NORMAL FORM FOR A PFAFFIAN

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1. Introduction. It is well known that "generally" (which is to is the say, usually) a Pfaffian, or 1-form,

$$\alpha = a_1(x)dx^1 + \cdots + a_n(x)dx^n$$

in $\boldsymbol{R}^n$ has one or the other of the two representations

1.1 $$\alpha = u^1du^2 + \cdots u^{2p-1}du^{2p} + \begin{cases} 0 \\ du^{2p+1} \end{cases}$$

in an appropriate coordinate system $(u^1, u^2, \cdots, u^n)$. Moreover, the last index ($2p$ or $2p + 1$) appearing in 1.1 is the rank r of the $n \times (n + 1)$ matrix

1.2 $$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & x_{2n} \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

in which a_{ij} is an abbreviation for $\partial a^i/\partial x^i - \partial a_j/\partial x^i$.

It goes without saying that this is regarded as a *local* proposition, indicating that if the rank of 1.2 were constant in some neighborhood of a point $\boldsymbol{P}_0$, then a smaller neighborhood of $\boldsymbol{P}_0$ and a curvilinear coordinate system valid on that neighborhood, could be found yielding the representation 1.1.

It is very probable that a satisfactory proof concerning the possibility of reducing a Pfaffian in this way exists in the literature[1]. Nevertheless, it should be pointed out that the accepted version is not exactly true (and this is part of our object in writing this paper.)

Consider the Pfaffian $ydx + 2xdy$ in ordinary $\boldsymbol{R}^2$. The Pfaffian matrix is

$$\begin{bmatrix} y & 2x \\ 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

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[1] For references to the older literature, see pp. 324-6 of E. Weber's article in the Encyklop. d. Math. Wiss. Band II, Erster Teil, Erste Hälfte (1.I) Teubner (1916). This article attributes to Frobenius a proof of the sort of proposition stated above, which we will therefore call the *accepted* version.

It has rank 2 everywhere. Indeed, this Pfaffian can be written

$$y^{-1}d(xy^2)\ ,$$

but this does not confirm the accepted version for points at which $y = 0$. (Actually, as our Theorem 2.1 shows, the representation $u\,dv$ is possible at any point different from the origin.) We will now indicate why there do not exist functions u and v each $\mathscr{C}^\infty$ in a neighborhood of $(0, 0)$ such that $y\,dx + 2xdy = u\,dv$. It is evident that $v = f(xy^2)$, whence $f(\lambda) = v(\lambda\varepsilon^{-2}, \varepsilon)$ for some $\varepsilon \neq 0$. This shows that f is $\mathscr{C}^\infty$ in some neighborhood of 0. From this we obtain $y^{-1}d(xy^2) = y\,dx + 2xdy = uf'(xy^2)d(xy^2)$ wherever $y \neq 0$, or $y^{-1} = u\,f'(xy^2)$ where $y \neq 0$, which shows that y^{-1} is bounded where $y \neq 0$— an absurdity. Thus the accepted version is defective.

The only explanation of this state of affairs is that Pfaff's problem problem is, by the authors mentioned in our bibliography at least, not regarded as an "advanced calculus" type of problem (or theorem.) This is made explicit by Thomas in his postulational approach; and it is made *evident* by the fact that a reading of the first line of his table on p. 45, shows that $\omega = ydx + 2x\,dy$ should have a canonical form pdq, since $0 \neq d\omega = dx \wedge dy$ and $\omega \wedge d\omega = 0$, if that theory really did apply.

Our proof, given below, shows that a sufficient additional assumption is that the Pfaffian does not vanish at P (and this is already implicitly assumed when r is odd.)

A person might imagine that it would be an easy task to glance at some rather elementary proof such as the first proof presented by Gours at in his book, and verify that the denominator of each quotient formed by Goursat does not vanish at P if the Pfaffian does not. But this proof is by *induction*, and it is apparent that if you lop terms off a Pfaffian you may find at some lower (even) dimension that the non-vanishing feature has been lost. The exercise of vigilance of this kind almost doubles the length of Goursat's proof. On the other hand, explicit use of Frobenius' theorem on involutory vector field systems, enables us to present a proof which is shorter than Goursat's.

Cartan, in his book (p. 57) sketches a theory of Pfaffian *equations*, which is to say that two Pfaffians differing by a factor which is only a function are regarded as equivalent. It appears that he permits these functions to have zeros. (Indeed if he did not, then his short proof on p. 57 would establish the accepted version).

