqq p < \infty$) the deficiency of $\lambda \in \sigma_c(T)$ is exactly the multiplicity of λ as an eigenvalue of T^* in l_q, where T^* denotes[1] the Toeplitz operator generated by the transpose of the matrix generating T.

In order to determine $\sigma_p(T)$ for a multidiagonal Toeplitz operator T, let us consider the formal system of equations $Tx = \lambda x$ in infinitely many unknowns $x_0, x_1, \cdots$. Under the assumption (4), this system may be viewed as a difference equation

$$\sum_{k=-l}^{m} c_{-k} x_{n+k} = \lambda x_n , \qquad n = 0, 1, 2, \cdots \tag{7}$$

whose solutions $(x_{-l}, x_{-l+1}, \cdots)$ are required to satisfy

$$x_{-l} = x_{-l+1} = \cdots = x_{-1} = 0 . \tag{8}$$

The multiplicity of λ as an eigenvalue of T in l_p is the number of linearly independent solutions to (7) and (8) which satisfy $\| x \|_p < \infty$.

The general solution to the difference equation (7) may be expressed in terms of the zeros of the "characteristic polynomial" $Q(z; \lambda)$ defined in (6). Let $z_1, z_2, \cdots, z_r$ denote the distinct zeros of Q arranged so that $0 < |z_1| \leqq |z_2| \leqq \cdots \leqq |z_r|$, and let μ_k be the multiplicity of z_k. Hence $\mu_1 + \cdots + \mu_r = l + m$.

According to the elementary theory of difference equations [6, Chap. VII], the general solution to (7) is

$$x_n = \sum_{k=1}^{r} P_k(n + l)\, z_k^{n+l} , \qquad n = -l, -l + 1, \cdots , \tag{9}$$

where

$$P_k(\xi) = \sum_{j=0}^{\mu_k - 1} \alpha_{kj} \xi^j \tag{10}$$

[1] It is well to note that T^*, which is equivalent to the Banach space adjoint of T, differs from the Hilbert space adjoint (when $p = 2$) by complex conjugation.

is an arbitrary polynomial of degree no greater than $\mu_k - 1$.

LEMMA 1. *Let* $w_0, w_1, \cdots, w_k$ *be distinct complex numbers of modulus one, and let* $a_0, a_1, \cdots, a_k$ *be arbitrary nonzero complex constants. Then the sequence*

$$t_n = \sum_{j=0}^{k} a_j w_j^n$$

has the property $\limsup_{n\to\infty} |t_n| > 0$.

Proof. Suppose, on the contrary, that $t_n \to 0$. Then

$$s_n = \sum_{j=1}^{k} b_j \zeta_j^n \to -1 , \tag{11}$$

where $b_j = a_j/a_0$ and $\zeta_j = w_j/w_0 \neq 1$. If (11) were true, then the arithmetic means

$$\sigma_N = \frac{1}{N}\sum_{n=1}^{N} s_n = \sum_{j=1}^{k} b_j \left[\frac{1}{N}\sum_{n=1}^{N}\zeta_j^n\right]$$

would also tend to -1 as $N \to \infty$. But

$$\left|\frac{1}{N}\sum_{n=1}^{N}\zeta_j^n\right| \leqq \frac{2}{N|1-\zeta_j|} \to 0 \text{ as } N \to \infty ,$$

$j = 1, \cdots, k$. This contradicts the conclusion $\sigma_N \to -1$, and proves the lemma.

LEMMA 2. *Let* $h = h(\lambda)$ *denote the number of distinct zeros of* $Q(z;\lambda)$ *which satisfy* $|z| < 1$. *Then* x *in* (9) *has finite* l_p *norm* $(1 \leqq p < \infty)$ *if and only if*

$$P_k(\xi) \equiv 0 , \qquad k = h+1, \cdots, r . \tag{12}$$

In fact, x_n *does not tend to zero if* (12) *fails to hold.*

Proof. Application of Minkowski's inequality shows easily that (12) implies $x \in l_p$. Conversely, suppose (12) is not satisfied, and let κ be the largest value of k for which $P_k(\xi) \not\equiv 0$. Thus $\rho = |z_\kappa| \geqq 1$. Let ω be the highest power of ξ which appears (with non-vanishing coefficient) in those polynomials $P_k(\xi)$ such that $|z_k| = \rho$, and let $k_1, k_2, \cdots, k_s$ be the values of k ($|z_k| = \rho$) for which ξ^ω appears in $P_k(\xi)$. Set $z_{k_\nu} = \rho e^{i\theta_\nu}$ ($\nu = 1, \cdots, s$), and construct the sequence

$$t_n = x_n(n+l)^{-\omega}\rho^{-(n+l)} = \sum_{\nu=1}^{s} A_\nu e^{i(n+l)\theta_\nu} + O\left(\frac{1}{n}\right) , \tag{13}$$

where, in the notation of (10), $A_\nu = \alpha_{k_\nu \omega} \neq 0$. By Lemma 1, t_n does not tend to zero; hence the same must be true for x_n.

