CHARACTERIZATIONS OF CONVOLUTION SEMIGROUPS OF MEASURES

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A problem of fundamental importance in the study of compact topological semigroups is that of classifying in an *intrinsic* way each of a certain class of such semigroups. Unfortunately, virtually nothing has been done along these lines, even for such geometrically pleasing semigroups as the affine semigroups introduced by the author and H. Cohen in [3]. It is the purpose of this note to rectify this situation, at least for several particular types of compact affine topological semigroups; namely, certain convolution semigroups of real valued regular Borel measures on compact topological semigroups. The author's interest in this problem dates back to the early papers of Peck [13] and Wendel [21], and to some unpublished work of Wendel. Since that time, quite a literature has developed as regards these semigroups (e.g., see the bibliography), but almost without exception these papers merely study the *properties* of the semigroups without making any attempt to abstract sufficiently many of their properties to characterize them.

If S is a compact Hausdorff space and $P(S)$ denotes the set of all nonnegative regular Borel measures on S of variation norm one, it is known that $P(S)$ is a convex set which is compact in the weak-* topology $\left(\text{a net } \{\mu_\alpha\} \text{ of measures in } P(S) \text{ converges weak-* to } \mu \in P(S) \text{ if } \int f d\mu_\alpha \to \int f d\mu, \text{ for each real continuous function } f \text{ on } S\right)$. In similar fashion, the unit ball $B(S)$ of real-valued regular Borel measures of norm $\leqq 1$ is a compact convex set. When S is endowed with a continuous associative multiplication, each of $P(S)$ and $B(S)$ becomes a compact affine topological semigroup relative to convolution multiplication (see [10]); when such is the case, we denote these semigroups by $\tilde{S}$ and $\tilde{\tilde{S}}$ respectively. Note that our use of the symbol $\tilde{\tilde{S}}$ differs from that of Glicksberg in [10], where $\tilde{\tilde{S}}$, denoted the ball semigroup of *complex* measures.

In § 2, the following three types of images of the sets $P(S)$ and $B(S)$ are determined:

(a) all *extremal* images of $P(S)$; i.e., all continuous affine images under mappings which preserve extreme points,

(b) all one-to-one affine bicontinuous images of $P(S)$, and

(c) all one-to-one affine bicontinuous images of $B(S)$. The common requirements in each of (a), (b), and (c) are that the image K be

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compact and convex, have a separating family of real continuous affine functions, and have a compact set of extreme points. In (b), the additional requirement is that K be a simplex in the sense of Choquet [2] or Loomis [12]. In (c), one must require the existence of a compact subset T of K and a point z in K such that K is "symmetric relative to z", $T \cup (2z - T)$ is the set of extreme points of K, the closed convex hull K_1 of T is a simplex, and there exists on K a continuous real affine function which vanishes at z and is one on K_1.

In § 3, the imposition of a topological semigroup structure on S (and consequently on $P(S)$ and $B(S)$ via convolution) enables us to use the results of § 2 to characterize

(a) all extremal homomorphic images of $\tilde{S}$,

(b) all one-to-one affine bicontinuous and isomorphic images of $\tilde{S}$,

and (c) all one-to-one affine bicontinuous and isomorphic images of $\tilde{\tilde{S}}$. The only requirement needed in addition to the corresponding ones in § 2 is that the set of extreme points of K in cases (a) and (b) be a topological semigroup, while case (c) requires that the set T be a topological semigroup and the point z be a zero of K.

In each of §§ 2 and 3 additional characterizations of some interest are given. In our use of the Choquet simplex condition we prefer the formulation Loomis gives in [12], and it is a pleasure to record here the author's indebtedness to Professor Loomis for recent conversations during a visit by him as consultant to a Banach Algebra seminar at Louisiana State University.

1. Preliminaries. Throughout this paper the letter K will denote a compact convex subset of some real Hausdorff topological vector space. A mapping f with domain K and range another such set is *affine* if $x, y \in K$ and $0 \leqq a \leqq 1$ imply $f(ax + (1-a)y) = af(x) + (1-a)f(y)$. The symbol $L(K)$ will be used for the set of all continuous real valued affine functions defined on K, and it is clear that $L(K)$ needs not in general distinguish points of K. If, however, the vector space containing K is locally convex, the set $L(K)$ will distinguish points, and thus the assumption of local convexity (which we do not make) would permit a considerable simplification in the statements of the theorems to follow. If $z \in K$, the symbol $L_0(K)$ denotes the subset of $L(K)$ consisting of those functions each of which vanishes at z. It is easy to see that $L(K)$ separates points of K if and only if $L_0(K)$ does. If each of $L(K)$ and $L_0(K)$ is given the supremum norm, then both bocome real Banach spaces and as such have adjoint spaces of real continuous linear functionals, denoted respectively by $L(K)^*$ and $L_0(K)^*$. In each of these spaces we make use of the weak-* topology to embed K. Explicitly, if $x \in K$, denote by $\bar{x}$ [and x'] respectively the element of

