DOUBLY INVARIANT SUBSPACES, II

MORISUKE HASUMI AND T. P. SRINIVASAN

1. Introduction. Let X be a locally compact Hausdorff space and μ a positive Radon measure on X. Let $\mathscr{H}$ be a separable Hibert space and let $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq +\infty)$ denote the space of $\mathscr{H}$-valued functions on X which are weakly measurable and whose norms are in scalar $L^p(d\mu)$. Call P a *measurable range function* if P is a function on X defined a.e. $(d\mu)$ to the space of orthogonal projections on $\mathscr{H}$ which is weakly measurable. We shall regard two range functions P, P' to be the same if $P(x) = P'(x)$ l.a.e., i.e. $P(x) = P'(x)$ a.e. on every compact subset of X. We shall denote by $\hat{P}$ the operator on $L^p_{\mathscr{H}}$ defined by $(\hat{P}f)(x) = P(x)f(x)$ l.a.e. Let A be a subalgebra of the algebra $C(X)$ of bounded continuous functions on X such that $A \cup \bar{A}$ (where the bar denotes complex conjugation) is weakly* dense in $L^\infty(d\mu)$. Say that a subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is *doubly invariant* if

(i) $\mathscr{M}$ is closed in $L^p_{\mathscr{H}}$ if $1 \leqq p < \infty$ and weakly* closed if $p = \infty$,

(ii) $\mathscr{M}$ is invariant under multiplication by functions in $A \cup \bar{A}$.

We shall refer to the following theorem as Wiener's theorem for $L^p_{\mathscr{H}}$:

THEOREM. *Every doubly invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq \infty)$ is of the form $\hat{P}L^p_{\mathscr{H}}$ for some measurable range function P (and trivially conversely); $\mathscr{M}$ determines P uniquely.*

For compact spaces X, Wiener's theorem was proved in [4] for arbitrary $\mathscr{H}$ for $p = 2$ and for the scalar $\mathscr{H}$ (the space of complex numbers) for arbitrary p. It was pointed out in [4] that the $L^2_{\mathscr{H}}$ theorem is true for locally compact spaces and the proof was outlined considering the real line as an example. It was also mentioned in [4] that the $L^2_{\mathscr{H}}$ theorem is a special case of a known theorem on rings of operators [2; p. 167, Théorème 1]. But the proof in [4] and the proof of the more general theorem in [2] implicitly assume the σ-finiteness of μ or at least of the separability of $L^2_{\mathscr{H}}$ (as opposed to the separability of $\mathscr{H}$). The theorem itself is true without this restriction not only for $p = 2$ but for all p and all (separable) $\mathscr{H}$ (not necessarily the scalar $\mathscr{H}$). Indeed the general $L^p_{\mathscr{H}}$ theorem is true even under the weaker assumption that the restriction of $A \cup \bar{A}$ to every compact subset K of X is L^2-dense in $L^2(d\mu \mid K)$, instead of being weakly* dense in L^∞. In this paper we prove this theorem

Received July 18, 1963. This work was done while both authors held visiting appointments in the University of California, Berkeley, and were respectively sponsored in part by the National Science Foundation, Grants NSF GP-2 and NSF G-18974.

(Theorem 4) in its full generality (with the above weaker assumption). This is done as follows: Using the techniques employed in [5] we first show in § 2 (Theorem 2) that a general class of subalgebras dense in L^2 is weakly* dense, which seems to be of independent interest. This enables us to reduce the L^2-density case to that of weak* density. To overcome the difficulties caused by the (possible) non-separability of $L^2_{\mathscr{H}}$ we extend in § 3 (Theorem 3) a theorem of Dunford-Pettis [1; p. 46, Corollaire 2] to apply to our setup. We finally use the $L^2_{\mathscr{H}}$ theorem for compact X in [4] and the broad techniques in [4] to complete the proof. As pointed out in [4], the $L^p_{\mathscr{H}}$ theorem for $p \neq 2$ is of special interest as it shows that the doubly invariant subspaces of $L^p_{\mathscr{H}}$ admit projections of norm 1 commuting with bounded (scalar) functions; as is well known, a closed linear subspace of a Banach space does not in general have any bounded projection at all. In the final section of the paper we extend a known theorem [2] on operators in $L^p_{\mathscr{H}}$ which commute with multiplication by bounded (scalar) functions (Theorem 5).

2. Weak* density of certain subalgebras of L^∞.

THEOREM 1. *Let (X, m) be a finite measure space. Any subalgebra $\mathscr{A}$ of $L^\infty(dm)$ which is conjugate-closed and dense in $L^2(dm)$ is weakly* dense in $L^\infty(dm)$.*[1]

The following three lemmas will lead to the proof of the theorem.

