CLIFFORD VECTORS

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In this paper we present a generalization of parallel vector fields in a Riemannian space. As it turns out, such fields exist in spaces of constant positive curvature.

Restricting ourselves to a Riemannian 3-space throughout, we need the oriented third-order tensor [3, p. 249]

$$\eta_{ijh} = [\operatorname{sgn}(g)g]^{1/2}\varepsilon_{ijh} \ .$$

whose covariant derivative vanishes [3, pp. 251–252]. The latter fact is best ascertained by the use of geodesic coordinates. If we write the determinant of the metric tensor with the aid of permutation symbols we also find without difficulty

$$g^{pq}\eta_{ijp}\eta_{khq} = g_{hj}g_{ik} - g_{hi}g_{jk} \ . \tag{1}$$

DEFINITION. *Let the direction of a vector field at any point be that of the unit vector V. The field is said to consist of Clifford vectors if*

$$V_{i,j} = L\eta_{ijh}V^h \ , \qquad L \neq 0 \ . \tag{2}$$

THEOREM. *If the Riemannian curvature K is constant and equal to L^2, the system of equations (2) is completely integrable. If, at any point, solutions of (2) exist in all directions, then $K = L^2 = const.$*

It is known that integrability conditions for (2) are obtained using covariant differentiation. Hence, on account of a Ricci identity [3, p. 83] and (1) we have

$$L_{,k}\eta_{ijh}V^h - L_{,j}\eta_{ikh}V^h + L^2(g_{hj}g_{ik} - g_{hk}g_{ij})V^h = R_{hijk}V^h \ . \tag{3}$$

If the Riemannian curvature is constant [3, p. 112],

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{4}$$

and conditions (3) are identically satisfied. This proves the first part of our theorem.

For proof of the second part we multiply (3) by $W^iV^jW^k$ and get

$$L^2(g_{hj}g_{ik} - g_{hk}g_{ij})V^hW^iV^jW^k = R_{hijk}V^hW^iV^jW^k \ .$$

Thus L^2 is the Riemannian curvature associated with the unit vectors

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V, W [3, p. 95]. Assume now that W is a solution of (2) and M the corresponding scalar factor. Then the above curvature is also equal to M^2. Continuing this process we conclude from Schur's theorem [3, p. 112] that the curvature is constant and because of (4) that $K = L^2$.

To conclude, we demonstrate a geometric property of Clifford vectors justifying the name chosen for them. Let $\boldsymbol{t}$ be the unit tangent to a geodesic and $\boldsymbol{U}$ a unit vector which undergoes a parallel displacement along the geodesic. Hence $U^i{}_{,j}t^j = 0$ and $\boldsymbol{U}$ remains in a plane passing through the geodesic [1, p. 161]. On the other hand, because of (2), $V_{i,j}t^it^j = 0$ which shows that a Clifford vector, propagaged along the geodesic, is inclined at a constant angle to it. Letting $\cos\theta = U^iV_i$, we see that

$$- \sin\theta d_s\theta = L\eta_{ijh}U^it^jV^h \ .$$

We now make the simplifying assumption that both $\boldsymbol{U}$ and $\boldsymbol{V}$ are perpendicular to $\boldsymbol{t}$. In this case the vector $\eta_{ijh}U^iV^h$ has the direction of $\boldsymbol{t}$ and using (1) we find its length to be $\sin\theta$. Thus $d_s\theta = \pm L$ and the Clifford vector rotates about the geodesic in either sense through an angle proportional to the displacement. This property may be used to define the Clifford parallels or paratactic lines in elliptic 3-space [2, p. 108].

References

1. A. Duschek, W. Mayer, *Lehrbuch der Differentialgeometrie*, Band II, Teubner, Leipzig und Berlin, 1930.

2. D. M. Y. Sommerville, *The elements of non-Euclidean geometry*, Dover, New York, 1958.

3. J. L. Synge, A. Schild, *Tensor calculus*, University of Toronto Press, Toronto, 1959.

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