$L(K)^*$ [of $L_0(K)^*$] for which $\bar{x}(l) = l(x)$ for all $l \in L(K)$ [$x'(l) = l(x)$ for all $l \in L_0(K)$]. It is obvious that these mappings are one-to-one if and only if $L(K)$ separates points of K, and that each is affine and continuous between K and its image, the latter given the relativized weak-* topology. The embedding $x \to \bar{x}$ was used by Loomis in [12] to formulate and extend Choquet's work [2]. Following Loomis, we say that K is a *simplex* if (i) $L(K)$ separates points of K and (ii) the truncated cone $T_r(K) = \{a\bar{z}: 0 \leqq a \leqq 1, z \in K\}$ determined by K in $L(K)^*$ is a lattice relative to the partial order: if $x, y \in T_r(K)$, then $\bar{x} \leqq \bar{y} \leqq \bar{y}$ means $\bar{y} - \bar{x} \in T_r(K)$. Our only contact with Loomis's work here (aside from borrowing some of his notation) is the use of his Theorem 6 to prove (when K has a compact set of extreme points) that K is a simplex if and only if K is the one-to-one affine bicontinuous image of some $P(S)$. The statement that K is an *affine semigroup* means (see [3]) there exists an associative separately affine multiplication on K; K is an *affine topological semigroup* if the multiplication function is also (doubly) continuous. The semigroups $\tilde{S}$ and $\tilde{\tilde{S}}$ are important examples of such semigroups, as are many semigroups of matrices. Another important class of such semigroups is the class of *group extremal* semigroups (the term is Wendel's), where by definition the compact affine topological semigroup K is *group extremal* if (i) it has an identity element and (ii) the set of elements with inverse coincides with the set of extreme points of K. Peck in [13] proved that each such semigroup has a zero, and Wendel (unpublished) observed that this result follows also from the fact that each such semigroup is the homomorphic image of some $\tilde{S}$, with S a compact group.

2. Affine images of $P(S)$ and $B(S)$. In this section of the paper K (as above) will be a convex compact set and $E(K)$ will denote its set of extreme points (a priori, possibly void). However, if $L(K)$ separates points of K, such is not the case; in fact, the Krein-Milman theorem holds for K. The first theorem gives conditions on K necessary and sufficient that K be the extremal image of some $P(S)$.

THEOREM 2.1. *Suppose that $L(K)$ separates points of K. Then K is the extremal image of some $P(S)$ if and only if $E(K)$ is compact.*

Proof. Suppose first that there exists a compact space S and an extremal mapping F (F is continuous affine onto and preserves extreme points) of $P(S)$ to K. By the Kelley-Arens theorem [1, Lemmas 3.1 and 3.2], the set of point measures on S is the set of extreme points of $P(S)$, hence is compact. Thus $F(E(P(S)))$ is compact and contains $E(K)$ (this inclusion holds always). Since F is extremal, the other inclusion is true also; i.e., $E(K)$ is compact.

Conversely, suppose $E(K)$ is compact, and let $S = E(K)$. By assumption, the embedding $\bar{K}$ of K in $L(K)^*$ is one-to-one bicontinuous affine and onto. We now define a mapping R on $P(S)$ onto $\bar{K}$ which is continuous affine and extremal. For $\mu \in P(S)$, and $l \in L(K)$, let $R_\mu(l) = \int_S l d\mu$. Fix a partition $\{E_i\}_{i=1}^n$ of S by Borel sets, $t_i \in E_i$, $1 \leqq i \leqq n$. Then $\sum_{i=1}^n l(t_i)\mu(E_i) = \sum_{i=1}^n \bar{t}_i(l)\mu(E_i) = \sum_{i=1}^n l(\bar{t}_i)\mu(E_i)$ (regarding l as a linear functional on $L(K)^*) = l(\sum_{i=1}^n \mu(E_i)\bar{t}_i)$. Since $\sum_{i=1}^n \mu(E_i) = 1$, $\mu(E_i) \geqq 0$, and $\bar{t}_i \in \bar{S}$, clearly the sum $\sum_{i=1}^n \mu(E_i)\bar{t}_i \in$ convex hull of $\bar{S}$. Since sums of the form $\sum_{i=1}^n l(t_i)\mu(E_i)$ converge to $\int_S l d\mu$, this implies that $R_\mu \in$ weak-* closed convex hull of $\bar{S} = \bar{K}$. It it clear that R maps the extreme points of $P(S)$ onto those of $\bar{K}$ and hence (since R is obviously continuous and affine) R maps $P(S)$ onto $\bar{K}$. This completes the proof.

The next theorem gives several different sets of necessary and sufficient conditions that K be the one-to-one affine bicontinuous image of a $P(S)$. It perhaps should be remarked that the requirement that K be a simplex can be stated without mentioning explicitly the embedding $\bar{K}$. We now do this, merely referring the reader to Loomis [12, Theorem 6] for the verification. The result will be stated as Lemma 2.1.

LEMMA 2.1. *Suppose that $L(K)$ separates points of K. Then K is a simplex if and only if given $\{a_i x_i\}_{i=1}^m$ and $\{b_j y_j\}_{j=1}^n$, where $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = 1$, $a_i \geqq 0$, $b_j \geqq 0$, $x_i, y_j \in K$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ and $\sum_{i=1}^m a_i x_i = \sum_{j=1}^n b_j y_j$, there exists $\{c_k z_k\}_{k=1}^p$ with $\sum_{k=1}^p c_k = 1$, $c_k \geqq 0$ and $z_k \in K$, $1 \leqq k \leqq p$, for which* (a) $\sum_{k=1}^p c_k z_k = \sum_{i=1}^m a_i x_i = \sum_{j=1}^n b_i y_j$, (b) $\{1, 2, \cdots, p\}$ *can be written as the pairwise disjoint union of sets $\{I_i\}_{i=1}^m$ and $\{J_j\}_{j=1}^n$, with $a_i = \sum_{k \in I_i} c_k$, $a_i x_i = \sum_{k \in I_i} c_k z_k$, $1 \leqq i \leqq m$, and $b_j = \sum_{k \in J_j} c_k$, $b_j y_j = \sum_{k \in J_j} c_k z_k$, $1 \leqq j \leqq n$.*