COROLLARY. *The multiplicity of* λ *as an eigenvalue of* T *in* $l_p\,(1 \leqq p < \infty)$ *is the number of linearly independent solutions* $(\alpha_{10}, \cdots, \alpha_{1,\mu_1-1}, \alpha_{20}, \cdots, \alpha_{h,\mu_h-1})$ *to the equations*

$$(14) \qquad \sum_{k=1}^{h}\left[\alpha_{k0} + \sum_{j=1}^{\mu_k-1} \alpha_{kj}(n+l)^j\right] z_k^{n+l} = 0\,, \quad n = -l, -l+1, \cdots, -1\,.$$

LEMMA 3. *Let* $h = h(\lambda)$ *be the number of distinct zeros of* $Q(z; \lambda)$ *in* $|z| < 1$, *and let* $u = u(\lambda)$ *be the number of distinct zeros on* $|z| = 1$. *Then* x *in* (9) *has finite* l_∞ *norm if and only if*

$$(15) \qquad \begin{aligned} P_k(\xi) &\equiv \alpha_{k0}\,, \qquad & k &= h+1, \cdots, h+u\,; \\ P_k(\xi) &\equiv 0\,, \qquad & k &= h+u+1, \cdots, r\,. \end{aligned}$$

Proof. Conditions (15) are obviously sufficient for the boundedness of x. To prove necessity, construct the sequence (13) as in the proof of Lemma 2. Once again, $\limsup_{n\to\infty} |t_n| > 0$. Thus $\limsup_{n\to\infty} |x_n| = \infty$ if either $\rho > 1$ or if $\rho = 1$ and $\omega > 0$; that is, if conditions (15) fail to hold.

COROLLARY. *The multiplicity of* λ *as an eigenvalue of* T *in* l_∞ *is the number of linearly independent solutions* $(\alpha_{10}, \cdots, \alpha_{h,\mu_h-1}, \alpha_{h+1,0}, \cdots, \alpha_{h+u,0})$ *to the equations*

$$(16) \qquad \sum_{k=1}^{h+u} \alpha_{k0} z_k^{n+l} + \sum_{k=1}^{h} \sum_{j=1}^{\mu_k-1} \alpha_{kj}(n+l)^j z_k^{n+l} = 0\,, \qquad n = -l, -l+1, \cdots, -1\,.$$

In the system (14) there are l equations in $\nu = \sum_{k=1}^{h} \mu_k$ unknowns. The rank of the coefficient matrix is $\min\{l, \nu\}$. Hence the familiar theorem "rank + nullity = dimension of domain" [2, p. 227] of linear algebra tells us that (14) has exactly $\nu - \min\{l, \nu\}$ independent solutions. But $\nu = s_1$, so the part of Theorem 1 concerning point spectrum is established for $1 \leqq p < \infty$.

Similarly, (16) has $\nu + u - \min\{l, \nu + u\}$ independent solutions. This proves the statement about point spectrum in l_∞.

As for $\sigma_c(T)$, the remarks made in the beginning of this section show that it is equivalent, for $1 \leqq p < \infty$, to study the point spectrum of T^* in l_q. The characteristic polynomial associated with T^* is

$$Q^*(z; \lambda) = \sum_{k=-m}^{l} c_k z^{m+k} - \lambda z^m\,.$$

Because of the identity

$$Q(z;\lambda) = z^{l+m}Q^*(1/z;\lambda)\ ,$$

the zeros of Q^* are just the reciprocals of those of Q. In particular, with obvious notation, $s_1^*(\lambda) = s_2(\lambda)$; $s_2^*(\lambda) = s_1(\lambda)$. Hence our previous results on $\sigma_p(T)$ can be applied to yield the part of Theorem 1 which concerns $\sigma_c(T)$. The l_∞ case presents a difficulty because l_∞^* is larger than l_1; we therefore obtain only the sufficient condition $s_2(\lambda) > m$ for a point λ to belong to $\sigma_c(T)$ in this case. For completeness, however, we have included in Theorem 1 a statement of Krein's more precise result for $\lambda \notin \Gamma$.

We now know that for every λ, the null-space of $(T - \lambda I)$ in l_p, $1 \leqq p \leqq \infty$, is of *finite* dimension. Thus we have only to apply Krein's general result to conclude that $\sigma_e(T) = \Gamma$. It is clear that $\lambda \in \Gamma$ if and only if $s_1(\lambda) + s_2(\lambda) < l + m$.

The assertion concerning $\rho(T)$ now follows by elimination.

4. Completely continuous perturbations. The preceding developments were founded upon the explicit formula (9) for solutions to a difference equation with constant coefficients, but it was the *asymptotic* behavior of the solutions which was of primary interest. For this reason, the theory can be partially recaptured for a more general class of operators leading to the consideration of linear difference equations whose variable coefficients converge as n becomes infinite. It is plausible that the solutions to such an equation should behave asymptotically as though the coefficients were constant, and indeed there are theorems of Poincaré and Perron [8] which affirm this.

