ANOTHER CHARACTERIZATION OF THE n-SPHERE AND RELATED RESULTS

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In [5] we defined an irreducible $B(J)$-cartesian membrane and an excluded middle membrane property EM, and used these to characterize the n-sphere. There the class $B(J)$ was of $(n-1)$-spheres contained in a compact metric space S. Since part of the proof does not depend upon the fact that elements of $B(J)$ are $(n-1)$-spheres, we consider the possibility of other entries in the class $B(J)$. Recent developments in this direction have been made by Bing in [2] and by Andrews and Curtis in [1]. In [3] and [4] Bing constructed a space B not homeomorphic with E^3, which has been called the dogbone space. By Theorem 6 of [2], the sum of two cones over the one point compactification $\bar{B}$ of B is homeomorphic with S^4. This sum of two cones over a common base X is called the suspension of X.

In [1] Andrews and Curtis showed that if α is a wild arc in S^n that the decomposition space S^n/α is not homeomorphic with S^n. They proved, however, that the suspension of S^n/α is always homeomorphic with S^{n+1} for any arc $\alpha \subset S^n$. The reader will easily see that a class $\bar{B}$ or of S^n/α as described will satisfy the conditions for a class $B(J)$ for which an n-sphere will have property EM.

The results below were obtained in considering such spaces, and Theorem 1 below is a weaker characterization of the n-sphere than is Theorem 2 of [5]. We find it difficult to determine the properties $J \in B(J)$ must have for S to have Property EM, as is shown by our Theorem 4 below.

I. Definition and basic properties. Let S always be a compact metric space and let $B(J)$ be a class of mutually homeomorphic subcontinua of S. We put conditions on this general class $B(J)$ in our theorems below.

We define a $B(J)$-cartesian membrane as we did in [5] and [6]. Let F be a compact subset of S containing $J \in B(J)$. Let M be a subcontinuum of F, $b \in M$ and C be homeomorphic to J. Denote by $(C \times M, b)$ the decomposition space [10: pp 273–274] of the upper semicontinuous decomposition of the cartesian product $C \times M$, where the only nondegenerate element is taken to be $C \times b$ (intuitively the decomposition space is a sort of generalized cone with vertex at the point $C \times b$). With this notation we give:

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DEFINITION 1. We say that F is a $B(J)$-*cartesian membrane* from b to J (or for brevity with base J) if and only if there is a homeomorphism h from $(C \times M, b)$ onto F for some M such that:

(i) for some $a \in M - b$, $J = h(C \times a)$,

(ii) for all $q \in M - b$, $h(C \times q) \in B(J)$, and

(iii) $h(C \times b) = b$.

If M is irreducible from a to b, then we prefix the above definition by *irreducible*. Whenever F is a $B(J)$-cartesian membrane and $F = h(C \times m, b)$, h is assumed to be a homeomorphism from $(C \times M, b)$ onto F with properties (i), (ii) and (iii). We say b is the *vertex* of F and J is the *base* of F.

The definition of $B(J)$-cartesian membrane is rather general; for example, a point or any continuum can be taken as a $B(J)$-cartesian membrane. We shall place restrictions on the space S to limit possibilities such as these when the need arises. The excluded middle membrane property of Theorem 2 in [5] is the following:

Property EM. We say that the space S has *Property EM* with respect to the class $B(J)$ if the following hold:

(1) The class $B(J)$ is not empty;

(2) For each $J \in B(J)$, $S = F_1 + F_2$ where F_1 and F_2 are irreducible $B(J)$-cartesian membranes with base J, such that $F_1 \not\subset F_2$ and $F_2 \not\subset F_1$ and whenever S is such a union and F_3 is any other $B(J)$-cartesian membrane containing J, then F_3 contains F_1 or F_2 but not both; and

(3) If $J \in B(J)$ and $p \in S - J$, then there exists a $B(J)$-cartesian membrane from p to J.

Below F, F', F_1 and F_2 are always irreducible $B(J)$-cartesian membranes.

We proved in [5] that when $B(J)$ is a class of $(n-1)$-spheres and $n > 1$ that:

(A) A necessary and sufficient condition that S be an n-sphere is that S have Property EM.