THEOREM 2.2. *The following conditions are mutually equivalent for K:* (1) *K is the one-to-one affine bicontinuous image of some $P(S)$,* (2) (a) *$E(K)$ is compact and* (b) *K is a simplex,* (3) (a) *$L(K)$ separates points of K,* (b) *$E(K)$ is compact, and* (c) *each continuous real function on $E(K)$ is extendable to be in $L(K)$.*

Proof. (1) $\rightarrow$ (2). Suppose F on $P(S)$ to K is one-to-one affine bicontinuous onto, where S is a compact Hausdorff space. It is easily verified then that $E(K)$ is compact and $L(K)$ separates points of K, for $P(S)$ has these properties. It thus remains to prove that the truncated cone $T_r(K)$ determined by $\bar{K}$ in $L(K)^*$ is a lattice. Let C be the truncated cone determined by $P(S)$ in the vector space of all real regular Borel measures on S. Clearly C itself is a lattice; we will now show that F has an extension G to all of C such that

(i) G is one-to-one and affine on C onto $T_r(K)$,

(ii) $G(0) = 0$, and

(iii) $\mu, \nu \in C$ implies $\mu \leqq \nu$ if and only if $G(\mu) \leqq G(\nu)$. If this can be proved, it will easily follow that $T_r(K)$ is a lattice. Thus, we proceed to the definition of G. If $a\mu \in C$, with $0 \leqq a \leqq 1$, $\mu \in P(S)$, define $G(a\mu) = aF(\mu)$. If $a\mu = b\nu$, with $a\mu, b\nu \in C$ and $f \equiv 1$ on S, then $a = a\int_S f d\mu = \int_S fd(a\mu) = \int_S fd(b\nu) = b\int_S fd\nu = b$; i.e., $a=b$. Thus, if $a = 0$, $a\mu = b\nu = 0$, and $G(a\mu) = aF(\mu) = 0 = bF(\nu) = G(b\nu)$. If $a \neq 0$, then $\mu = \nu$, so $G(a\mu) = aF(\mu) = bF(\nu) = G(b\nu)$; i.e., G is well defined on C into $T_r(K)$. Clearly G is onto $T_r(K)$. To show G is one-to-one, let $G(a\mu) = G(b\nu)$. Then $aF(\mu) = bF(\nu)$; i.e., $a\bar{x} = b\bar{y}$, with $x, y \in K$. If l is identically one on K then $l \in L(K)$ and $a = al(x) = a\bar{x}(l) = b\bar{y}(l) = bl(y) = b$; i.e., $a = b$, and $aF(\mu) = aF(\nu)$. Since $a = 0$ implies $a\mu = 0 = b\nu$, we can assume $F(\mu) = F(\nu)$, whence $\mu = \nu$ and $a\mu = b\nu$. It is obvious that (and this has been proved already) $G(0) = 0$, so it remains only to verify that G extends F (this is clear), that G is affine, and (iii) holds. To prove G is affine, let $a\mu, b\nu \in C$, $0 \leqq c \leqq 1$, and let $d = ca + (1-c)b$. If $d \neq 0$, then

$$\begin{aligned} G(ca\mu + (1-c)b\nu) &= G\Big[d\Big(\frac{ca}{d}\mu + \frac{(1-c)b\nu}{d}\Big)\Big] \\ &= d\Big[F\Big(\frac{ca}{d}\mu + \frac{(1-c)b\nu}{d}\Big)\Big] = d\Big[\frac{ca}{d}F(\mu) + \frac{(1-c)b}{d}F(\nu)\Big] \\ &= caF(\mu) + (1-c)bF(\nu) = cG(a\mu) + (1-c)G(b\nu) . \end{aligned}$$

If $d = 0$, then $\int_S 1d(ca\mu + (1-c)b\nu) = ca\int_S 1d\mu + (1-c)b\int_S 1d\nu = ca + (1-c)b = 0$, hence $ca\mu + (1-c)b\nu = 0$, and $ca = (1-c)b = 0$. The desired result easily follows. Now, suppose $\phi, \psi \in C$, with $\phi \leqq \psi$. Then $(1/2)0 + (1/2)\psi = (1/2)\psi = (1/2)\phi + (1/2)(\psi - \phi)$, whence $(1/2)G(\psi) = (1/2)G(0) + (1/2)G(\psi) = (1/2)G(\phi) + (1/2)G(\psi - \phi)$ and this implies $G(\psi) - G(\phi) = G(\psi - \phi) \in T_r(K)$; i.e., $G(\phi) \leqq G(\psi)$. To conclude (1) $\rightarrow$ (2), we suppose $G(\phi) \leqq G(\psi)$, with $\phi, \psi \in C$. Then there is $\alpha \in C$ such that $G(\alpha) = G(\psi) - G(\phi)$, so $G(\psi) = G(\alpha) + G(\phi)$. From this, it follows that $G(\psi/2) = (1/2)G(\psi) = G((\alpha + \phi)/2)$ and since G is one-to-one, $(\psi/2) = ((\alpha + \phi)/2)$. Thus $\psi = \alpha + \phi$, $\psi - \phi = \alpha \in C$, and $\phi \leqq \psi$.

