MINIMUM PROBLEMS OF PLATEAU TYPE IN THE BERGMAN METRIC SPACE

KYONG T. HAHN

Dedicated to my teacher Professor C. Loewner on his seventieth birthday

1. Introduction. In this paper we are concerned with the existence of minimal surfaces with respect to the B-area (see § 4) and related problems in a bounded domain D in the space $\boldsymbol{C}^2$ of two complex variables z_1, z_2.

Let $\boldsymbol{K}_D(z, \bar{z})$, $z = (z_1, \cdots, z_n)$, be the Bergman kernel function of a bounded domain D in the space $\boldsymbol{C}^n$ of n complex variables. Throughout this paper, we assume $\boldsymbol{K}_D(z, \bar{z})$ has the boundary value infinity at every point on the boundary of D. The kernel $\boldsymbol{K}_D(z, \bar{z})$ enables us to define the Bergman metric

$$ds_D^2(z) = \sum_{\mu,\nu=1}^{n} T_{\mu\bar{\nu}}(z, \bar{z}) dz_\mu d\bar{z}_\nu, \; T_{\mu\bar{\nu}} = \frac{\partial^2 \log \boldsymbol{K}_D}{\partial z_\mu \partial \bar{z}_\nu}, \tag{1.1}$$

which is invariant with respect to pseudo-conformal mappings [4, pp. 51–53]. Using (1.1) we construct (see § 2) the complete Bergman metric space (D, d) over D and state a theorem for complete Riemannian spaces that for any two points in D, there exists a minimal curve with respect to d which connects the two points.

In § 3 we show that, if D is a plane domain bounded by finitely many boundary components $b_1, b_2, \cdots, b_n$, then there exists a minimal closed curve with respect to d among those curves which are homotopic to a fixed inner boundary component, say b_1, in $\overline{D(b_1)}$ (see § 3 for notation). If D is doubly connected, there exists a unique minimal closed curve in D. Furthermore, we prove a distortion theorem which gives bounds for the Bergman lengths of the minimal closed curves.

Analogous results are obtained in the case of two complex variables replacing the length by the B-area.

For a closed Jordan curve Γ in a complete metric space (D, d), we ask whether there exists a minimal surface with respect to the B-area which spans Γ. Answers to this question which constitute the main result of this paper are given in § 4.

As a generalization of § 3, we consider a domain D which is topologically equivalent to a product domain of the form $D_1 \times D_2$,

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where D_k is a bounded domain as considered in § 3. When does there exist a minimal closed surface with respect to the B-area among those surfaces which are homotopic to T_1 in $\overline{D(T_1)}$ (see § 5 for notation)?

Answers are given in § 5. Distortion theorems for the minimal surfaces are given in § 6.

2. The Bergman metric space. A (continuous) curve c in D is said to be *regular* if it admits a regular (parametric) representation, i.e., there exists a continuously differentiable representation

$$G \mid I: z_k = G_k(t), k = 1, 2, \cdots, n, t \in I = [a, b] , \tag{2.1}$$

and dG_k/dt never vanish simultaneously at any $t \in I$. A curve c in D is said to be *piecewise regular* if it admits a piecewise regular representation, i.e., there exists a partition Δ: $a = t_0 < t_1 < \cdots t_{m-1} < t_m = b$ such that $G \mid [t_{k-1}, t_k]$ is regular for $k = 1, 2, \cdots, m$.

For a piecewise regular curve c given by (2.1), we define

$$L_D(c) = \int_a^b \left[\sum_{\mu,\nu=1}^{n} T_{\mu\bar{\nu}}(G(t), \overline{G(t)}) \frac{dG_\mu}{dt} \frac{\overline{dG_\nu}}{dt} \right]^{1/2} dt . \tag{2.2}$$

$L_D(c)$ is independent of the choice of piecewise regular representations of c. $L_D(c)$ will be called the *Bergman length* of c.

For any two points z^1 and z^2 in D, we define a distance function d by

$$d(z^1, z^2) = \inf_c L_D(c) , \tag{2.3}$$

where c runs over all piecewise regular curves which connect z^1 and z^2. Then the following theorem holds [15, § 16].

THEOREM 2.1. *d satisfies all the axioms for a metric and the metric space (D, d) is topologically equivalent to the metric space (D, ρ) with the Euclidean metric ρ. Moreover, the metric space (D, d) is finitely connected in the sense that every pair of points in D can be connected by a curve of finite Bergman length.*

The metric space (D, d) will be called the *Bergman metric space* over D. The significance of this metric space is that all metric properties are invariant under pseudo-conformal mappings.