We observed in our proof of (A) that if S had Property EM with respect to a class of mutually homeomorphic continua, we were able to prove:

(B) That whenever $S = F_1 + F_2$ where F_1 and F_2 have base J, $F_1 \cdot F_2 = J$;

(C) If $F = h(C \times M, b)$ was an irreducible $B(J)$-cartesian membrane, then M was always a simple continuous arc with b as endpoint; and

(D) If $S = F_1 + F_2$ where F_1 and F_2 have base J and F_3 is any other irreducible $B(J)$-cartesian membrane with base J, then $F_1 = F_3$ or $F_2 = F_3$.

In the first paragraph of the proof of Theorem 2 of [5], (D) appeared easily as result (R_1); then by a long proof we showed that $F_1 \cap F_2 = J$, which is (B) above, and we note this long proof only depends upon J being a continuum, not on J being an $(n-1)$-sphere. Finally, the following argument show that (C) holds. Let $S = F_1 + F_2$, where F_1 and F_2 are irreducible $B(J)$-cartesian membranes with base J. By (B) $F_1 \cdot F_2 = J$, and so every element of $B(J)$ separates S. Then if $F_1 = h(C \times M, b)$ where M is irreducible from a to b, and if $z \in M - a - b$, $h(C \times z) \in B(J)$ by (ii) of Definition 1 above. Hence $h(C \times z)$ separates S, and therefore separates F_1. This implies z separates M, and so M is a simple continuous arc, as desired in (C).

II. Characterization of the n-sphere, for $n > 1$. We give now several lemmas that will enable us to characterize the n-sphere.

NOTATION. For a subset K of S, we will use $cl(K)$ to denote the closure of K in S, and for an open subset U of S, we will use $Fr(U)$ to denote the set $cl(U) - U$.

LEMMA 1. *If S has Property EM, then S is homogeneous.*

Proof. Let $x, y \in S$, $x \neq y$, and let J be an element of $B(J)$ such that $J \subset S - x - y$. By (3) of Property EM there exists an irreducible $B(J)$-cartesian membrane $F = h(C \times M, x)$ from x to J and by (D) and (2) of Property EM, $S = F + F'$, where F' has base J. Now by (B) each $J' \in B(J)$ separates S, hence by (ii) of Definition 1, some $J_0 = h(C \times q)$ separates x from y. Then by (2) of Property EM, $S = F_1 + F_2$ where F_1 and F_2 have base J_0. From (D) and (3) of Property EM there exists h_1 and h_2 such that $F_1 = h_1(C \times M_1, x)$ and $F_2 = h_2(C \times M_2, y)$. From (C) M_1 and M_2 are simple continuous arcs and x y are endpoints of M_1 and M_2 respectively. Hence from (B) there exists a homeomorphism from S onto S that carries x onto y; therefore S is homogeneous [10: p 378].

A topological space X is *invertible* [7] if for each nonempty open set U in X there is a homeomorphism h of X onto itself such that $h(X - U)$ lies in U.

LEMMA 2. *If S has Property EM then S is invertible.*

Proof. For any open set U in S and any point $x \in U$, some $J \in B(J)$ separates x from $Fr(U)$; then if $S = F_1 + F_2$ where F_1 and F_2 have base J, we can find a homeomorphism as in Lemma 1, that maps S onto S such that F_1 maps onto F_2 and F_2 maps onto F_1, hence $(S - U)$ into U.

THEOREM 1. *Let $n > 1$ and let each element of $B(J)$ contain a point at which it is locally euclidean of dimension $(n-1)$. Then S is an n-sphere if and only if S has Property EM.*

Proof of the sufficiency. Let $J \in B(J)$ and let x be an element of J at which J is locally euclidean of dimension $(n-1)$. Let U be an open $(n-1)$-cell neighborhood of x in J. Let $F = h(C \times M, b)$ have base J. By (C), M is an arc, and if V is an open subinterval of M containing a point y, $h(U \times V)$ is an open n-cell neighborhood of $h(x, y)$ in F. Since $h(U \times V)$ misses J, $h(U \times V)$ is open in $F - J$, and hence in S. By Lemma 1, S is homogeneous; hence every element of S has an open n-cell neighborhood, and so S is n-manifold. Doyle and Hocking in Theorem 1 of [7], have shown that if S is an invertible, n-manifold, then S is an n-sphere; hence by Lemma 2, S is an n-sphere.