(2) $\rightarrow$ (1). Let $S = E(K)$ and define R on $P(S)$ into $L(K)^*$ as in the proof of Theorem 1; thus R is an extremal mapping of $P(S)$ onto $\bar{K}$, and there remains only the proof that R is one-to-one. Let $\mu, \nu \in P(S)$ with $R_\mu = R_\nu$. Extend μ and ν to the Borel sets of K by defining them to be zero at Borel sets missing S (call these extensions $\bar{\mu}$ and $\bar{\nu}$), and note then that $R'_{\bar{\mu}} = R'_{\bar{\nu}}$, where, for example, $R'_{\bar{\mu}}(l) = \int_K ld\bar{\mu}$, all $l \in L(K)$. The mapping R' is the resultant mapping

used by Loomis in [12], and in Theorem 6 he proved that (since $\bar{K}$ a lattice clearly implies the set of all subelements of $\bar{x} = R'_{\bar{\mu}} = R'_{\bar{\nu}}$ is a lattice) there exists a unique extremal measure whose resultant is $\bar{x}$. Since $S = E(K)$ is known here to be compact, this says [12, p. 517] that $\mu = \nu$. Thus R is one-to-one, and (1) is proved.

(2) $\rightarrow$ (3). As was seen in the proof of (2) $\rightarrow$ (1), the mapping R on $P(S)$ to $\bar{K}$ is one-to-one bicontinuous and affine onto, where $S = E(K)$. If f is continuous real valued on S, denote by h the restriction to $P(S)$ of the linear functional on the space of all real regular Borel measures determined by $f: h(\mu) = \int_S f d\mu$, all $\mu \in P(S)$. Then $x \in E(K)$ implies (denote by $\dot{x}$ the point measure on S determined by x) $\bar{f}(x) \equiv h(R^{-1}(\bar{x})) = h(\dot{x}) = \int_S f d\dot{x} = f(x)$; i.e., $\bar{f}$ extends f to be in $L(K)$.

(3) $\rightarrow$ (1). If $\mu, \nu \in P(S)$ and $R_\mu = R_\nu$, where $S = E(K)$, and f is continuous real valued on S, let $\bar{f}$ be its extension to K to be continuous and affine. Then $\int_S f d\mu = \int_S \bar{f} d\mu = \int_S \bar{f} d\nu = \int_S f d\nu$, i.e., μ and ν are equal as functionals on the space of real continuous functions on S. The Riesz theorem then implies $\mu = \nu$ as measures. This concludes the proof of Theorem 2.

REMARK 2.1. It is easy to verify that in the preceding theorem the condition (c) of part (3) may be replaced by (c'): each continuous real function f on $S = E(K)$ is *uniquely* extendable to $\bar{f} \in L(K)$. It follows then that $f \rightarrow \bar{f}$ is an isometric isomorphism of $C(S)$ onto $L(K)$, where $C(S)$ is the space of real continuous functions on S, and each space is given the supremum norm.

We conclude § 2 now with our characterization of all one-to-one affine and bicontinuous images K of real unit balls of measures. The conditions given here (in Theorem 2.3) are quite natural with possibly one exception: the requirement that $L_0(K)$ contain a function which is identically one on T seems somewhat artificial. However, some remarks regarding this condition are made following the proof of the theorem, and these may help place the condition in proper perspective.

THEOREM 2.3. *The following conditions are mutually equivalent for* K:

(1) *K is the one-to-one affine bicontinuous image of some* $B(S)$,

(2) (a) *there exists $z \in K$ and compact $T \subset K$ such that $x \in K$ implies $2z - x \in K$ and $E(K) = T \cup (2z - T)$* (b) *if K_1 denotes the closed convex hull of T, then K_1 is a simplex,* (c) *$L(K)$ separates points of K and $L_0(K)$ contains a function which is identically one on K_1,*

(3) (a) *part* (a) *of* (2) *holds and* (b) $L(K)$ *separates points of* K *and each continuous function* f *on* T *is extendable to an* $\bar{f} \in L_0(K)$.

Proof. (1) $\rightarrow$ (2). Let F be a one-to-one affine bicontinuous mapping of $B(S)$ onto K, where S is a compact Hausdorff space. Let $z = F(0)$, $T =$ the image under F of all measures determined by the points of S. If $x \in K$, there exists $\mu \in B(S)$ such that $F(\mu) = x$. Then $2z - x = 2F(0) - F(\mu) = F(2 \cdot 0 - \mu) = F(-\mu) \in K$. The Kelley-Arens theorem [1, Lemmas 3.1 and 3.2] says $\underset{\cdot}{S} \cup (-\underset{\cdot}{S})$ is the set of extreme points of $B(S)$, where $\underset{\cdot}{S}$ is the set of point measures. Hence $E(K) = F[\underset{\cdot}{S} \cup (-\underset{\cdot}{S})] = F(\underset{\cdot}{S}) \cup F(-\underset{\cdot}{S}) = T \cup (2z - T)$. Thus (a) of (2) is verified. Using the Kelley-Arens result again ($\underset{\cdot}{S}$ is the set of extreme points of $P(S)$), $K_1 = F(P(S))$ hence Theorem 2.2 implies K_1 is a simplex. Since $B(S)$ has a separating family of continuous real affine functions vanishing at 0 and contains one which is identically one on $P(S)$, part (c) of (2) follows easily. Thus, (2) is proved.