We define the length (generalized) of a continuous curve c in D in the following way: For a partition $\Delta(I) = \{I_1, I_2, \cdots, I_m\}$, $I_k = [t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, of I, we define

$$\sigma(G; \Delta(I)) = \sum_{k=1}^{m} \sigma(G; I_k), \quad \sigma(G; I_k) = d(G(t_k), G(t_{k-1})) .$$

Further, we define

$$\mathscr{L}_D(c) = \sup_{\Delta} \sigma(G; \Delta(I)) ,$$

where Δ runs over all possible partitions of I. Then $\mathscr{L}_D(c)$ is independent of the choice of continuous representations of c. Clearly, the functional $\mathscr{L}_D$ is lower semi-continuous, i.e.,

$$\mathscr{L}_D(c) \leqq \liminf_{k\to\infty} \mathscr{L}_D(c_k), \text{ if } c_k \to c .$$

Further, for every piecewise regular curve c, $L_D(c) = \mathscr{L}_D(c)$ [15, § 16].

If $\mathscr{L}_D(c) < \infty$, c is said to be *rectifiable*. A curve is said to be *completely degenerated* if there is a representation $G \mid I$ such that G is constant on I. A representation $G \mid I$ is said to be *normal* if $\mathscr{L}_D(G; [t, t']) = t' - t$, for $t, t' \in I, t < t'$.

Let c be a rectifiable curve which is not completely degenerated. Then c admits a normal representation $G \mid [0, \mathscr{L}_D(c)]$, If we set $F(t) = G(t\mathscr{L}_D(c))$, $t \in I_0$, $I_0 = [0, 1]$, then $F \mid I_0$ is also a representation of c. Such a representation $F \mid I_0$ is called a *reduced representation* of c. For a closed curve, F is defined on $(-\infty, \infty)$ and is periodic of period 1. It is, therefore, enough to consider F on I_0. If $F \mid I_0$ is a reduced representation of a curve c, then the inequality

$$d(F(t), F(t')) \leqq \mathscr{L}_D(c) \mid t - t' \mid \tag{2.4}$$

holds for every $t, t' \in I_0$.

A metric space is called *complete* if every bounded infinite subset contains a limit point in the metric space. If D is homogeneous, (D, d) is always complete. Further, for every bounded generalized analytic polyhedron $D, (D, d)$ is complete. This is a result of S. Kobayashi (see [11] for details). For domains D in the space C^n, $n \leqq 2$, Bergman has shown that the distance from a point in D to the boundary becomes infinite under certain hypothesis on the boundary of D [1], [6, Chap. III]. It is clear, in this case, that the metric space (D, d) is complete. Without going into great details in this direction, we shall assume in the sequel that the metric space (D, d) is always complete.

A curve K in (D, d) which connects z^1 and z^2 in D is called a *minimal curve* between z^1 and z^2 if $\mathscr{L}_D(K) \leqq \mathscr{L}_D(c)$ for all curves c connecting z^1 and z^2.

THEOREM 2.2. *For any two points z^1 and z^2, $z^1 \neq z_2$, in (D, d), there exists a minimal curve K between z^1 and z^2. Further, the* *minimal curves are analytic* (see [10] or [15, § 17]).

3. The existence of a minimal closed curve in a plane domain and its distortion theorem. We consider a multiply connected bounded domain D in the space $\boldsymbol{C}^1$ bounded by N Jordan closed curves $b_1, b_2, \cdots, b_n$, where b_n is the outer boundary component. Let (D, d) be the Bergman metric space derived from the Bergman metric

$$ds_D^2(z) = \boldsymbol{K}_D(z, \bar{z}) \, |dz|^2 . \tag{3.1}$$

It is assumed that (D, d) is complete. Then all the previous considerations, lemmas and theorems can be carried over to this case. We fix an inner boundary component of D, say b_1. Without loss of generality, we may assume b_1 to be a circle.

Let $\mathfrak{R}(D; b_1)$ be the class of all closed continuous curves c in D which are homotopic to b_1 in $\overline{D(b_1)}$, where $D(b_1)$ is a ring domain bounded by b_1 and b_n which contains the domain D, and $\overline{D(b_1)}$ is the closure of $D(b_1)$. A curve $K(D; b_1)$ in $\mathfrak{R}(D; b_1)$ which satisfies the condition $\mathscr{L}_D(K(D; b_1)) \leqq \mathscr{L}_D(c)$ for all $c \in \mathfrak{R}(D; b_1)$, will be called a *minimal closed curve* of D with respect to b_1. Due to the completeness of $(D; d)$ and the behavior of $\boldsymbol{K}_D(z, \bar{z})$ (described on page 943) on the boundary of D, we have

THEOREM 3.1. *There exists a minimal closed curve $K(D; b_1)$ of the domain D with respect to b_1. Further, it is analytic.*