LEMMA 1. *Let $\mathscr{B}$ be a conjugate-closed subalgebra of $L^\infty(dm)$ which contains constants and is closed in $L^\infty(dm)$. Then $\mathscr{B}$ is closed for absolute values.*

Proof. Let $f \in \mathscr{B}$, $0 \leqq f \leqq 1/2$, say. Then $f^{\frac{1}{2}} = (1 - (1 - f))^{\frac{1}{2}}$ can be expressed as the sum of a convergent series in $L^\infty(dm)$ whose terms come from $\mathscr{B}$; it follows that $f^{\frac{1}{2}} \in \mathscr{B}$ for all non-negative $f \in \mathscr{B}$. Since $\mathscr{B}$ is conjugate-closed, the lemm follows.

LEMMA 2. *Let (X, m) be a finite measure space and A a subalgebra of $L^\infty(dm)$ such that $A \cup \bar{A}$ is dense in $L^2(dm)$. Then every closed subspace $\mathscr{M}$ of $L^2(dm)$ which is invariant under multiplication by functions in $A \cup \bar{A}$ is of the form $C_S L^2(dm)$ for some measurable subset S of X (where C_S denotes the characteristic function of S).*

Proof. Let $\mathscr{B}$ be the closed subalgebra of $L^\infty(dm)$ generated by $A \cup \bar{A}$ and the constants. Then $\mathscr{M}$ is clearly invariant under multi-

[1] A weaker result was proved in [5].

plication by functions in $\mathscr{B}$. By Lemma 1, $\mathscr{B}$ is closed for absolute values. Let q be the orthogonal projection of the constant function 1 on $\mathscr{M}$. Then $1 - q \perp \mathscr{M}$. Since $\mathscr{M}$ is invariant under multiplication by function in $\mathscr{B}$, it follows that

$$\int fqdm = \int f\,|\,q\,|^2 dm \tag{2.1}$$

for all $f \in \mathscr{B}$. Let Y be any measurable subset of X and let $\{f_n\}$ be a sequence of functions from $\mathscr{B}$ which converges to C_Y in $L^2(dm)$. Since $|f_m - f_n| \in \mathscr{B}$, we have from (2.1)

$$\int |f_m - f_n|\,|\,q\,|^2 dm = \int |f_m - f_n|\,qdm$$

and the last integral is less than $\left(\int |f_m - f_n|^2 dm\right)^{\frac{1}{2}} \times \left(\int |\,q\,|^2 dm\right)^{\frac{1}{2}}$. It follows that $\{f_n\,|\,q\,|^2\}$ is a Cauchy sequence in $L^1(dm)$. Hence $f_n\,|\,q\,|^2 \to C_Y\,|\,q\,|^2$ in $L^1(dm)$; in particular,

$$\int f_n\,|\,q\,|^2 dm \to \int_Y |\,q\,|^2 dm\ . \tag{2.2}$$

Since $f_n \to C_Y$ in $L^2(dm)$, $f_n q \to C_Y q$ in $L^1(dm)$ and thus

$$\int f_n q dm \to \int_Y qdm\ . \tag{2.3}$$

It follows from (2.1)–(2.3) that $\int_Y |\,q\,|^2 dm = \int_Y qdm$ for all measurable subsets Y; hence $|\,q\,|^2 = q$ a.e. Thus $q = C_S$ a.e. for some $S \subset X$.

Because of invariance, $C_S L^2(dm) \subset \mathscr{M}$. If the inclusion were strict, let $g \in \mathscr{M} \ominus C_S L^2(dm)$. Then $g \perp C_S \mathscr{B}$ also $C_{S'} \in \mathscr{M}^{\perp}$ (where $S' = X - S$) and $\mathscr{M}^{\perp}$ is also invariant along with $\mathscr{M}$ under multiplications by functions in $\mathscr{B}$. So $g \perp C_{S'}\mathscr{B}$. It follows that $g \perp \mathscr{B}$ and because of density of $\mathscr{B}$ in $L^2(dm)$, we have $g = 0$ a.e. Thus $\mathscr{M} = C_S L^2(dm)$.

LEMMA 3. *Let (X, m) and A be as in Lemma 2. Then every closed subspace of $L^1(dm)$ which is invariant under multiplication by functions in $A \cup \bar{A}$ is of the form $C_S L^1(dm)$ for some measurable subset S.*

Proof. This follows from Lemma 2 above and Theorem 7 in [4].