(2) $\rightarrow$ (3). Since $T \subset E(K)$, it is obvious that $T \subset E(K_1)$. On the other hand, the closed convex hull of T is K_1, hence T (being closed) contains $E(K_1)$; i.e., $T = E(K_1)$. Consider the embeddings K', K_1', and $-K_1'$ in $L_0(K)^*$ of (respectively) K, K_1, and $(2z - K_1)$. Since K_1' and $-K_1'$ are compact convex sets whose union contains $T' \cup -T' = E(K')$, it is clear that the convex hull of $K_1' \cup -K_1'$ is compact (and convex) and thus coincides with K'. Now K_1' is a simplex (by Theorem 2.2) with $E(K_1) = T'$, so each continuous real function on T' can be extended to be continuous and affine on K_1'. This fact together with the fact that $x \rightarrow x'$ is one-to-one affine and bicontinuous on K onto K' reduces the problem to proving that each continuous affine f on K_1' extends to a continuous affine function on K' which vanishes at $z' = 0 \in L_0(K)^*$. Fix such an f, and let $ax' + (1-a)(-y') \in K'$, where $0 \leqq a \leqq 1$, $x, y \in K_1$ (note that K' is the union of the line segments $[p, q]$, with $p \in K_1'$, $q \in -K_1'$). Define $\bar{f}[ax' + (1-a)(-y')] = af(x') - (1-a)f(y')$. We show first that $\bar{f}$ is well defined. Let $ax' + (1-a)(-y') = bw' + (1-b)(-t')$, with $0 \leqq a, b \leqq 1$, $x, y, w, t \in K_1$, and let $l_0 \in L_0(K)$ be one on K_1. Then $l_0(ax' + (1-a)(-y')) = al_0(x) + (1-a)l_0(-y) = a + (1-a)(-1)$, since $l_0 \equiv -1$ on $2z - K_1$. Similarly, $l_0(bw' + (1-b)(-t')) = b + (1-b)(-1)$ so $2a - 1 = 2b - 1$, and $a = b$. But then $ax' + (1-a)(-y') = aw' + (1-a)(-t')$, hence $ax' + (1-a)t' = aw' + (1-a)y'$. Since f is affine, $af(x') + (1-a)f(t') = af(w') + (1-a)f(y')$, so $af(x') - (1-a)f(y') = af(w') - (1-a)f(t')$; i.e., $\bar{f}$ is well defined. That $\bar{f}$ extends f follows from $\bar{f}(x') = f(1 \cdot x' + 0 \cdot (-x')) = 1 \cdot f(x') - 0 \cdot f(-x') = f(x')$. To prove $\bar{f}$ is continuous on K', let $\{a_\alpha x_\alpha' + (1 - a_\alpha)(-y_\alpha')\}$ be a net in K' converging (weak-*) to $ax' + (1-a)(-y')$. The net $\{(a_\alpha, x_\alpha', -y_\alpha')\}$ in the compact space $[0, 1] \times K_1' \times (-K_1')$ has a subnet, say $\{(a_\beta, x_\beta', -y_\beta')\}$, converging in the product

space to $(b, w, -t)$. It follows that $a_\beta x'_\beta + (1 - a_\beta)(-y'_\beta) \to bw' + (1 + b)(-t')$, whence $bw' + (1 - b)(-t') = ax' + (1 - a)(-y')$, and $a=b$. Thus, $ax' + (1 - a)(-y') = aw' + (1 - b)(-t')$. But then (as above) $af(x')+(1-a)f(t')=af(w')+(1-a)f(y')$, whence $\bar{f}[a_\beta x'_\beta+(1-a_\beta)(-y'_\beta)] = a_\beta f(x'_\beta) - (1 - a_\beta)f(y'_\beta) \to af(w') - (1-a)f(t') = af(x') - (1-a)f(y') = \bar{f}[ax' + (1 - a)(-y')]$; i.e., $\bar{f}$ is continuous. Note also that $\bar{f}(0) = \bar{f}[(1/2)x' + (1/2)(-x')] = (1/2)f(x') - (1 - (1/2))f(x') = 0$. Finally, we show $\bar{f}$ is affine. To this end, let $ax' + (1 - a)(-y')$ and $bw' + (1 - b)(-t') \in K'$, $0 \leqq c \leqq 1$. Then $\phi = c[ax' + (1 - a)(-y')] + (1 - c)[bw' + (1 - b)(-t')] = cax' + (1 - c)bw' + c(1 - a)(-y') + (1 - c)(1-b)(-t')$. If $d = ca + (1-c)b$, then $1 - d = c(1-a) + (1-c)(1-b)$. If then $d \neq 0$, $d \neq 1$,

$$\begin{aligned}
\bar{f}(\phi) &= \bar{f}\Big[d\Big\{\frac{ca}{d}x' + \frac{(1-c)b}{d}w'\Big\} \\
&\quad + (1-d)\Big\{\frac{c(1-a)}{1-d}(-y') + \frac{(1-c)(1-b)}{1-d}(-t')\Big\}\Big] \\
&= d\Big[f\Big(\frac{ca}{d}x' + \frac{(1-c)b}{d}w'\Big)\Big] \\
&\quad - (1-d)\Big[f\Big(\frac{c(1-a)}{1-d}y' + \frac{(1-c)(1-b)}{1-d}t'\Big)\Big] \\
&= caf(x') + (1-c)bf(w') - c(1-a)f(y') \\
&\quad - (1-c)(1-b)f(t') = c[af(x') - (1-a)f(y')] \\
&\quad + (1-c)[bf(w') - (1-b)f(t')] \\
&= c\bar{f}[ax' + (1-a)(-y')] + (1-c)\bar{f}[bw' + (1-b)(-t')] \,.
\end{aligned}$$

Each of the cases $d = 0$ and $d = 1$ is resolved into easily handled sub-cases, and the arguments will be omitted. This completes the proof of (2) $\to$ (3).