The proof of the necessity is identical to that of Theorem 2 in [5].

Because 0-spheres are not connected the above proof does not hold for $n = 1$. We refer the reader to Theorem 1 of [5] for a characterization of the 1-sphere by an excluded middle membrane principle.

III. Related results.

LEMMA 3. *If S has Property EM then S is locally connected.*

Proof. We note that if F is an irreducible $B(J)$-cartesian membrane with base J, then $F - J$ is an open connected set in S, and proceed as in the proof of Lemma 2.

LEMMA 4. *If S has Property EM and $J \in B(J)$ then J is locally connected.*

Proof. Let $S = F_1 + F$ where F_1 and F have base J and $F = h(C \times M, b)$, where M is an arc from a to b; and $h(C \times a) = J$ as in (1) of Definition 1. Since S is locally connected, the open set $F - J - b$ is locally connected. We define $f(h(c, m)) = h(c, a)$, where $h(c, m)$ is a point in $F - J - b$; then f is a projection onto J and can easily be proved to be continuous and open. Since $F - J - b$ is locally connected and local connectedness is preserved under open, continuous mappings, J is locally connected.

THEOREM 2. *If S has Property EM and $J \in B(J)$, then J contains a 1-sphere.*

Proof. Let $J \in B(J)$, and $F = h(C \times M, b)$ have vertex $b = h(C \times b)$ and base J. Since J is locally connected, C must contain an arc I;

and by (C), M is an arc. Then the set $E' = h(I \times M, b)$ is a closed 2-cell contained in F. Let E be any subset of E' that is homeomorphic to euclidean 2-space E^2.

Let b_i $(i = 1, 2, \cdots)$ be a sequence converging to b in M. Then the half open intervals $M_i = bb_i - b_i$ form a basis of open sets in M at b, and the sets $U_i(b) = h(C \times M_i, b)$ form a basis of open sets in F at b. These open sets have the property that $Fr(U_i(b))$ is homeomorphic to J.

Choose $x \in E$, then $x \notin J$. By the homogeneity of S there exists a basis of open sets $U_i(x)$ which have the property that their boundaries are homeomorphic to J. Now fix i such that $U = U_i(x) \cdot E$ has a compact closure in E. Let V be the component of U that contains x. Since E is locally connected, V is open in E. Also $Fr(V) \subset Fr(U_i(x))$; therefore without loss of generality we can think of $Fr(V)$ as being a subset of J. Let V' be a component of $E - cl(V)$. Then V' is an open connected subset of E and $Fr(V') \subset Fr(V)$. Since $Fr(V')$ is closed and $Fr(V)$ compact, $Fr(V')$ is compact. By Theorem 25 of [10: p 176], $Fr(V')$ is a continuum. Then by Theorem 28 of [10: p 178], $Fr(V')$ is not disconnected by the omission of any point.

Let $r, s \in Fr(V')$, and let Y be an arc from r to s in J. Let $q \in Y - r - s$; now q does not separate r from s in $Fr(V')$; hence q does not separate r from s in J; then there exists an arc Y' from r to s in J that does not contain q, and $Y + Y'$ must contain a 1-sphere.

REMARK. Since J is locally connected, J is arcwise connected and as such cannot be an indecomposable continuum; by Theorem 2, J cannot be hereditarily unicoherent. A simple proof using the Brouwer Invariance of Domain Theorem [9: p. 95] will show that J cannot be a closed n-cell.

LEMMA 5. *Let S be an n-sphere having Property EM with respect to some $B(J)$.* (1) *If G is an $(n-2)$-sphere in $J \in B(J)$, then $J - G$ is not connected*; (2) *if E is a closed $(n-2)$-cell in J, then $J - E$ is connected.*

Proof. (1) Suppose $J - G$ is connected. Let $S = F_1 + F_2$ where F_1 and F_2 have base J; by (B) and (C) we can find h_1 and h_2 such that $F_1 = h_1(J \times M_1, b_1)$, $F_2 = h_2(J \times M_2, b_2)$ and $h_1 | (J \times a) = h_2 | (J \times a)$ where M_1 and M_2 are arcs from a to b_1 and a to b_2 respectively. Then $K = h_1((J - G) \times (M_1 - b_1)) + h_2((J - G) \times (M_2 - b_2))$ is connected. But $S - K = h_1(G \times M_1, b_1) + h_2(G \times M_2, b_2)$ is an $(n-1)$-sphere is S and must disconnect S by the Jordan Separation Theorem [9: p. 101].