Proof of Theorem 1. Let $\mathscr{M} = \left\{f \in L^1(dm)\colon \int fgdm = 0 \text{ for all } g \in \mathscr{A}\right\}$. Then $\mathscr{M}$ is $\mathscr{A}$-invariant, meaning invariant under multiplication by functions in $\mathscr{A}$ and Lemma 3 applies for $\mathscr{M}$ (with $\mathscr{A}$

replacing A). Thus $\mathscr{M} = C_S L^1(dm)$ for some S, so $\mathscr{M} \cap L^2(dm) = C_S L^2(dm)$. But $\mathscr{M} \cap L^2(dm) = L^2(dm) \ominus \mathscr{A}$. Since $\mathscr{A}$ is dense in $L^2(dm)$ by assumption, it follows that $C_S = 0$ a.e. Therefore $\mathscr{M} = \{0\}$ and the theorem follows.

REMARK. One of the corollaries of Theorem 1 is the "uniqueness" of the Fourier coefficients of any function in $L^1(G)$, for a compact Abelian group G. The characters are dense in $L^2(G)$ so that the subspace $\mathscr{A}$ of their finite linear combinations is weakly* dense in $L^\infty(dm)$ by Theorem 1 and the uniqueness follows.

We now extend Theorem 1 to infinite measure spaces. For convenience we state the result in terms of Radon measures on locally compact spaces. We have

THEOREM 2. *Let X be a locally compact Hausdorff space and μ a positive Radon measure on X. Let $\mathscr{A}$ be a subalgebra of the algebra of bounded continuous functions on X such that*

(i) *$\mathscr{A}$ is conjugate-closed,*

(ii) *$\mathscr{A} \mid K$ is dense in $L^2(d\mu \mid K)$ for every compact subset K of X. Then $\mathscr{A}$ is weakly* dense in $L^\infty(d\mu)$.*

Proof. Let $\mathscr{M} = \left\{f \in L^1(d\mu): \int fg d\mu = 0 \text{ for all } g \in \mathscr{A}\right\}$. If we show that $\mathscr{M} = \{0\}$, the theorem is proved. Now $\mathscr{M}$ is clearly a closed subspace of $L^1(d\mu)$ and is $\mathscr{A}$-invariant. We need the following lemma which will be proved below.

LEMMA 4. *Every closed $\mathscr{A}$-invariant subspace $\mathscr{M}$ of $L^1(d\mu)$ is of the form $C_S L^1(d\mu)$ for some measurable subset S* (where $\mathscr{A}$ is as in Theorem 2).

Assuming Lemma 4, the main theorem follows at once. For, since $\mathscr{M} = C_S L^1(d\mu)$, $\mathscr{A} \subset \mathscr{M}^\perp = C_{S'} L^\infty(d\mu)$. If $\mu(S) > 0$, then S contains a compact subset K of positive measure. Since $\mathscr{A} \subset C_{S'} L^\infty(d\mu)$, $\mathscr{A} \mid K = \{0\}$, contradicting the density of $\mathscr{A} \mid K$ in $L^2(d\mu \mid K)$. Hence $\mu(S) = 0$, so $\mathscr{M} = \{0\}$, completing the proof of the theorem.

Proof of Lemma 4. Let $\mathscr{M}_K = C_K \mathscr{M}$, $\mathscr{A}_K = C_K \mathscr{A}$ and $\mu_K = C_K \mu$. We shall identify $L^p(d\mu \mid K)$, $L^p(d\mu_K)$ and $C_K L^p(d\mu)$ which are clearly mutually isometrically isomorphic. Each $\mathscr{M}_K$ is closed and $\mathscr{A}_K$-invariant in $L^1(d\mu_K)$, so by Lemma 3, $\mathscr{M}_K = C_{S(K)} L^1(d\mu_K)$ for some $S(K) \subset K$. If $K' \supset K$, compact, then

$$\begin{aligned}C_{S(K)}L^1(d\mu) &= C_{S(K)}L^1(d\mu_K) = \mathscr{M}_K = C_K C_{K'}\mathscr{M}\\ &= C_K C_{S(K')}L^1(d\mu_{K'}) = C_{S(K')\cap K}L^1(d\mu_{K'})\\ &= C_{S(K')\cap K}L^1(d\mu)\,,\end{aligned}$$

so that $S(K) = S(K') \cap K$ (modulo null sets).