2. The reduction of Pfaffians. The solution of "Pfaff's Problem" lies in the following theorem.

2.1 THEOREM. *Let* $\alpha = a_1dx^1 + \cdots + a_ndx^n$ *be a Pfaffian with*

$\mathscr{C}_\infty$ components $a_1, \cdots, a_n$ defined in the neighborhood of the origin 0 in $\boldsymbol{R}^n$. Let

$$a_{ij} = \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i},$$

and suppose that the rank of the matrix

2.11
$$\begin{bmatrix} a_1 & \cdots & a_n \\ a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

is constant in a neighborhood of 0. Let that constant be r. If r is even,

2.12 *suppose that the $a_1, \cdots, a_n$ do not all vanish at 0.*

Then there is a coordinate system $(u^1, \cdots, u^n)$ defined in a neighborhood of 0 in $\boldsymbol{R}^n$ such that

2.13
$$\alpha = u^1du^2 + \cdots + u^{2p-1}du^{2p} + \begin{cases} 0 & \text{if } r = 2p \\ du^{2p+1} & \text{if } r = 2p+1 \,. \end{cases}$$

We observe first some invariance properties of the rank of 2.11.

2.2 Prop. *Let $\boldsymbol{P}$ be a point of $\boldsymbol{R}^n$ and let $cls\,(d, \boldsymbol{P})$ be the class of α at $\boldsymbol{P}$, namely the linear dimension of the class of vectors X at $\boldsymbol{P}$ such that*

2.21
$$\langle \alpha, X \rangle = 0$$

and

2.22
$$\langle d\,\alpha; X, Y \rangle = 0 \text{ for all vectors } Y \text{ at } \boldsymbol{P}\,.$$

Then $cls\,(\alpha, \boldsymbol{P})$ equals the rank of 2.11 at $\boldsymbol{P}$.

The proof of 2.2 is by simple linear algebra.

Thus $cls\,(\alpha, \boldsymbol{P})$ may be calculated as the rank of

2.23
$$\begin{bmatrix} b_1, & \cdots, & b_n \\ b_{11}, & \cdots, & b_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1}, & \cdots, & b_{nn} \end{bmatrix}$$

where $\alpha = b_1 dy^1 + \cdots + b_n\, dy^n$ in some other coordinate system, and $b_{ij} = (\partial b_i/\partial y^j) - (\partial b_j/\partial y^i)$.

Let $\alpha^{(1)} = \alpha$, $\alpha^{(2)} = d\alpha$, $\alpha^{(3)} = \alpha \wedge d\alpha$, $\alpha^{(4)} = d\alpha \wedge d\alpha$, $\alpha^{(5)} = \alpha \wedge d\alpha$, $\cdots$ *etc.*

2.24 Prop. (cf. Cartan, p. 58). *The class of α at a point $\boldsymbol{P}$ is the greatest integer r for which $\alpha^{(r)}(\boldsymbol{P}) \neq 0$.*

Proof. Suppose $\boldsymbol{P}$ is the origin, which we shall denote by 0. A theorem of linear algebra [cf. Cartan, p. 12] shows that a coordinate system $(y^1, \cdots, y^n)$ can be found such that $d\alpha$ *at the origin* has the form $dy^1 \wedge dy^2 + \cdots + dy^{2p-1} \wedge dy^{2p}$. In this coordinate system, α has components $b_1, \cdots, b_n$, and

$$d\alpha = \sum_{i,j} \frac{\partial b_i}{\partial y^j}\, dy^j \wedge dy^i = \frac{1}{2} \sum_{i,j} b_{ij}\, dy^j \wedge dy^i \,.$$

The n-by-n array of the $b_{ij}(0)$ is thus as follows:

$$\left[\begin{array}{cc:cc:c:cc:c}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
\hdashline
 & & 0 & -1 & & & & \\
 & & 1 & 0 & & & & \\
\hdashline
 & & & & \ddots & & & \\
\hdashline
 & & & & & 0 & -1 & \\
 & & & & & 1 & 0 & \\
\hdashline
 & & & & & & & \\
\end{array}\right]$$

with p such boxes situated on the diagonal (note $2p \leqq n$) and zeros in the unmarked places. This array has rank $2p$. Considering the way in which it enters into 2.23 we see that 2.23 has rank $2p$ precisely if $g_{2p+1}(0), b_{2p+2}(0), \cdots, b_n(0)$ are 0, and rank $2p+1$ otherwise.