Proof. Let $\gamma = \inf_c \mathscr{L}_L(c)$, where c runs over the class $\mathfrak{R}(D; b_1)$. Then $0 < \gamma < \infty$. There exists a minimizing sequence $\{c_k\}$ of rectifiable curves in $\mathfrak{R}(D; b_1)$. Let $G_k \mid I_0$ be the reduced representation of c_k. By (2.4), we have

$$d(G_k(t), G_k(t')) \leqq \mathscr{L}_D(c_k) \, |t - t'| \text{ for each } k ,$$

and $\{\mathscr{L}_D(c_k)\}$ has an upper bound δ which is finite. We choose an M such that $M^{1/2} > \delta / l(b_1)$, $l(b_1)$ is the Euclidean length of b_1. Then no c_k lies completely in $D - D_M$, $D_M = [z \mid \boldsymbol{K}_D(z, \bar{z}) \leqq M]$. Let

$$\rho = \max_{z_1 \, z_2 \in D_M} d(z_1, z_2) ,$$

then for every pair of positive integers p and q, we have $d(G_p(t), G_q(t)) < \rho + 2\delta$, $0 \leqq t \leqq 1$. Hence, we can select a subsequence $\{G_{k_i}\}$ of $\{G_k\}$ which converges uniformly to a continuous function G^0 on I_0. Let K be the closed curve whose representation is given by $G^0 \mid I_0$. Since $c_{k_i} \to K$, and by the lower semi-continuity of $\mathscr{L}_D$, we obtain $\mathscr{L}_D(K) = \gamma$. The analyticity of K is obvious.

THEOREM 3.2. *Every doubly connected domain has a unique minimal closed curve. It is analytic.*

Proof. We shall show first that annulus $Q = [z \mid r < |z| < 1]$ has a unique minimal closed curve given by $c_0 = [z \mid |z| = r^{1/2}]$. Let $P_1 = [z \mid r < |z| < r^{1/2}]$, $P_2 = [z \mid r^{1/2} < |z| < 1]$. If $c \cap P_1 = \phi$, it is immediate that $L_Q(c_0) \leqq L_Q(c)$, since the kernel function $\boldsymbol{K}_Q(z, \bar{z})$[1] assumes its minimum on c_0. If $c \cap P_2 = \phi$, by the conformal mapping $\zeta = r/z$, we have $\tilde{c} \cap P_1 = \phi$, where $\tilde{c}$ is the image curve of c under $\zeta = r/z$. Since $L_Q(\tilde{c}) = L_Q(c)$, $L_Q(c_0) \leqq L_Q(c)$ follows. If $c \cap P_1 \neq \phi$ and $c \cap P_2 \neq \phi$, we obtain two closed curves c_1, c_2 consisting of the subarcs of c and c_0 and such that $c_1 \cap P_2 = \phi$, $c_2 \cap P_1 = \phi$. By the previous arguments, $L_Q(c_i) \geqq L_Q(c_0)$, $i = 1, 2$. Since $L_Q(c_1) + L_Q(c_2) = L_Q(c) + L_Q(c_0)$, we have $L_Q(c_0) \leqq L_Q(c)$. Let D be a doubly connected domain. Then D can be mapped by a univalent analytic function $f(z)$ onto Q. It is clear that $f^{-1}(c_0)$ is the unique minimal closed curve of D with respect to the inner boundary component by the univalency of $f(z)$.

We consider a domain D in the z-plane which is bounded by $b_1 = [z \mid |z| = r]$, $b_N = [z \mid |z| = 1]$, and $(N-2)$ closed Jordan curves $b_2, \cdots, b_{N-1}$. The curves $b_2, \cdots, b_{n-1}$ lie in the domain bounded by b_1 and b_N.

Let $A_1 = [z \mid r < |z| < 1]$, $A_2 = [z \mid |z - a| < \rho, |z| > r]$[2], be exterior and interior domains of comparison for D, respectively, i.e., $A_1 \supset D \supset A_2$. Then

$$(3.2) \qquad L_{A_1}(K(A_1)) \leqq L_D(K(D)) \leqq L_{A_2}(K(A_2)) \,,$$

where $K(A_1)$, $K(A_2)$ and $K(D)$ are minimal closed curves of A_1, A_2 and D with respect to b_1, respectively. It is an immediate consequence of the fact that if $B \subset A$, then $\boldsymbol{K}_B(z, \bar{z}) \geqq \boldsymbol{K}_A(z, \bar{z})$ for $z \in B$. The linear transformation

$$(3.3) \qquad w = \frac{z - (a + \rho d)}{\rho - d(z - a)} \,, \qquad 0 < |a| \leqq \rho - r \,,$$

maps A onto $Q_R = [z \mid R < |z| < 1]$, where R is given by

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1 A simple computation shows that the kernel function of Q,

$$\boldsymbol{K}_Q(z, z) = \frac{1}{\pi |z|^2}\left[\wp(2\log|z|;\ -2\log r,\ 2\pi i) + \frac{\zeta(\pi i;\ -2\log r,\ 2\pi i)}{\pi i}\right]$$

(see [**9**], [**18**]), where $\wp$ and ζ are the Weierstrass elliptic functions, assumes its minimum on c_0.