Let $\mathscr{K}$ denote the set of all continuous functions with compact support and let σ be the linear functional on $\mathscr{K}$ defined by

$$\sigma(\varphi) = \int_{S(K)} \varphi d\mu \tag{2.4}$$

for $\varphi \in \mathscr{K}$ where K is any compact subset containing the support of φ. Then σ is well-defined and is continuous in the L^1-norm, so can be uniquely extended to a bounded linear functional on $L^1(d\mu)$, which we again denote by σ. Let σ be realized by the L^∞-function g so that

$$\sigma(f) = \int fg d\mu \tag{2.5}$$

for all $f \in L^1(d\mu)$. From (2.4) and (2.5) it is easy to see that $g \mid K = C_{S(K)}$ a.e. for every compact subset K; so we may assume $g = C_S$ for some measurable S with $S \cap K = S(K)$ (modulo null sets). Now

$$C_K C_S L^1(d\mu) = C_{S\cap K}L^1(d\mu) = C_{S(K)}L^1(d\mu) = \mathscr{M}_K = C_K\mathscr{M}$$

for all compact K. Since for any $f \in L^1(d\mu)$, $C_K f \to f$ in $L^1(d\mu)$, it follows from the above that $C_S L^1(d\mu) = \mathscr{M}$.

REMARK. The assumption that $\mathscr{A}$ is an algebra is crucial in both Theorems 1 and 2; the conclusion would be false if $\mathscr{A}$ were merely a linear subspace satisfying the rest of the assumptions. The following example shows that, in the locally compact case for instance, a conjugate-closed linear subspace of $L^\infty(d\mu)$ may be weakly* dense on every compact subset but not on the whole space.

Let X be a locally compact space and μ a non-finite Radon measure on X. Let $f \in L^1(d\mu)$ be real and have a support of infinite μ-measure. Then the support is non-compact. Let $\mathscr{A} = \left\{g \in L^\infty(d\mu) \colon \int gfd\mu = 0\right\}$. Then $\mathscr{A}$ is clearly not weakly* dense in $L^\infty(d\mu)$. But if g is any continuous function with compact support which is "orthogonal" to $\mathscr{A}$, then g must be in the linear span of f in $L^1(d\mu)$. It follows from our assumption on f that g is the zero function. Hence $\mathscr{A}$ is weakly* dense on every compact subset.

3. Dunford-Pettis theorem. Let X denote a locally compact Hausdorff space and μ a positive Radon measure on X. Let E be a separable Banach space and $\mathscr{K}_E$ denote the space of continuous functions from X into E with compact support. For $1 \leqq p < \infty$, let $\mathscr{F}_E^p$ be the space of all functions f from X into E with

$$N_p(f) = \left(\int_X^* \|f(x)\|^p \, d\mu(x)\right)^{1/p} < \infty$$

where $\int^*$ denotes the upper integral. $\mathscr{F}_E^p$ is then a locally convex space with respect to the seminorm N_p. Let $\mathscr{L}_E^p$ denote the closure of $\mathscr{K}_E$ in $\mathscr{F}_E^p$ and let $L_E^p = \mathscr{L}_E^p / \mathscr{N}_E^p$ where $\mathscr{N}_E^p$ is the set of all functions $f \in \mathscr{L}_E^p$ with $N_p(f) = 0$. Then L_E^p is a Banach space with the norm induced by N_p in the obvious way.

Denote by $\mathscr{L}_{E^*}^\infty$ the space of all weakly* measurable functions f on X to the dual E^* of E such that $\|f(x)\| \leqq A < \infty$ l.a.e. ($\|f(x)\| \leqq A$ a.e. on every compact subset). For $f \in \mathscr{L}_{E^*}^\infty$ let

$$N_\infty(f) = \sup_K (\text{ess.}\sup_{x \in K} \|f(x)\|)$$

where K ranges over all compact subsets of X. Then N_∞ is a seminorm which makes $\mathscr{L}_{E^*}^\infty$ a locally convex space. Let $L_{E^*}^\infty$ be the quotient of $\mathscr{L}_{E^*}^\infty$ by the space of all functions in $\mathscr{L}_{E^*}^\infty$ which vanish l.a.e. Then $L_{E^*}^\infty$ is a Banach space.

The following theorem is well-known (cf. for instance [1; p. 46, Corollaire 2]):

THEOREM (Dunford-Pettis). *Let F be a separable Banach space. For $f \in L_{F^*}^\infty$ and $g \in L^1(d\mu)$, let*

$$w_f(g) = \int_X g f d\mu \, .$$

Then $w_f(g) \in F^*$ and the mapping $f \to w_f$ induces an isometric isomorphism from $L_{F^*}^\infty$ onto $\mathscr{L}(L^1, F^*)$, the space of bounded linear maps from $L^1(d\mu)$ to F^*.