Returning to the form of $d\alpha$ at 0, we see that $\alpha^{(t)}(0) = 0$ for $t > 2p+2$. Indeed,

$$\alpha^{(2p+1)}(0) = [b_{2p+1}(0) dy^{2p+1} + \cdots + b_n(0) dy^n] \wedge dy^1 \wedge dy^2 \wedge \cdots dy^{2p} \,.$$

This is 0 or non-0 according to whether $b_{2p+1}(0) = \cdots = b_n(0) = 0$, or not. This proves 2.24.

We will now prove 2.2 by induction, assuming it true for each Pfaffian β whose class is less than r. We suppose, then, that $\alpha = a_i dx^i$ (using the summation convention in this proof) is of class r in a neighborhood U of 0, in $\boldsymbol{R}^n$.

The *constancy* of $cls\,(\alpha,.)$ in a neighborhood V of 0 enables one to find $n - r$ vector fields $X_{r+1}, \cdots X_n$ defined on a neighborhood W of 0 such that at each point $\boldsymbol{P}$ of W, any vector X such that 2.21 and 2.22 hold, is representable in the form

$$\lambda^{r+1}X_{r+1}(\boldsymbol{P}) + \cdots + \lambda^n X_n(\boldsymbol{P}) .$$

Let *sing-sol* $(\alpha, 0)$ designate the class of vector fields X representable in the form

$$f^{r+1}X_{r+1} + \cdots + f^n X_n$$

where $f^{r+1}, \cdots, f^n$ are $\mathscr{C}^\infty$ functions on some neighborhood of 0. (We will stop naming these neighborhoods). The term *sing-sol* refers to the fact that these vectors are both *solvers* of α (2.21) and singular for $d\,\alpha$ (2.22). It is easy to see that if X and Y are vector fields in sing-sol $(\alpha, 0)$ then so is $[X, Y]$. Accordingly, we have here an *involutory distribution* so that one may assert [cf. Chevalley, p. 89. The theorem holds equally well in the $\mathscr{C}^\infty$ situation. Incidentally, 2.1 holds equally well in the analytic situation.] that there is a coordinate system $y^1, \cdots, y_n$ in a neighborhood of 0 such that sing-sol $(\alpha, 0)$ is generated by the vector fields

2.25 $$\frac{\partial}{\partial y^{r+1}}, \frac{\partial}{\partial y^{r+2}}, \cdots, \frac{\partial}{\partial y^n} .$$

Let $\alpha = b_1\,dy^1 + \cdots + b_n\,dy^n$. Let $X = \partial/\partial y^s$ where $s > r$. Then 2.21 holds and tells us that

2.26 $$b_s = 0 \textit{ for } s > n .$$

Let $Y = \partial/\partial y^i$. Thus 2.22 holds. Thus

$$0 = \langle db_k \wedge dy^k;\, X,\, Y\rangle = \begin{vmatrix} X(b_k) & X(y^k) \\ Y(b_k) & Y(y^k) \end{vmatrix}$$

$$= \frac{\partial b_k}{\partial y^s}\,\delta^k_i - \frac{\partial b_k}{\partial y^i}\,\delta^k_s = \frac{\partial b_i}{\partial y^s} - \frac{\partial b_s}{\partial y^i} = \frac{\partial b_i}{\partial y^s} .$$

Therefore

2.27 $$\alpha = b_1 dy^1 + \cdots + b_r dy^r$$

where $b_1, \cdots, b_r$ *depend only on* $y^1, \cdots, y^r$.

Our reduction problem is thus obviously reduced to the case $n = r$. We therefore start all over again, supposing $n = r$.

We consider first the case in which the skew symmetric matrix, the r by r minor of 2.11

$$[a_{ij}]_{i,j=1,\cdots,r}$$

has a *nonzero determinant*. Then r is necessarily *even*: $r = 2p$. One can then solve the equations

2.28 $$a_{ij}Y^j = a_i \qquad (\textit{summation convention!})$$

for the functions $Y^1, \cdots, Y^r$ which are not all 0 by 2.12. Let Y be the vector field

$$Y^j \frac{\partial}{\partial x^j} .$$

There is a coordinate system $y^1, \cdots, y^r$ such that $Y = \partial/\partial y^r$ [cf. Chevalley, p. 89, Lemma 1].