2 Here we choose a and ρ in such a way that $|z - a| < \rho$ contains b_1 but no other b_k, $k = 2, \cdots, N$, and A_2 to be the largest among such domains.

$$R = \left[\frac{r^2 - (a + \rho d)^2}{(\rho + ad)^2 - r^2 d^2}\right]^{1/2},$$
(3.4)
$$d = \frac{r^2 - a^2 - \rho^2 + [(r^2 - a^2 - \rho^2)^2 - 4a^2\rho^2]^{1/2}}{2a\rho}.$$

Since $L_{A_2}(K(A_2)) = L_{Q_R}(K(Q_R))$, using (3.2), we obtain

THEOREM 3.3. *$E(r) \leqq (1/2)L_D(K(D)) \leqq E(R)$, where R is given by (3.4) and*

$$E(r) = [\pi\mathfrak{P}(\log r; -2\log r, 2\pi i) - i\zeta(\pi i; -2\log r, 2\pi i)]^{1/2},$$

$\mathfrak{P}$ and ζ are the Weierstrass elliptic functions.

The estimation of the bounds for the Bergman lengths of the minimal closed curves in Theorem 3.3 seems to be done only for a special domain. However, every multiply connected domain can always be mapped onto such a domain by a conformal mapping. Therefore, if we know the geometry of a given domain D, combining the various distortion theorems in the theory of conformal mappings and the result in Theorem 3.3, we can obtain various bounds for the Bergman lengths of the minimal curves for quite general domains.

4. The existence of a minimal surface which spans a given closed curve in (D, d). A surface S in the space C^2 is said to be *continuously differentiable* if it admits a continuously differentiable representation

$$G \mid Q_0 : z_k = G_k(u_1, u_2),\ k = 1, 2,\ (u_1, u_2) \in Q_0 = [0 \leqq u_1, u_2 \leqq 1].$$

A surface S is said to be *piecewise continuously differentiable* if it admits a piecewise continuously differentiable representation $G \mid Q_0$, i.e., there exists a partition $\varDelta = \{\varDelta_1, \varDelta_2, \cdots, \varDelta_m\}$ of Q_0 by rectilinear triangles $\varDelta_k$ such that $G \mid \varDelta_k$ is continuously differentiable, $k = 1, 2, \cdots, m$. The *ordinary B-area element* at a point (z_1, z_2) on a piecewise continuously differentiable surface S is defined by the equation [6, Chap. XI]

$$db_S(z) = \left|\frac{\partial(G_1, G_2)}{\partial(u_1, u_2)}\right| du_1 du_2. \tag{4.1}$$

The ordinary area element of S is given by the equation

(4.2)
$$da_S(z) = [g_{11}g_{22} - (Re\, g_{12})^2]^{1/2} du_1 du_2,$$
$$g_{\alpha\beta} = \sum_{i=1}^{2} \frac{\partial G_i}{\partial u_\alpha} \overline{\frac{\partial G_i}{\partial u_\beta}},\ \alpha, \beta = 1, 2.$$

Further (4.1) can also be written in the following form,

$$db_S(z) = [g_{11}g_{22} - |g_{12}|^2]^{1/2}\, du_1 du_2 \,. \tag{4.1$'$}$$

Therefore, $da_S(z) \geqq db_S(z)$ at every point $z \in S$; the equality holds if and only if $Im\, g_{12} = 0$.

For a piecewise continuously differentiable surface S, the ordinary B-area is defined and given by the equation

$$b(S) = \iint_{Q_0} \left| \frac{\partial(G_1, G_2)}{\partial(u_1, u_2)} \right| du_1 du_2 \,. \tag{4.3}$$

$b(S)$ is independent of the choice of piecewise continuously differentiable representations $G \mid Q_0$ of S. A surface S is said to be *analytic* if it admits an analytic representation $G \mid Q_0$, i.e., $\partial G_k/\partial \bar{w} = 0$, $k = 1, 2$, $w = u_1 + iu_2$.

For an analytic or an anti-analytic surface S, $b(S) = 0$. It is also clear that $b(S) = 0$ if and only if the tangent plane of S at every point is an analytic plane. A simple computation shows the following lemma:

LEMMA 4.1. *Thc following three conditions are equivalent*:

1) $b(S) = a(S)$,
2) $\dfrac{\partial(G_1, \bar{G}_1)}{\partial(u_1, u_2)} + \dfrac{\partial(G_2, \bar{G}_2)}{\partial(u_1, u_2)} = 0$ *at each point on* S,
3) $\oint_c \bar{G}_1 dG_1 + \bar{G}_2 dG_2 = 0$ *for every closed curve* c *on* S.