We need the following variant of the Dunford-Pettis theorem:

THEOREM 3. *Let E, F be separable Banach spaces. For any bounded linear map u of L_E^1 into F^* there exists a function Φ from X into $\mathscr{L}(E, F^*)$ such that*

(i) *$\langle \Phi(x)s, t\rangle$ is measurable for every $s \in E$, $t \in F$,*

(ii) *$N_\infty(\Phi) < \infty$, and*

(iii) *$u(f) = \int_X \Phi(x) f(x) d\mu(x)$ for every $f \in L_E^1$ with $\|u\| = N_\infty(\Phi)$.*

Conversely, any function Φ satisfying (i) *and* (ii) *defines a bounded linear map u satisfying* (iii).

Proof. Only the direct part needs a proof. First we note that $\mathscr{L}(E, F^*)$ can be regarded as the strong dual of the projective tensor product $E \hat{\otimes} F$. Indeed, the strong dual of $E \hat{\otimes} F$ is canonically identified with the space $B(E, F)$ of bounded bilinear forms on $E \times F$ and $\mathscr{L}(E, F^*)$ is canonically isomorphic with $B(E, F)$. Since E, F are separable, so is $E \hat{\otimes} F$ and therefore $\mathscr{L}(E, F^*)$ can be regarded as the strong dual of a separable Banach space.

Let u be a bounded linear map of L^1_E into F^*. Then u induces a bounded bilinear form $\tilde{u}$ on $L^1 \times E$ into F^* by $\tilde{u}(f, s) = u(f \otimes s)$ for $f \in L^1$, $s \in E$. For any fixed $f \in L^1$, $s \to \tilde{u}(f, s)$ is a bounded linear map of E into F^* which we shall denote by u_f. Then u_1: $f \to u_f$ is a bounded linear map from L^1 into $\mathscr{L}(E, F^*)$ with $\| u_1 \| = \| u \|$. By the Dunford-Pettis theorem, there exists a function Φ: $X \to \mathscr{L}(E, F^*)$ such that

(i) $\langle \Phi(x)s, t \rangle$ is measurable for each $s \in E$, $t \in F$

(ii) $N_\infty(\Phi) = \| u_1 \|$, and

(iii) $u_1(f) = u_f = \int_X f(x)\Phi(x)d\mu(x)$.

Hence

$$u(f \otimes s) = \tilde{u}(f, s) = u_f(s) = \int_X f\Phi s d\mu$$
$$= \int_X \Phi(f \otimes s)d\mu \ .$$

Because of the continuity of u, the theorem follows.

4. Doubly invariant subspaces. In this section we prove Wiener's theorem in the general setup. Let as usual X denote a locally compact Hausdorff space, μ a positive Radon measure on X, $\mathscr{H}$ a separable Hilbert space and $\mathscr{K}_{\mathscr{H}}$ the space of continuous functions from X into $\mathscr{H}$ with compact support. Let A be a subalgebra of the algebra of bounded continuous functions on X and $\mathscr{A}$ denote the algebra generated by $A \cup \bar{A}$ and the constants. A subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is clearly invariant under multiplication by functions is $A \cup \bar{A}$ if and only if it is $\mathscr{A}$-invariant. We recall that $\mathscr{M}$ is *doubly invariant* if

(i) $\mathscr{M}$ is closed in $L^p_{\mathscr{H}}$ if $1 \leqq p < \infty$ and weakly* closed if $p = \infty$,

(ii) $\mathscr{M}$ is $\mathscr{A}$-invariant.

Then we have

THEOREM 4. *If $\mathscr{A} \mid K$ is dense in $L^2(d\mu \mid K)$ for every compact subset K, then every doubly invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq \infty)$ is of the form $\hat{P}L^p_{\mathscr{H}}$ for some measurable range function P; $\mathscr{M}$ determines P uniquely.*

Proof. We divide the proof into three parts; in the first and the second we assume $\mu(X) < \infty$ and the proof is an imitation of that of the scalar case in [4]. In the last part we treat the case of arbitrary measure spaces and an indication of the proof in this case was given in the proof of Theorem 2.

(i) $\mu(X) < \infty$, $1 \leqq p \leqq 2$. By Theorem 2, $\mathscr{A}$ is weakly* dense in $L^\infty(d\mu)$ and in this case the theorem has been proved in [4] for $p = 2$. Let $1 \leqq p < 2$ and $\mathscr{N} = \mathscr{M} \cap L^2_{\mathscr{H}}$. Then $\mathscr{N}$ is a doubly invariant subspace of $L^2_{\mathscr{H}}$ and so $\mathscr{N} = \hat{P}L^2_{\mathscr{H}}$ for some measurable range function P. We wish to show that $\mathscr{M} = \hat{P}L^p_{\mathscr{H}}$.