The proof of (2) is similar to that of (1).

THEOREM 3. *A necessary and sufficient condition that S be a 3-sphere is that S have Property EM if and only if $B(J)$ is a collection of 2-spheres.*

Proof. The sufficiency follows from Theorem 2 of [5].

By Theorem 2, every $J \in B(J)$ contains a 1-sphere, and by (1) of Lemma 5 every 1-sphere in J separates J. By (2) of Lemma 5 no proper subcontinuum of a 1-sphere in J separates J; and by Lemma 4, J is locally connected; therefore by Zippin's Characterization in [11: p. 88] J is a 2-sphere. The rest follows from Theorem 2 of [5].

We need Hypothesis:

(H 1) If F_c, F_b and F'' are irreducible $B(J_0)$-cartesian membranes with base J_0 then $F_c + F_b + F''$ is contained in some E^3;

(H 2) If $S_x = F_x + F''$ is a 2-sphere in E^3, x is vertex of $B(J_0)$-cartesian membrane F_x and $t'_\alpha = h_c(c_\alpha \times M'', x)$ $(c_\alpha \in C)$ is a projecting arc from x to J through a point $y \in \operatorname{int}(S_x, E^3)$, (the interior of S_x in E^3), then $t'_\alpha - x \subset \operatorname{int}(S_x, E^3)$; if $q \in \operatorname{int}(S_x, E^3) \cdot J = J'$, then $q \notin \operatorname{cl}(J - J')$.

THEOREM 4. *Let S have Property EM, let* (H 1) *and* (H 2) *hold and let there exist a region R in S such that $J \cdot R$ contains a 1-sphere J_0 and $R \cdot J$ is embedded in the euclidean E^2; let there exist $q \in J - R$. Then J contains a closed 2-cell with J_0 as boundary.*

Proof. By (2) of Property *EM* there exist irreducible $B(J)$-cartesian membranes such that $S = h(C \times M, b) + h'(C \times M', b')$ where $h \mid (C \times a) = h' \mid (C \times a)$ and M, M' are arcs from a to b and a to b' respectively; since $J \supset J_0$, there exists $C_0 \subset C$ homeomorphic to J_0; let $h(C_0 \times M, b) = F_b$ and $h'(C_0 \times M', b') = F''$, where then F_b and F'' are irreducible $B(J_0)$-cartesian membranes from J_0 to b and b' respectively. Let $S_b = F_b + F''$; by Theorem 2 of [5], S_b is a 2-sphere.

By hypothesis there exists $q \in J - R$; thus $q \notin S_b$, and so by (H 2) the projecting arc from b to q does not contain a point of $\operatorname{int}(S_b, E^3)$; let c be an element of this projecting arc. By (3) of Property *EM*, there exists an irreducible $B(J_0)$-cartesian membrane $F_c = h_c(C_0 \times M_c, c)$ with base J_0, a subset of an irreducible $B(J)$-cartesian membrane $h_c(C \times M_c, c)$ from c to J; by the choice of c, $h_c(C \times M_c, c) = h(C \times M, b)$ and thus $S_c = F_c + F''$ is a 2-sphere.

Since $c \notin \operatorname{int}(S_b, E^3)$, there exists a region R' about c such that $cl(R') \cdot S_b = \phi$; then by Lemma 3 of [6] there exists an irreducible $B(J)$-cartesian membrane $F_{0c} = h_c(C \times M'_c, c)$, for $M'_c \subset M_c$, such that $F_c \cdot R' \supset F_{0c}$.

Let $\{t_{\alpha c}\}$ be the class of all projecting subarcs from c to J which are contained in $(S_c - (F_{0c} - J'_c)) + \operatorname{int}(S_c, E^3) - (F_{0c} - J'_c)$, where J'_c is the base of F_{0c}; that is $t_{\alpha c}$ is an arc from J to F_{0c} in and on S_c.