Let D be a bounded domain in the space $\boldsymbol{C}^2$ on which (D, d) is complete. The quantity

$$dB_D(z) = [\boldsymbol{K}_D(z, \bar{z})]^{1/2}\, db_S(z),\ z = (z_1, z_2)\,, \tag{4.4}$$

is invariant with respect to pseudo-conformal mappings and a monotone decreasing functional of D [6]. $dB_D(z)$ is called the *invariant B-area element* of S. For a piecewise continuously differentiable surface S in D, the invariant B-area of S is defined and given by the equation

$$B_D(S) = \iint_{Q_0} [\boldsymbol{K}_D(G, \bar{G})]^{1/2} \left| \frac{\partial(G_1, G_2)}{\partial(u_1, u_2)} \right| du_1 du_2 \tag{4.5}$$

and is independent of the choice of piecewise continuously differentiable representations $G \mid Q_0$ of S.

A surface S in D is said to satisfy the *condition* (L) *with respect to the metric* d if there exists a representation $G \mid Q_0$ of S for which there exists a constant $L(S) > 0$ depending only on S and satisfying the inequality

$$\mathscr{L}_D(G; \sigma(w_1, w_2)) \leqq L(S)\, |w_1 - w_2| \tag{4.6}$$

for every pair of points w_1; w_2 in Q_0; here $\sigma(w_1, w_2)$ is the line segment that joins w_1 and w_2 in Q_0, $w_k = u_1^{(k)} + iu_2^{(k)}$, $k = 1, 2$.

It is clear that $G \mid \partial(Q_0)$, where $\partial(Q_0)$ is the boundary of Q_0, is a representation of the boundary curve Γ of S and that Γ is rectifiable. It is also clear that every continuously differentiable surface S satisfies the condition (L) with respect to d.

We shall say that a surface S is *of class* $C'\mathfrak{R}(L, N, \Gamma)$ if S admits a continuously differentiable representation

$$G \mid Q_0 : z_k = G_k(w) , \quad k = 1, 2, w \in Q_0 ,$$

which satisfies the following conditions:

(a) for a fixed positive constant L, $L(S) \leqq L$,

(b) for a fixed positive constant N,

$$\left| \frac{\partial G(w_1)}{\partial u_j} - \frac{\partial G(w_2)}{\partial u_j} \right| \leqq N \mid w_1 - w_2 \mid , \quad j = 1, 2, G = (G_1, G_2) ,$$

for every pair of points w_1, w_2 in Q_0,

(c) S spans a preassigned closed Jordan curve Γ in D in such a way that G is a one-to-one mapping on $\partial(Q_0)$.

A surface S_m is called *minimal surface* of the class $C'\mathfrak{R}(L, N, \Gamma)$ if $B_D(S_m) \leqq B_D(S)$ for all $S \in C'\mathfrak{R}(L, N, \Gamma)$.

THEOREM 4.1.[3] *For each L and N for which the class $C'\mathfrak{R}(L, N, \Gamma)$ is not empty, there exists a minimal surface S_m in the class.*

Proof. Let $\inf_S B_D(S) = \gamma$, where S runs over all surfaces in $C'\mathfrak{R}(L, N, \Gamma)$. Then $0 \leqq \gamma < \infty$. Hence, there exists a minimizing sequence $\{S_n\}$. Let $G^n \mid Q_0$ be a representation of S_n which satisfies conditions (a), (b) and (c). From (a) it follows that for any pair of positive integers p, q,

$$d(G^p(w), G^q(w)) \leqq 2\text{-}2^{3/2} L .$$

Therefore, $\{G^n(w)\}$ is equi-bounded. The equi-continuity of $\{G^n(w)\}$ follows from the inequality

$$d(G^n(w), G^n(w')) \leqq L \mid w - w' \mid \qquad \text{for any } w, w' \in Q_0 \text{ and all } n. \tag{4.7}$$

Hence, we can select a subsequence $\{G^m(w)\}$ of $\{G^n(w)\}$ which converges uniformly to a continuous function $G^0(w)$ defined in Q_0. Let $G^0 \mid Q_0$ define a surface S_0. Then it is clear that S_0 spans Γ in such a way that G^0 is a one-to-one mapping on $\partial(Q_0)$. The family $\{\partial G^m/\partial u_j\}$

[3] Replacing (a) by the condition (a′) on page 951, a result similar to Theorem 4.1 can be given (see Corollary 2).