Let A be an operator on $l_p\,(1 \leqq p \leqq \infty)$ whose associated matrix (a_{jk}) has the property $a_{jk} = 0$ for $j - k > l$ or for $k - j > m$. (Once again, l and m are non-negative integers.) Such an operator A will be called a *multidiagonal operator*—it need not be a Toeplitz operator. If, in addition, $a_{j+l,j} \neq 0$ and $a_{j,j+m} \neq 0\,(j = 0, 1, \cdots)$, we shall say that A has *diagonal index* (l, m).

It is not difficult to prove [1, p. 63], for $1 < p < \infty$, that the completely continuous multidiagonal operators are precisely the ones whose diagonal sequences tend to zero:

$$\lim_{j\to\infty} a_{j+k,j} = 0\ , \qquad\qquad k = -m, -m+1, \cdots, l\ .$$

Suppose now that A is a multidiagonal operator with diagonal index (l, m), and suppose

$$\lim_{j\to\infty} a_{j,j+k} = c_{-k},\ k = -l, -l+1, \cdots, m;\ c_l \neq 0,\ c_{-m} \neq 0\ . \tag{17}$$

A is therefore the sum of a Toeplitz operator and a completely continuous operator. The equation $Ax = \lambda x$ may again be regarded as a difference equation

$$\sum_{k=-l}^{m} a_{n,n+k}x_{n+k} = \lambda x_n , \qquad n = 0, 1, \cdots \tag{18}$$

whose solutions $(x_{-l}, x_{-l+1}, \cdots)$ are required to satisfy

$$x_{-l} = x_{-l+1} = \cdots = x_{-1} = 0 . \tag{19}$$

The solutions to (18) can no longer be written down explicitly, but under the hypothesis (17) their asymptotic behavior is determined by the zeros of the characteristic polynomial $Q(z; \lambda)$, as defined in (6). We shall content ourselves with assuming that the zeros of $Q(z; \lambda)$ have distinct moduli:

$$0 < |z_1| < |z_2| < \cdots < |z_{l+m}| .$$

A theorem of Poincaré [8, p. 305] then asserts that the difference equation (18) has a basis of solutions $x_n = u_n^{(j)}$ for which

$$\lim_{n\to\infty} u_{n+1}^{(j)}/u_n^{(j)} = z_j , \qquad j = 1, 2, \cdots, l + m . \tag{20}$$

Assuming further that $\lambda \notin \Gamma$, so that no $|z_j| = 1$, one readily uses (20) to prove, as in § 3, that the multiplicity of λ as an eigenvalue of A in l_p $(1 \leqq p \leqq \infty)$ is the number of linearly independent solutions $(b_1, \cdots, b_s)$ to the equations

$$\sum_{k=1}^{s} b_k u_n^{(k)} = 0 , \qquad n = -l, -l + 1, \cdots, -1 , \tag{21}$$

where $s = s_1(\lambda)$ is the number of z_j inside $|z| < 1$. The precision of our final result is curtailed by the apparent fact that the coefficient matrix of (21) may contain linearly dependent rows, even if $s \geqq l$. We phrase the theorem in terms of the winding number $n(\lambda)$ of the curve Γ about a point $\lambda \notin \Gamma$.

THEOREM 2. *Let A be a multidiagonal operator in l_p $(1 \leqq p \leqq \infty)$ with diagonal index (l, m), and suppose* (17) *is satisfied. Suppose further that $\lambda \notin \Gamma$ and that the $l + m$ zeros of $Q(z; \lambda)$ have distinct moduli. Then*

1. *$n(\lambda) < 0$ implies $\lambda \in \sigma_p(A)$ with multiplicity $\geqq |n|$.*
2. *$n(\lambda) > 0$ implies $\lambda \in \sigma_c(A)$ with deficiency $\geqq n$.*

Roughly speaking, the theorem says that the point and compression spectra of a multidiagonal Toeplitz operator, together with multiplicity and deficiency, are not diminished by the addition of a completely continuous operator of the same multidiagonal form. The hypothesis that the zeros have distinct moduli means geometrically that for no r

$(0 < r < \infty)$ does a curve

$$w = \sum_{k=-l}^{m} c_{-k} r^k e^{ik\theta} \qquad (0 \leqq \theta < 2\pi)$$

have a double-point or a cusp at λ.

The conclusion (1) remains valid if only the zeros in $|z| > 1$ have distinct moduli. Dually, (2) is still true if only the zeros in $|z| < 1$ have distinct moduli.

5. Remarks. It is interesting to contrast the spectrum of a Toeplitz operator with that of a completely continuous operator. The spectrum of the latter is a discrete set of points which can cluster only at the origin. Only the origin may belong to the essential spectrum, while any other point of the spectrum belongs simultaneously to σ_p and to σ_c, with equal multiplicity and deficiency. (This is the content of the Fredholm alternative.) For Toeplitz operators the situation is radically different. According to Krein's theorem, a point $\lambda \notin \Gamma$ cannot belong both to $\sigma_p(T)$ and to $\sigma_c(T)$. Our Theorem 1 shows that for *multidiagonal* Toeplitz operators (and for $p < \infty$), the point and compression spectra are in fact disjoint. It is a reasonable conjecture that this is true generally.

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