Let $Z' = \cup t_{\alpha c}$ and let $Z = Z' \cdot J$. Suppose $Z' = Z'_1 + Z'_2$ separate [11: p. 8]. Since each $t_{\alpha c}$ is connected, each is contained wholly in Z'_1 or in Z'_2; this is also true of J_0 and so of $F_c - F_{0c}$; so let $Z'_1 \supset F_c - F_{0c} \supset J_0$.

By Theorem 5.37 of [11: p. 66] S_c is arcwise accessible from the embedding E^3; hence there exists an arc cb' such that $cb' - c - b' \subset \operatorname{int}(S_c, E^3)$. But cb' contains a point of $\operatorname{int}(S_b, E^3)$ and a point c of $S - \operatorname{int}(S_b, E^3) - S_b$; hence cb' contains some $v \in S_b$, because by the Jordan-Brouwer Separation Theorem [11: Theorem 5.23, p. 63] S_b separates E^3 into two domains. Hence by (2) of Property EM there exists a projecting arc from c to J through v, and so some $t_{\alpha c} \supset v$ and $Z' \supset t_{\alpha c}$. Let $Z_i = Z'_i \cdot Z (i = 1, 2)$, where by agreement $Z_1 \supset J_0$. By hypothesis $J \cdot R$ is contained in some euclidean E^2, and so let E be the 2-cell bounded by J_0 in this E^2. Thus $J_0 + E \supset Z$, and because of v above $E \cdot Z \neq \phi$. If $j \in J \cdot E$, by (H 2) the projecting arc cj is such that $cj - c \subset \operatorname{int}(S_c, E^3)$. Thus $j \in Z$, and so $Z = J_0 + J \cdot E = Z_1 + Z_2$ separate. Hence $J = (Z_1 + (J - E)) + Z_2$ separate, which is a contradiction, since J is a continuum. Therefore Z and Z' are connected. By Lemma 4 J is locally connected, and so by (H2) Z is also.

Since Z is closed, Z contains all of its boundary points in the space J. By the Torhorst Theorem [10: p. 191, Theorem 42], the boundary of any complementary domain of Z in E must be a 1-sphere J'_0. Using J'_0 in place of J_0, one obtains a 2-sphere S'_c with poles c and b' and with J'_0 as a base in S'_c. Thus an arc bc' above exists such that $cb' - c - b' \subset \operatorname{int}(S'_c, E^3)$ and there exists a point $v \in S_b \cdot cb'$; also there exists $t_{\alpha c}$ as above, now contained in the $\operatorname{int}(S'_c, E^3)$; hence an endpoint of $t_{\alpha c}$ is an element of $\operatorname{int}(J'_0, E^2)$; thus a point of Z is in the complementary domain above of Z in E, which is a contradiction. Therefore $Z = E$, and so J contains a closed 2-cell.

If (H 1) and (H 2) hold, J cannot be a plane universal curve.

REFERENCES

1. J. J. Andrews and M. L. Curtis, *n-Space modulo an arc*, Annals of Math., **75** (1962), 1-7.

2. R. H. Bing, *The cartesian product of certain nonmanifold and a line is E^4*, Annals of Math., **70** (1959), 399-412.

3. ———, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Annals of Math., **65** (1957), 484-500.

4. ———, *The cartesian product of a certain nonmanifold and a line is E^4*, Bull. Amer. Math. Soc., **64** (1958), 82-84.

5. R. F. Dickman, L. R. Rubin and P. M. Swingle, *Characterization of n-spheres by an excluded middle membrane principle*, Mich. Math. Jour., **11** (1964), 53-59.

6. ———, *Irreducible continua and generalization of hereditarily unicoherent continua*

by means of membranes, to appear.
7. P. H. Doyle and J. G. Hocking, *A characterization of euclidean n-space*, Mich. Math. Jour., **7** (1960), 199–200.
8. ———, *Invertible spaces*, Amer. Math. Monthly **68** (1961), 959–965.
9. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press (1948).
10. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications **13** (1962).
11. R. L. Wilder, *Topology of Manifolds*, Amer. Math. Soc. Colloquium Publications **32** (1948).

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