of continuous functions $\partial G^m/\partial u_j$ is equi-bounded and equi-continuous by (b) for $j = 1, 2$. Therefore, we can select a subsequence $\{G^{m_i}(w)\}$ of $\{G^m(w)\}$ which converges uniformly to $G^0(w)$ and such that $\{\partial G^{m_i}/\partial u_j\}$ converges uniformly to a continuous function $\partial G^0/\partial u_j$ for $j = 1, 2$. This implies that S_0 is a continuously differentiable surface. In order to show $S_0 \in C'\mathfrak{R}(L, N, \Gamma)$, let c_{m_i} and c_0 be the image curves of a line segment $\sigma(w_1, w_2)$ which connects two points w_1 and w_2 in Q_0 under $G^{m_i}(w)$ and $G^0(w)$, respectively. Then c_{m_i} converges to c_0 and, hence, $\varliminf_{i\to\infty} L_D(c_{m_i}) \geqq L_D(c_0)$ by the lower semi-continuity of L_D. Since $L_D(c_{m_i}) \leqq L\,|\,w_1 - w_2\,|$ for all m_i, $L_D(c_0) \leqq L\,|\,w_1 - w_2\,|$. It is clear that $G^0(w)$ satisfies (b). Since the functional B_D is lower semi-continuous in $C'\mathfrak{R}(L, N, \Gamma)$ and $S_0 \in C'\mathfrak{R}(L, N, \Gamma)$, we have $B_D(S_0) = \gamma$. Thus S_0 is a minimal surface in the class $C'\mathfrak{R}(L, N, \Gamma)$.

REMARK. In the case that Γ lies on an analytic plane π and the portion $\tilde{\pi}$ of $\pi \cap \mathrm{D}$ enclosed by Γ is simply connected, $\tilde{\pi}$ is a minimal surface of $C'\mathfrak{R}(L, N, \Gamma)$ with some L and N, and $B_D(S_0) = 0$. In general, if there exists an analytic surface S in D which spans Γ, then S is a minimal surface with some L and N, and $B_0(S) = 0$.

Let $C'\mathfrak{R}(N, \Gamma)$ be the class of continuously differentiable surfaces in the space $\boldsymbol{C}^2$ which span a preassigned Jordan closed curve Γ in $\boldsymbol{C}^2$ and satisfy the condition (b). Then (b) implies condition (a) with respect to the Euclidean metric ρ for every surface in $C'\mathfrak{R}(N, \Gamma)$. Since $\boldsymbol{C}^2$ is complete with respect to ρ, the following corollary follows by the same procedure as in Theorem 4.1.

COROLLARY 1. *In the class $C'\mathfrak{R}(N, \Gamma)$, there exists a minimal surface S'_m in the sense that*

$$b(S'_m) \leqq b(S) \text{ for all } S \in C'\mathfrak{R}(N, \Gamma)\ .$$

Let $C'\mathfrak{R}_\alpha(N, \Gamma)$ be the class of continuously differentiable surfaces S in D which satisfy conditions (b), (c) and

(a′) for a preassigned real number α, $0 \leqq \alpha \leqq 1$,

$$\frac{db_S(z)}{da_S(z)} \geqq \alpha \text{ at every point } z \in S\ . \tag{4.8}$$

We notice that the class $C'\mathfrak{R}_\alpha(N, \Gamma)$ is motone decreasing with respect to α.

COROLLARY 2. *For a fixed $\alpha > 0$ and N for which $C'\mathfrak{R}_\alpha(N, \Gamma)$ is not empty there exists a minimal surface in the class.*

Proof. The B-areas $B_D(S_n)$ of S_n which belong to a minimizing sequence $\{S_n\}$ have a fixed uppper bound. Therefore, condition (a') ensures the existence of an $M > 0$ such that every S_n lies completely in D_M, $D_M = [z \mid \boldsymbol{K}_D(z, \bar{z}) \leqq M]$. This implies condition (a) with some L, which depends on α and N. Hence, the corollary follows from the theorem.

5. The existence of minimal closed surfaces in (D, d). Let D_k be a domain in the space of one complex variable z_k bounded by n_k closed curves $b_1^{(k)}, b_2^{(k)}, \cdots, b_{n_k}^{(k)}$. Here $b_{n_k}^{(k)}$ is the outer boundary component of D_k and $b_1^{(k)}$ is an inner boundary component, which is a circle, i.e., $b_1^{(k)} = [z_k \mid |z_k| = r_k]$.

Let D be a domain in the space C^2 which is topologically equivalent to the product domain $\tilde{D} = D_1 \times D_2$, and T_1 the topological image of $\tilde{T}_1 = b_1^{(1)} \times b_1^{(2)}$. A surface S in D which is homotopic to T_1 in $\overline{D(T_1)}$, where $D(T_1)$ is the topological image of $\tilde{D}(\tilde{T}_1) = D_1(b_1^{(k)}) \times D_2(b_2^{(k)})$ (see § 3 for notation), is a closed surface of the torus type and, hence, admits a doubly periodic representation

$$G \mid R^2 : z_k = G_k(u_1, u_2),\ k = 1, 2,\ (u_1, u_2) \in R^2\ ,$$
$$R^2 = (-\infty < u_1, u_2 < +\infty)\ ,$$

of periods 1. For our purposes, therefore, it is enough to consider G on the unit square Q_0 as a representation of S.