Using the original coordinates, it is easy to see from 2.28, that for our Y

2.29 $$\langle d\,\alpha, Y, Z\rangle = \langle \alpha, Z\rangle \textit{ for every } Z .$$

Let $Z = \partial/\partial y^i$. Expanding 2.29 (as in the lines between 2.26 and 2.27 above) one obtains

$$b_i = \frac{\partial b_i}{\partial y^r} - \frac{\partial b_r}{\partial y^i} .$$

For $i = r$ this says that $b_r = 0$, and this in turn shows that $\partial/\partial y\,(b_i e^{-y}) = 0$. (Here y is an abbreviation for y^r.) Thus $\alpha = e^y\beta$ where

$$\beta = h_1 dy^1 + \cdots + h_{r-1}dy^{r-1}$$

and

2.3 $$h_1, \cdots, h_{r-1} \textit{ depend only on } y^1, \cdots, y^{r-1} .$$

We will show that β satisfies the conditions of 2.1 with r replaced by $r - 1$. Since $\alpha = e^y\beta$ we have $d\alpha = e^y(dy \wedge \beta + d\beta)$. Now $\alpha^{(2p)}(0) \neq 0$ so, in a neighborhood of 0,

$$0 \neq (dy \wedge \beta + d\beta) \wedge \cdots \wedge (dy \wedge \beta + d\beta) \qquad (p \textit{ factors}) ,$$

whence

$$0 \neq p\,dy \wedge \beta \wedge d\,\beta \wedge \cdots \wedge d\,\beta + d\,\beta \wedge \cdots \wedge d\beta$$

or

$$0 \neq p\,dy \wedge \beta^{(2p-1)} + \beta^{(2p)} .$$

Now $\beta^{(2p)}$ certainly is 0, because it has $2p = r$ differentials $dy^1, \cdots, dy^{r-1}$ in it, so that there will be repetitions. Thus $\beta^{(r-1)} \neq 0$ in a neighborhood of 0, while $\beta^{(r)} = 0$ as we just observed. Now 2.24 shows that $cls\,(\beta, 0) = r - 1$. Thus 2.1 applies and

$$\beta = u^1du^2 + \cdots + du^{2p-1}$$

in some coordinate system $u^1, \cdots, u^n$. Thus

$$\alpha = v^1du^2 + v^3du^4 + \cdots + v\,du^{2p-1} .$$

These functions are independent because $\alpha^{(2p)} \neq 0$. Hence the desired reduction has been achieved.

The remaining case is where the determinant of $[a_{ij}]$ vanishes at 0. In this case its rank is less than r, but it cannot be less than $r - 1$, because the rank of 2.11 is r. Hence *its rank* is $r - 1$ and so $r - 1$ is *even*[2], as the rank of an antisymmetric matrix is even. In particular, the rank of $[a_{ij}]$ cannot be r, so that $det\ [a_{ij}] = 0$. However $[a_{ij}]$ has at least one $r - 1$ rowed minor whose determinant is nonzero. It follows that there are functions $Y^1, \cdots, Y^r$ such that

2.31 $$a_i Y^i = 1 \quad and \quad a_{ij} Y^i = 0 .$$

Let $Y = Y^j\,\partial/\partial x^j$. Properties 2.31 translate into

2.32 $$\langle \alpha, Y \rangle = 1 \text{ and } \langle d\,\alpha; Y, Z \rangle = 0 \text{ for every } Z .$$

We now choose coordinates $y^1, \cdots, y^r$ so that $Y = \partial/\partial y^r$. Using 2.32 in the same way as before, we find that

$$\alpha = b_1dy^1 + \cdots + b_rdy^r$$

where

2.33 $$b_r = 1 \text{ and } b_1, \cdots, b_{r-1} \text{ depend only on } y^1, \cdots, y^{r-1} .$$

Let $b_1dy^1 + \cdots + b_{r-1}\,dy^{r-1} = \beta$. Then $\alpha = \beta + dy^r$. Now $r = 2p + 1$ and $0 \neq \alpha^{(r)} = \alpha \wedge \alpha^{(2p)} = \alpha \wedge \beta^{(2p)}$. Hence $\beta^{(r-1)} \neq 0$. On the other hand $\beta^{(r)} \equiv 0$. Thus $r - 1$ is clearly the rank of the matrix 2.11 for β. We must, however, verify that $\beta(P) \neq 0$, because $cls\,(\beta)$ is even. This might in fact not be true! We can, in such a case write

$$\alpha = (b_1 + 1)dy^1 + \cdots + b_{r-1}\,dy^{r-1} + d(y^r - y^1) = \gamma + dy ,$$

where y stands for $y^r - y^1$, and the last equality itself defines γ. Then $\gamma(P) \neq 0$, and otherwise γ has all the properties of β that have

[2] [cf. Cartan, p. 13]

already been established for the induction argument. Thus β (or γ) has the form $u^1dv^1 + \cdots + u^pdv^p$ and α can easily be brought into the desired from.

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