(3) $\to$ (1). Define R on $B(T)$ into $L_0(K)^*$ as usual: for $\mu \in B(T)$ and $l \in L_0(K)$, let $R_\mu(l) = \int_T l d\mu$. By an argument similar to one used before, R maps $B(T)$ onto the weak-* closed convex symmetric hull in $L_0(K)^*$ of T'. This set is K', and as before R is affine and continuous, so the proof that R is one-to-one is all that remains. Let $\mu, \nu \in B(T)$ and $R_\mu = R_\nu$. Then if f is continuous on T, it has an extension $\bar{f}$ to be in $L_0(K)$. But then $\int_T f d\mu = \int_T \bar{f} d\mu = R_\mu(\bar{f}) = R_\nu(\bar{f}) = \int_T \bar{f} d\nu = \int_T f d\nu$, and $\mu = \nu$ as functionals on the real continuous functions on T. The Riesz theorem completes the proof that $\mu = \nu$, and thus the theorem is concluded.

LEMMA 2.2. *Suppose K is compact convex, with $z \in K$ and $T \subset K$ such that $x \in K$ implies $2z - x \in K$ and T is compact. Let further*

$L_0(K)$ separate points of K, $E(K) = T \cup (2z - T)$, and K_1 be the closed convex hull of T. The following conditions are then mutually equivalent:

(1) *$L_0(K)$ contains l_0 which is one on K_1,*

(2) *$0 \leqq a, b \leqq 1$, $x, y, w, t \in K_1$ and $ax' + (1-a)(-y') = bw' + (1-b)(-t')$ imply $a = b$,*

(3) *each $l \in L(K_1)$ can be extended to an $\bar{l} \in L_0(K)$.*

Proof. The implication (1) $\to$ (3) is part of the proof of (2) $\to$ (3) of the previous theorem. If (3) holds, then since $L(K_1)$ contains the function which is constantly one on K_1, clearly (1) holds; i.e., (3) $\to$ (1). The proof that (1) $\to$ (2) is also in the proof of (2) $\to$ (3) of Theorem 2.3, so it remains only to show that (2) $\to$ (1). This proof, however, is also found in (2) $\to$ (3) of the previous theorem, for all that was needed to extend $f \in L(K_1)$ to $\bar{f} \in L_0(K)$ was condition (2) of the present lemma. In particular, then, the function identically one on K_1 is extendable; i.e., (1) holds.

REMARK 2.2. Given the hypotheses of Lemma 2.2., each of (1) through (3) of that lemma is equivalent to the geometric condition: *Let C be the cone $\{ax' : a \geqq 0,\ x \in K_1\}$ in $L_0(K)^*$ determined by K_1'. Then 0 is not in K_1' and each $\phi \neq 0$ in C is uniquely representable as $\phi = ax'$, for some $a > 0$ and $x \in K_1$.* The proof of this statement is quite easy, as follows. Let $ax' = by'$, with $a, b > 0$, $x, y \in K_1$. Then, by (1) of Lemma 2.2, $a = al_0(x) = l_0(ax') = l_0(by') = bl_0(y) = b$, hence $a = b$ and thus $x' = y'$. Clearly, 0 is not in K_1'. Conversely, if this geometric condition obtains, let $0 \leqq a, b \leqq 1$, $x, y, w, t \in K_1$ and $ax' + (1-a)(-y') = bw' + (1-b)(-t')$. Then $ax' + (1-b)t' = bw' + (1-a)y'$. Let $d = a - b + 1$, and note that $b - a + 1 = 2 - d$. Thus if $d \neq 0$, $d \neq 2$, we have that $d[(a/d)x' + ((1-b)/d)t'] = 2 - d[(b/(2-d))w' + ((1-a)/(2-d))y']$, hence (by the condition) $d = 2 - d$. Thus $d = 1$, and $a = b$. Note that if $d = 0$, then $a + 1 = b \leqq 1$ implies $a \leqq 0$, so $a = 0$ and $b = 1$. But then $-y' = w'$, which says $(1/2)(w' + y') = 0 \in K_1'$. If $d = 2$, then $b + 1 = a \leqq 1$ implies $b = 0$ and $a = 1$. Then $x' = -t'$, and again $0 \in K_1'$. This completes the proof.

3. Affine homomorphic and isomorphic images of $\tilde{S}$ and $\tilde{\tilde{S}}$. In this section we are interested in homomorphic and isomorphic (as well as affine) images K of the convolution semigroups $\tilde{S}$ and $\tilde{\tilde{S}}$. The essential difficulties involved in the characterizations we obtain have already been solved in § 2, and the additional requirements are (primarily) that (a) K be a compact affine topological semigroup and (b) $E(K)$ or T be a compact topological semigroup.

The following lemma takes care of most of the additional difficulties encountered when one requires a topological semigroup structure on S and K.

LEMMA 3.1. *Let K be a compact affine topological semigroup, L a norm closed linear subspace of $L(K)$ separating points of K, and $T \subset E(K)$ be a compact sub-semigroup of K. Denote by $x \to x^0$ the embedding of K into L^*, giving L^* the weak-* topology determined by pointwise convergence on L, and let A* (let B) *respectively denote the closed convex symmetric hull of T^0 (the closed convex hull of T^0). Then:*

(1) *If R on $P(T)$ into L^* is defined (for $\mu \in P(T), l \in L$) by $R_\mu(l) = \int_T l d\mu$, then R is a continuous affine homomorphism of $\widetilde{T}$ onto B.*

(2) *If A is contained in K^0 and has 0 as a zero and Q on $B(T)$ into L^* is defined (for $\mu \in B(T)$ and $l \in L$) by $Q_\mu(l) = \int_T l d\mu$, then Q is a continuous affine homomorphism of $\widetilde{\widetilde{T}}$ onto A. Note that R and Q are the mappings of Theorem* 2.1 *and* 2.3 *respectively, if $L = L(K)$ and $L_0(K)$ respectively.*