We shall say that a closed surface S is *of class* $C'\mathfrak{K}_\alpha(N, T_1)$ if S is homotopic to T_1 in $\overline{D(T_1)}$ and admits a continuously differentiable representation $G \mid Q_0$ satisfying condition (a') and (b) in § 4. By the same procedure as in Corollary 2 of Theorem 4.1, we can prove the following theorem for any fixed $\alpha > 0$.

THEOREM 5.1. *For each N for which the class $C'\mathfrak{K}_\alpha(N, T_1)$ is not empty, there exists a minimal closed surface $S_m(D)$ in the class.*

Let $D'\mathfrak{K}(\tilde{D}, \tilde{T}_1)$ be the class of all closed surfaces S of the form $S = c_1 \times c_2$ in D, where c_k is a piecewise continuously differentiable closed curve in D_k which is homotopic to $b_1^{(k)}$ in $\overline{D_k(b_1^{(k)})}$. For each $S \in D'\mathfrak{K}(\tilde{D}, \tilde{T}_1)$, we have $B_{\tilde{D}}(S) = L_{D_1}(c_1) \cdot L_{D_2}(c_2)$. It follows from the fact that $\boldsymbol{K}_{\tilde{D}}(z, \bar{z}) = \boldsymbol{K}_{D_1}(z_1, \bar{z}_1) \cdot \boldsymbol{K}_{D_2}(z_2, \bar{z}_2)$ [7]. Therefore, the following is an immediate consequence of Theorem 3.1.

THEOREM 5.2. *There exists a minimal surface $S_m(\tilde{D})$ of the class $D'\mathfrak{K}(\tilde{D}, \tilde{T}_1)$. It is given by $K(D_1) \times K(D_2)$, where $K(D_k)$ is a minimal closed curve of D_k with respect to $b_1^{(k)}$.*

Let $A = A_1 \times A_2$, where A_k is a doubly connected plane domain in the z_k-plane. Let $D'\mathfrak{R}_1(A, T)$ be the class of piecewise continuously differentiable closed surfaces S in A which are homotopic to $T = b_1^{(1)} \times b_2^{(2)}$ in $\bar{A}$, where $b_1^{(k)}$ is the inner boundary component of A_k, and satisfy the condition $da_S(z) = db_S(z)$.[4] Then the following theorem holds:

THEOREM 5.3. *There exists a unique minimal closed surface in the class* $D'\mathfrak{R}_1(A, T)$. *It is given by* $K(A_1) \times K(A_2)$, *where* $K(A_k)$ *is a minimal closed curve of* A_k *with respect to* $b_1^{(k)}$.

Proof. Let $A = Q = Q_1 \times Q_2$, $Q_k = [z_k \mid r_k < |z_k| < 1]$. We shall show $S_m(Q) = K(Q_1) \times K(Q_2)$ is a unique minimal closed surface of $D'\mathfrak{R}_1(Q, T)$. Let $P_{1k} = [z_k \mid r_k < |z_k| < r_k^{1/2}]$, $P_{2k} = [z_k \mid r_k^{1/2} \leqq |z_k| < 1]$. If $S \in P_{21} \times P_{22}$, it is immediate that $B_Q(S) \geqq B_Q(S_m)$. For any $S \in D'\mathfrak{R}_1(Q, T)$, S can be replaced by a surface $\tilde{S} \in D'\mathfrak{R}_1(Q, T)$ with $B_Q(S) = B_Q(\tilde{S})$ and lying in $P_{21} \times P_{22}$ by the pseudo-conformal mapping $z_k^* = r_k/z_k$, $k = 1, 2$. Thus, $B_Q(S) \geqq B_Q(S_m)$ for every $S \in D'\mathfrak{R}_1(Q, T)$. There exists a univalent analytic function $f_k(z_k)$ which maps A_k onto Q_k. Therefore, the pseudo-conformal mapping $w_k = f_k^{-1}(z_k)$ maps A onto Q and, hence, $S_m(Q)$ onto $S_m(A)$, $S_m(A) = K(A_1) \times K(A_2)$. The uniqueness of $S_m(A)$ is clear.