For any $f \in \mathscr{M}$ let $f_1(x) = \|f(x)\|^{1-(p/2)}$ and $f_2(x) = f_1(x)^{-1}f(x)$ (of course $f_2(x) = 0$ if $f_1(x) = 0$). Then $f_1 \in L^s(d\mu)$ where $(1/s) + (1/2) = (1/p)$ and $f_2 \in L^2_{\mathscr{H}}$. Let $\mathscr{N}_2$ be the doubly invariant subspace of $L^2_{\mathscr{H}}$ generated by f_2. Then $\mathscr{N}_2 = \hat{P}_2L^2_{\mathscr{H}}$ for a measurable range function P_2. Here we may assume that $P_2(x) = 0$ for those x for which $f_1(x) = 0$. For any $\varphi \in \mathscr{K}_{\mathscr{H}}$

$$f_1\hat{P}_2\varphi \in f_1\hat{P}_2L^2_{\mathscr{H}} = f_1\mathscr{N}_2 \subset \mathscr{M} .$$

On the other hand, since $s > 2$,

$$f_1\hat{P}_2\varphi \in L^s_{\mathscr{H}} \subset L^2_{\mathscr{H}}$$

as $f_1 \in L^s$, $\hat{P}_2\varphi$ is bounded and $\mu(X) < \infty$. Hence

$$f_1\hat{P}_2\varphi \in \mathscr{M} \cap L^2_{\mathscr{H}} = \mathscr{N} = \hat{P}L^2_{\mathscr{H}} .$$

This means that $\hat{P}\hat{P}_2f_1\varphi = \hat{P}_2f_1\varphi$ for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. So, $P_2(x) \leqq P(x)$ l.a.e. Thus we have $\mathscr{N}_2 = \hat{P}_2L^2_{\mathscr{H}} \subset \hat{P}L^2_{\mathscr{H}}$. Hence

$$f = f_1f_2 \in f_1\mathscr{N}_2 \subset f_1\hat{P}L^2_{\mathscr{H}} \subset \hat{P}L^p_{\mathscr{H}} ;$$

the last inclusion resulting from the fact that $f_1 \in L^s$ where $(1/s) + (1/2) = (1/p)$. This shows that $\mathscr{M} \subset PL^p_{\mathscr{H}}$.

Since $\mathscr{M} \supset \mathscr{N} = \hat{P}L^2_{\mathscr{H}}$, we have $\mathscr{M} \supset \hat{P}\mathscr{K}_{\mathscr{H}}$. But $\mathscr{K}_{\mathscr{H}}$ is dense in $L^p_{\mathscr{H}}$ and $\hat{P}$ is L^p-continuous. So $\mathscr{M} \supset \hat{P}L^p_{\mathscr{H}}$ and we have $\mathscr{M} = PL^p_{\mathscr{H}}$.

(ii) $\mu(X) < \infty$, $2 < p \leqq \infty$. Let $\mathscr{M}' = \{f \in L^q_{\mathscr{H}} : f \perp \mathscr{M}\}$ where $(1/q) + (1/p) = 1$. Then $1 \leqq q < 2$ and $\mathscr{M}'$ is doubly invariant in $L^q_{\mathscr{H}}$. Hence by (i) $\mathscr{M}' = \hat{P}'L^q_{\mathscr{H}}$ for some measurable range function P'. Then it is easy to see that $\mathscr{M} = \hat{P}L^p_{\mathscr{H}}$ where $P(x) = I - P'(x)$, I denoting the identity operator on $\mathscr{H}$.

(iii) $\mu(X)$ not necessarily finite, $1 \leqq p \leqq \infty$. Consider any compact subset K of X. Let $\mathscr{M}_K = C_K\mathscr{M}$, $\mathscr{A}_K = C_K\mathscr{A}$ and $\mu_K = C_K\mu$. We shall identify $L^p_{\mathscr{H}}(d\mu \mid K)$, $L^p_{\mathscr{H}}(d\mu_K)$ and $C_KL^p_{\mathscr{H}}(d\mu)$ which are obviously mutually isometrically isomorphic and denote any of them by $L^p_{\mathscr{H}}(K)$. Now $\mathscr{M}_K$ is a doubly invariant subspace of $L^p_{\mathscr{H}}(d\mu_K)$ (with $\mathscr{A}_K$ replacing $\mathscr{A}$) and $\mathscr{A}_K$ is dense in $L^2(d\mu_K)$. Hence by (i) and (ii) above, $\mathscr{M}_K = \hat{P}_K L^p_{\mathscr{H}}(K)$. We extend P_K to the whole of X by defining $P_K(x) = 0$ outside of K.