Proof. The statements regarding Q and R (except for those involving the homomorphism properties) are proved exactly as in Theorems 2.1 and 2.3. It therefore suffices, for example, to prove that Q is a homomorphism, so let $\mu, \nu \in B(T)$. Suppose first $\mu = \sum_{i=1}^m a_i\mu_i$, $\nu = \sum_{j=1}^n b_j\nu_j$, with $\sum_1^m a_i = 1 = \sum_1^n b_j$ and $a_i, b_j \geqq 0$, μ_i, ν_j extreme points of $B(T)$, all i and j. Then $\mu\nu = \sum_{i,j} a_ib_j\mu_i\nu_j$, so $Q_{\mu\nu} = \sum_{i,j} a_ib_jQ_{\mu_i\nu_j}$. If both μ_i and ν_j are point measures determined respectively by t_i and $s_j \in T$, then $Q_{\mu_i\nu_j} = Q_\alpha$, where α is the point measure determined by $t_is_j \in T$. But then $Q_\alpha(l) = \int_T l d\alpha = l(t_is_j) = (t_is_j)^0(l) = (t_i^0 \cdot s_j^0)(l) = (Q_{\mu_i} \cdot Q_{\nu_j})(l)$; i.e., $Q_{\mu_i\nu_j} = Q_{\mu_i} \cdot Q_{\nu_j}$. Now if $\alpha, \beta \in B(T)$, then $-\alpha \cdot \beta = -(\alpha \cdot \beta) = \alpha \cdot (-\beta)$ and $(-\alpha)\cdot(-\beta) = \alpha \cdot \beta$. Thus if $\mu_i(\nu_j)$ is each the minus of a point measure, say $\mu_i = -\alpha_i$, $\nu_j = -\beta_j$, then $Q_{\mu_i\nu_j} = Q_{\alpha_i\beta_j} = Q_{\alpha_i} \cdot Q_{\beta_j} = x^0 \cdot y^0$, with $x, y \in T$. On the other hand $Q_{\mu_i} \cdot Q_{\nu_j} = (-x^0)\cdot(-y^0) = (2\cdot 0 - x^0)(2 \cdot 0 - y^0) = 4\cdot 0 - 2\cdot 0 - 2 \cdot 0 + x^0 \cdot y^0 = x^0 \cdot y^0$; i.e., $Q_{\mu_i\nu_j} = Q_{\mu_i} \cdot Q_{\nu_j}$. If (say) μ_i and α_j are point measures, with $\nu_j = -\alpha_j$, then $Q_{\mu_i\nu_j} = -Q_{\mu_i\alpha_j} = -(Q_{\mu_i} \cdot Q_{\alpha_j}) = -(x^0 \cdot y^0)$, with $x, y \in T$. On the other hand $Q_{\mu_i} \cdot Q_{\nu_j} = x^0 \cdot (-y^0) = x^0 \cdot (2 \cdot 0 - y^0) = 2 \cdot 0 - (x^0 \cdot y^0) = -(x^0 \cdot y^0)$; i.e., in all cases, $Q_{\mu_i\nu_j} = Q_{\mu_i} \cdot Q_{\nu_j}$. Thus, $Q_{\mu\nu} = \sum_{i,j} a_ib_jQ_{\mu_i\nu_j} = (\sum_i a_iQ_{\mu_i}) \cdot (\sum_j b_jQ_{\nu_j}) = Q_\mu \cdot Q_\nu$. Suppose next that $\mu = \sum_{i=1}^n a_i\mu_i$, with μ_i extreme points of $B(T)$. Then $\mu\nu = \sum_{i=1}^n a_i\mu_i\nu$, so $Q_{\mu\nu} = \sum_{i=1}^n a_iQ_{\mu_i\nu}$. Let $\{\nu_a\}$ be a net of convex combinations of extreme points of $B(T)$ converging to ν; then $\mu_i\nu_a \to_a \mu_i\nu$, so Q continuous implies $Q_{\mu_i} \cdot Q_{\nu_a} = Q_{\mu_i\nu_a} \to_a Q_{\mu_i\nu}$. Since $Q_{\mu_i} \cdot Q_{\nu_a} \to_a Q_{\mu_i} \cdot Q_\nu$, it follows that $Q_{\mu_i\nu} = Q_{\mu_i} \cdot Q_\nu$, $1 \leqq i \leqq n$, hence $Q_{\mu\nu} = \sum_{i=1}^n a_i(Q_{\mu_i} \cdot Q_\nu) =$

$(\sum_{i=1}^{n} a_i Q_{\mu_i}) \cdot Q_\nu = Q_\mu \cdot Q_\nu$. Now let μ, ν be arbitrary, $\{\mu_a\}$ a net of convex combinations of extreme points converging to μ. Then, by the preceding, $Q_{\mu_a} \cdot Q_\nu = Q_{\mu_a \nu} \to_a Q_{\mu\nu}$, while $Q_{\mu_a} \cdot Q_\nu \to_a Q_\mu \cdot Q_\nu$. Thus, Q is a homomorphism and the argument for R is similar, though simpler.

THEOREM 3.1. *Suppose K is a compact affine topological semigroup with $L(K)$ separating points of K. Then K is the extremal homomorphic image of the convolution semigroup $\tilde{T}$ of measures over some compact Hausdorff semigroup T if and only if $E(K)$ is a compact semigroup.*

Proof. It is now obvious that the extremal homomorphic image of a $\tilde{T}$ has a compact semigroup of extreme points. For the converse, use Theorem 2.1 and Lemma 3.1, letting the L of Lemma 3.1 be $L(K)$, $T = E(K)$, and letting the mapping be the R of Lemma 3.1, part (1).