6. Bounds for the *B*-areas of minimal closed surfaces in the space (D, d). Using the method of exterior and interior domains of comparison, various bounds for the B-areas of minimal surfaces can be obtained. As we have considered in § 3, let D_k be bounded by $b_1^{(k)} = [z_k \mid |z_k| = r_k]$, $b_{n_k}^{(k)} = [z_k \mid |z_k| = 1]$, and $(n_k - 2)$ closed Jordan curves $b_2^{(k)}, \cdots, b_{n_k - 1}^{(k)}$, which lie in the domain bounded by $b_1^{(k)}$ and $b_{n_k}^{(k)}$. Let $A_{1k} = [z_k \mid r_k < |z_k| < 1]$, $A_{2k} = [z_k \mid |z_k - a_k| < \rho_k, |z_k| > r_k]$, $0 < |a_k| \leqq \rho_k - r_k$, be exterior and interior domains of comparison for D_k, respectively. Then $A_j = A_{j1} \times A_{j2}$ can be used as exterior and interior domains of comparison of $D = D_1 \times D_2$, i.e., $A_1 \supset D \supset A_2$. Let $S_m(\tilde{D})$ and $S_m(A_j)$ be minimal surfaces of the classes $D'\mathfrak{R}(\tilde{D}, \tilde{T}_1)$ and $D'\mathfrak{R}(A_j, \tilde{T}_1)$, respectively. Then $B_{A_1}(S_m(A_1)) \leqq B_{\tilde{D}}(S_m(D)) \leqq B_{A_2}(S_m(A_2))$. Using this inequality, we have the following distortion theorem for minimal surfaces of the class $D'\mathfrak{R}(\tilde{D}, \tilde{T}_1)$.

THEOREM 6.1. $\prod_{k=1}^{2} E(r_k) \leqq (1/4) B_{\tilde{D}}(S_m(\tilde{D}, \tilde{T}_1)) \leqq \prod_{k=1}^{2} E(R_k)$, *where* R_k *is given in* (3.4) *with the corresponding subscript* k *and* $E(r)$ *is given in Theorem* 3.3.

By a construction of an interior domain of comparison for D in

[4] This is the case $\alpha = 1$ in (a′) (see (4.8) and Lemma 4.1).

Theorem 5.1, we can also obtain a distortion theorem for minimal surfaces $S_m(D)$ in Theorem 5.1 which gives us an upper bound. Suppose an interior domain of comparison for D is given by A_{2k}, then we have

THEOREM 6.2. *$B_D(S_m(D)) \leqq 4 \prod_{k=1}^{2} E(R_k)$, where R_k and $E(r)$ are given as in Theorem* 6.1.

REMARK. For the product domain $Q = Q_1 \times Q_2$ of two annuli Q_1 and Q_2, $K(Q_1) \times K(Q_2)$ is not necessarily a minimal surface for the class $D'\mathfrak{K}_\alpha(Q, T)$ for a fixed $\alpha, 0 < \alpha < 1$.

BIBLIOGRAPHY

1. S. Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math., **169** (1933), 1-42, and **172** (1934), 89-128.
2. ———, *Sur quelques propriétés des transformations par un couple de fonctions de deux variables complexes*, Rend. Della R. Accad. Naz. Del Lincei, **19** (ser. 6) (1934), 474-478.
3. ———, *Zur Theorie von Pseudokonformen Abbildungen*, Recueil Math., nouv., **1** (43), (1936), 76-96.
4. ———, *Sur les functions orthogonales de plusieurs variables complexes*, Mém. des Sciences Math., **106** (1947).
5. ———, *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Mém. des Sciences Math., **108** (1948).
6. ———, *The kernel function and conformal mapping*, Math. Surveys 5, Amer. Math. Soc., New York, 1950.
7. H. J. Bremermann, *Holomorphic continuation of the kernel function and the Bergman metric in several complex variables*, in Lectures on functions of a complex variable ed. by W. Kaplan, Univ. of Michigan Press, 1955.
8. H. Busemann, and M. Mayer, *On the foundations of calculus of variations*, Trans. Amer. Math. Soc., **49** (1941), 173-198.
9. B. Epstein, *The kernel-function and conformal invariants*, J. Math. and Mech., **7** (1958), 925-936.
10. D. Hilbert, *Über das Dirichletsche Princip*, Jahr. Dtsch. Math.-Verein., **8** (1900), 184-188.
11. S. Kobayashi, *Geometry of Bounded Domains*, Trans. Amer. Math. Soc., **92** (1959), 267-290.
12. C. B. Morrey Jr., *Quasi-convexity and the lower semi-continuity of multiple integrals*, Pacific J. Math., **2**, (1952), 25-53.
13. ———, *The parametric variational problem for double integrals*, Communications on Pure and Applied Math., **14** (1961), 569-575.
14. K. Reinhardt, *Über Abbildungen durch analytische Funktionen zweier Veranderlicher*, Math. Ann., **83** (1921), 211-255.
15. W. Rinow, *Die Innere Geometrie der Metrischen Räume*, Die Grundlehren der Math. Wissenschaften **105** (1961).
16. J. M. Stark, *On distortion in pseudo conformal mappings*, Pacific J. Math., **6** (1956), 565-582.
17. ———, *Minimum problems in the theory of pseudo-conformal transformations*

and their application to estimation of the curvature of the invariant metric, Pacific J. Math., **10** (1960), 1021-1038.
18. K. Zarankiewicz, *Über ein numerisches Verfahren zur konformen Abbildung zweifach zusammenhangender Gebiede,* Zeits. f. ang. Math. u. Mech. **14** (1934), 97-104.

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