For any two compact subsets K_1, K_2 with $K_1 \supset K_2$ we have

$$\begin{aligned}\hat{P}_{K_2} L^p_{\mathscr{H}} &= \hat{P}_{K_2} L^p_{\mathscr{H}}(K_2) = \mathscr{M}_{K_2} = C_{K_2} C_{K_1} \mathscr{M} = C_{K_2} \hat{P}_{K_1} L^p_{\mathscr{H}}(K_1) \\ &= \hat{P}_{K_1} C_{K_2} L^p_{\mathscr{H}}(K_1) = \hat{P}_{K_1} C_{K_2} L^p_{\mathscr{H}} .\end{aligned}$$

Hence $P_{K_2} = P_{K_1} C_{K_2}$ a.e. It follows from this that the map σ: $\mathscr{K}_{\mathscr{H}} \to \mathscr{H}$ given by

$$\sigma(\varphi) = \int_X P_K(x) \varphi(x) d\mu(x) ,$$

where K is any compact subset containing the support of φ, is well-defined. σ is clearly continuous with respect to the $L^1_{\mathscr{H}}$-norm and so can be uniquely extended to the whole of $L^1_{\mathscr{H}}$ to be continuous. We shall denote the extended map by $\tilde{\sigma}$. By Theorem 3 there exists a weakly measurable bounded operator-valued function Φ: $X \to \mathscr{L}(\mathscr{H}, \mathscr{H})$ such that

$$\tilde{\sigma}(f) = \int_X \Phi(x) f(x) d\mu(x)$$

for all $f \in L^1$. Then, since $\tilde{\sigma}$ entends σ, it is obvious that

$$\Phi \mid K = P_K \text{ a.e.}$$

for every compact set K; so there exists a measurable range function P such that $\Phi = P$ l.a.e.

We assert that $\mathscr{M} = \hat{P} L^p_{\mathscr{H}}$. *This follows from the fact that* $C_K \mathscr{M} = C_K \hat{P} L^p_{\mathscr{H}}$ for every compact set K and every $f \in \mathscr{M}$ is the L^p-limit (or the weak* limit if $p = \infty$) of $C_K f$. This completes the proof.

The uniqueness of P (for a given $\mathscr{M}$) follows from the uniqueness established in [4] for finite measure spaces.

5. **Decomposable operators.** Let X, μ, A and $\mathscr{A}$ be as in § 4 and let T be an operator in $L^p_{\mathscr{H}}$ bounded if $1 \leqq p < \infty$ and in addition weakly* continuous if $p = \infty$. Clearly T commutes with multiplication by functions in $A \cup \bar{A}$ if and only if it commutes with functions in $\mathscr{A}$, and any operator T which operates pointwise (l.a.e.), meaning

$$(Tf)(x) = T(x) f(x) \text{ l.a.e.}$$

for an operator-valued function $T(x)$, clearly has this property. We wish to prove the following converse.

THEOREM 5. *If T is a bounded (and weakly* continuous, if*

$p = \infty$) *linear map from* $L^p_{\mathscr{H}}$ *into* $L^p_{\mathscr{H}}$ ($1 \leqq p \leqq \infty$) *which commutes with multiplication by functions in* $\mathscr{A}$, *then there exists an operator-valued function* $T(x)$ *defined* a.e. *with* $T(x) \in \mathscr{L}(\mathscr{H}, \mathscr{H})$ *which is weakly measurable and uniformly bounded such that*

$$(Tf)(x) = T(x)f(x) \text{ a.e. } ((Tf)(x) = T(x)f(x) \text{ l.a.e. } if\ p = \infty)$$

This theorem is usually stated for $L^2_{\mathscr{H}}$ [2; p. 162, Theoreme 1] and as far as we are aware, the existing proofs require $L^2_{\mathscr{H}}$ to be separable. We use the variant of Dunford-Pettis theorem established by us in § 3 to get around the difficulties that may be caused by non-separability (we of course assume that the Hilbert space $\mathscr{H}$ is separable).