COROLLARY 3.1.1. *Suppose K is a compact affine topological semigroup with $L(K)$ separating points of K. Then K is group extremal (i.e., K has an identity and $E(K)$ is a compact group) if and only if K is the affine continuous homomorphic image of some $\tilde{T}$, with T a compact topological group.*

Proof. Suppose first that K has an identity element and $E(K)$ is a compact group. Now by Wendel's theorem [3, Theorem 1] the maximal group T containing the identity is contained in $E(K)$, hence $E(K) = T$. Now the mapping R of Theorem 3.1 and Lemma 3.1 (since T is a group and R is a homomorphism) is extremal, so K is the affine continuous and homomorphic image under R of $\tilde{T}$. If this condition holds, then K is the extremal image of a semigroup of measures over a compact group, hence $E(K)$ is a compact group and K has an identity. This completes the proof.

REMARK 3.1. The group extremal semigroups of the preceding corollary are known to always have a zero [13, 3]. Familiar examples of such semigroups are the closed unit disc of complex numbers, with ordinary complex multiplication, and the interval $[-1, 1]$ of reals, with ordinary multiplication.

In the following theorems S will always be a compact semigroup.

THEOREM 3.2. *The following conditions are mutually equivalent for the compact affine topological semigroup K:*

(1) *K is the one-to-one affine bicontinuous and isomorphic image of some probability semigroup $\tilde{S}$,*

(2) (a) *$E(K)$ is a compact topological semigroup, and* (b) *K is a simplex,*

(3) (a) *$L(K)$ separates points of K,* (b) *$E(K)$ is a compact semigroup, and* (c) *each continuous real function f on $E(K)$ is extendable to $\bar{f} \in L(K)$.*

Proof. If (1) holds, then Theorem 2.2 implies everything claimed in (2) save the statement that $E(K)$ is a semigroup, and this follows because $E(K)$ is the isomorphic image of a semigroup, namely S. Now the implication (2) $\rightarrow$ (3) follows directly from (2) and Theorem 2.2. To conclude the proof of the theorem, we show that (3) $\rightarrow$ (1). Here again we use Lemma 3.1, letting $T = E(K)$, $L = L(K)$, and the mapping be the R of Lemma 3.1. Note this function is the same as that used in (3) $\rightarrow$ (1) of Theorem 2.2. The result of applying Lemma 3.1 and Theorem 2.2 is that K is the one-to-one bicontinuous affine and isomorphic image of $\tilde{T}$ under R.

THEOREM 3.3. *Let K be a compact affine topological semigroup. The following conditions are mutually equivalent.*

(1) *K is the one-to-one affine bicontinuous and isomorphic image of some real unit ball semigroup $\approx\!\!\!\!\!S$,*

(2) *the same as* (2) *of Theorem* 2.3 *except for the additional requirements that the z and T of that theorem be a zero for K and a semigroup, respectively,*

(3) *the same as* (3) *of Theorem* 2.3 *except requiring additionally that the z and T of that theorem be a zero for K and a semigroup, respectively.*

Proof. The only conditions which need to be checked (in virtue of Theorem 2.3) are those involving the semigroup structures on the spaces involved. Thus, in (1) $\rightarrow$ (2), the zero of $\overset{\approx}{S}$ maps onto z (hence z is a zero for K) and $\underset{\cdot}{S}$ maps onto T (hence T is a semigroup). The implication (2) $\rightarrow$ (3) follows immediately from Theorem 2.3 and the additional assumptions on z and T. To prove, finally, that (3) $\rightarrow$ (1), note that z maps into $0 \in L_0(K)^*$ under the embedding K into K'; thus, 0 is a zero for K'. Further, T maps onto T' and the closed convex symmetric hull A of T' in $L_0(K)^*$ is K'. In Lemma 3.1, then, we take $L = L_0(K)^*$, and let Q on $B(T)$ onto K' be as in that lemma. Then Lemma 3.1 together with Theorem 2.3 insure that K' is the one-to-one affine bicontinuous and isomorphic image of $\overset{\approx}{T}$, so also then is K. This completes the proof.

REMARK 3.2. A simple example illustrating the last two theorems may be constructed in the plane, as follows. Let the K of Theorem

3.3. be all pairs (x, y) of reals such that $|x| + |y| \leqq 1$, and let $T = \{i, j\}$, where $i = (1, 0)$, $j = (0, 1)$. Define $i^2 = i$, $j^2 = j$, $ij = ji = j$. The multiplication (on the entire plane) is defined as follows: $(ai + bj)(ci + dj) = aci + (ad + bc + bd)j$. Then K_1 is $\{(x, y): x, y \geqq 0, x + y = 1\}$, a simplex, and is affinely isomorphic with $[-1, 1]$ with usual multiplication. K itself is, of course, a unit ball semigroup of measures, with $z = (0, 0)$.

Examples of similar nature could be constructed on any finite simplex, of course, the requirements being that multiplications of a suitable nature be defined on the set of vertices. It is clear that exactly n distinct geometric figures exist in n-space on which probability semigroup structures can be defined; namely, the n simplexes each with i vertices, $2 \leqq i \leqq n + 1$. Thus the number of distinct probability semigroups in n-space is $= \sum_{i=2}^{n+1} A(i)$, where $A(i)$ is the number of distinct associative multiplications on a set of i elements (isomorphic and anti-isomorphic semigroups are identified).

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