Proof of Theorem 5. We first consider the case $1 \leqq p < \infty$, for convenience we assume that T is bounded by 1. Let $f \in L^p_{\mathscr{H}}$. Then

$$\int_X \| (Tf)(x) \|^p \, d\mu(x) \leqq \int_X \| f(x) \|^p \, d\mu(x) \, .$$

Since T commutes with multiplication by functions in $\mathscr{A}$, this yields

$$\int_X | \alpha(x) |^p \, \| (Tf)(x) \|^p \, d\mu(x) \leqq \int_X | \alpha(x) |^p \, \| f(x) \|^p \, d\mu(x)$$

for all $\alpha \in \mathscr{H}$. From the weak* density of $\mathscr{A}$ in L^∞, it follows that

$$\| (Tf)(x) \| \leqq \| f(x) \| \text{ a.e.}$$

If $L^p_{\mathscr{H}}$ is separable, we can obtain $T(x)$ by an explicit construction. In the general case we argue as follows:

Define a map u: $\mathscr{K}_{\mathscr{H}} \rightarrow \mathscr{H}$ by setting

$$u(\varphi) = \int_X (T\varphi)(x) d\mu(x) \, , \qquad \varphi \in \mathscr{K}_{\mathscr{H}} \, .$$

Then u is continuous with respect to the $L^1_{\mathscr{H}}$-norm on $\mathscr{K}_{\mathscr{H}}$ because

$$\left\| \int_X (T\varphi)(x) d\mu(x) \right\| \leqq \int_X \| (T\varphi)(x) \| \, d\mu(x)$$
$$\leqq \int_X \| \varphi(x) \| \, d\mu(x) \, .$$

Since $\mathscr{K}_{\mathscr{H}}$ is dense in $L^1_{\mathscr{H}}$, u can be extended by continuity to the whole $L^1_{\mathscr{H}}$ without increasing its norm. We denote the extended map also by u. By Theorem 3 there exists a function $\Phi(x)$ from X into $\mathscr{L}(\mathscr{H}, \mathscr{H})$ such that Φ is weakly measurable, uniformly bounded with $\| \Phi(x) \| \leqq \| u \| \leqq 1$ and

$$u(f) = \int_X \Phi(x) f(x) d\mu(x)$$

for every $f \in L^1_{\mathscr{H}}$. Thus for any $\varphi \in \mathscr{K}_{\mathscr{H}}$

$$\int_X (T\varphi)(x)d\mu(x) = u(\varphi) = \int_X \Phi(x)\varphi(x)d\mu(x) .$$

Since T commutes with multiplication by functions in $\mathscr{A}$ and every $\alpha \in \mathscr{A}$ is continuous, we get

$$\begin{aligned}\int_X \alpha(x)\Phi(x)\varphi(x)d\mu(x) &= \int_X \Phi(x)\alpha(x)\varphi(x)d\mu(x) \\ &= \int_X (T\alpha\varphi)(x)d\mu(x) = \int_X \alpha(x)(T\varphi)(x)d\mu(x) .\end{aligned}$$

By the weak* density of $\mathscr{A}$ in L^∞, this implies

$$(T\varphi)(x) = \Phi(x)\varphi(x) \text{ a.e.}$$

for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. If $\hat{\Phi}$ denotes the operator in $L^p_{\mathscr{H}}$ defined by

$$(\hat{\Phi}f)(x) = \hat{\Phi}(x)f(x) \text{ a.e.,}$$

then we have $T\varphi = \hat{\Phi}\varphi$ for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. Since both T and $\hat{\Phi}$ are bounded in $L^p_{\mathscr{H}}$ and $\mathscr{K}_{\mathscr{H}}$ is dense in $L^p_{\mathscr{H}}$, it follows that $T = \hat{\Phi}$. Now we have only to put $\Phi(x) = T(x)$ in order to get the theorem.

If $p = \infty$ and T is bounded and weakly* continuous, then the transposed map T^* of T maps $L^1_{\mathscr{H}}$ into $L^1_{\mathscr{H}}$. Since T^* commutes with multiplication by functions in $\mathscr{A}$, T^* is expressed by an operator-valued function which is weakly measurable and uniformly bounded. Therefore T is also a uniformly bounded and weakly measurable operator-valued function $T(x)$. In this case, we clearly have

$$(Tf)(x) = T(x)f(x) \text{ l.a.e.}$$

for all $f \in L^\infty_{\mathscr{H}}$.

References

1. N. Bourbaki, *Intégration*, Chap. 6: Intégration vectorielle, Hermann, Paris, 1960.
2. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
3. N. Dunford and B. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc., **47** (1940), 323-392.
4. T. P. Srinivasan, *Doubly invariant subspaces*, Pacific J. Math., **14** (1964), 691-697.
5. ———, *Simply invariant subspaces and generalized analytic functions* (to appear).

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

IBARAKI UNIVERSITY
MITO, JAPAN

PANJAB UNIVERITY
CHANDIGARH, INDIA