COVERING SPACES OF PARACOMPACT SPACES

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Introduction. Let $\tilde{X}$ and X be two Hausdorff spaces and f a continuous[1] mapping of $\tilde{X}$ into X. We say that f is a covering mapping if f maps $\tilde{X}$ onto X and there exist an open covering[1] $\mathscr{V}$ of X having the following property:

(1) For every $V \in \mathscr{V}$, $f^{-1}[V]$ is a union of a family $\mathscr{F}(V)$ consisting of pairwise-disjoint open sets each of which is mapped homeomorphically onto V by f.

The pair $(\tilde{X}, f)$ is called a covering space of X.

If X is a metric space, nothing can be said, in general, about the diameters of the elements of the covering $\mathscr{V}$ of X, the diameters of the elements of $\mathscr{F}(V)$, $V \in \mathscr{V}$, or any isometric properties of f, as can be seen from the following example:

EXAMPLE 1. Let $\tilde{X}$ be the real line with the usual metric, X the unit circle $|z| = 1$ in the complex Z-plane with length of minor arc as the distance between two points and finally f the function: $f(\tilde{x}) = e^{i|\tilde{x}|\tilde{x}}$.

Then $(\tilde{X}, f)$ is a covering space of X, if $\mathscr{V}$ is the set of arcs of length one. Now, let V be the unit spherical region (i.e. the arc of length one) with $z = 1$ as centre. One can easily see that $f^{-1}[V]$ consists of intervals of the form $2\kappa\pi - 1 < x^2 < 2\kappa\pi + 1$ and the infinum of their diameters is zero. Thus if $\tilde{V} \in \mathscr{F}(V)$, $f \mid \tilde{V}$ has in general no isometric properties. But it is easily seen that the metric in $\tilde{X}$ can be changed (without changing the topology of $\tilde{X}$) in such a way that $f \mid \tilde{V}$ will be an isometry for every $\tilde{V} \in \mathscr{F}(V)$ and every $V \in \mathscr{V}$. This leads to the following problem:

Problem. Let $(\tilde{X}, f)$ be a covering space of a metrisable space X. Does there exist a metric $\tilde{\rho}$ in $\tilde{X}$ and a metric ρ in X, inducing the topologies of $\tilde{X}$ and X respectively and such that the family $\mathscr{S}$ of unit spherical regions in (X, ρ) has the following property:

(A) For every $S \in \mathscr{S}$, $f^{-1}[S]$ is a union of a family $\mathscr{F}(S)$, consisting of pairwise-disjoint unit spherical regions in $(\tilde{X}, \tilde{\rho})$ each of

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[1] In this paper all mappings and functions are assumed to be continuous, and all coverings to be open. The qualifying adjectives are omitted accordingly.

which is mapped isometrically onto S by f?

In this paper we give a positive answer to this question for locally-connected spaces (Part 1). If X is not locally-connected, it may happen that no metrics $\tilde{\rho}$ and ρ can be found, so that (A) will be satisfied (Example 2). But similar results are also valid in general spaces (not necessarily locally connected). (Theorem 3, Part 2)

Part 1. Covering spaces of locally-connected paracompact spaces. In this part we deal with paracompact uniform spaces. It contains Theorem 1, providing a solution to a problem analogous to the original one stated in the introduction for paracompact locally-connected uniform spaces. From this theorem a solution to the original problem concerning locally-connected metric spaces is derived (Theorem 2).

LEMMA 1. *Let $(\tilde{X}, f)$ be a covering spaces of a locally-connected paracompact space X. There exists a covering $\mathscr{V}$ of X having property* (1) *and satisfying the following condition*:

(2) *For every V and W of $\mathscr{V}$, each element of $\mathscr{F}(V)$ intersects at most one element of $\mathscr{F}(W)$.*

Proof. Let $\mathscr{U}$ be a covering of X by connected open sets having property (1). Let $\mathscr{V}$ be a Δ-refinement[2] of $\mathscr{U}$ consisting of connected open sets. It is clear that $\mathscr{V}$ has property (1). It has to be shown that (2) holds. In fact, suppose there exist elements V and W of $\mathscr{V}$, an element $\tilde{W}$ of $\mathscr{F}(W)$ and two distinct elements $\tilde{V}_1$ and $\tilde{V}_2$ of $\mathscr{F}(V)$ such that $\tilde{V}_1 \cap \tilde{W} \neq \phi$ and $\tilde{V}_2 \cap \tilde{W} \neq \phi$. Thus $V \cap W \neq \phi$ and since $\mathscr{V}$ is a Δ-refinements of $\mathscr{U}$, it follows that $V \cup W$ is contained in some element U of $\mathscr{U}$. The sets $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{W}$, being homeomorphic with connected sets, are connected, and since $\tilde{V}_1 \cap \tilde{W}$ and $\tilde{V}_2 \cap \tilde{W}$ are not empty we have that the set $\tilde{V}_1 \cup \tilde{V}_2 \cup \tilde{W}$ is connected. By $V \cup W \subset U$ we have that the set $\tilde{V}_1 \cup \tilde{V}_2 \cup \tilde{W} \subset f^{-1}[U]$, and since $f^{-1}[U]$ is a union of disjoint open sets, this set, being connected, is contained in one and only one element $\tilde{U}$ of $\mathscr{F}(\tilde{U})$. Then, however, $f(\tilde{V}_1) = f(\tilde{V}_2)$ contradicts the fact that $f \mid \tilde{U}$ is a homeomorphism.

THEOREM 1. *Let $(X, \mathscr{A})$ be a locally-connected paracompact uniform space with $\mathscr{A}$ as the maximal uniformity*[3], *and $\tilde{X}$ a Hausdorff space and f a mapping of $\tilde{X}$ onto X.*

[2] We recall that a Δ-refinement $\mathscr{V}$ of a covering $\mathscr{U}$ of space X is a covering of X having the property that the union of the elements of $\mathscr{V}$ which contain a fixed element x of X is contained in some element of $\mathscr{U}$. For the proof of the existence of such refinement in paracompact spaces see [3].

[3] I.e., the uniformity consisting of all neighbourhoods of the diagonal in $X \times X$. The equivalence of paracompactness and the existence of such uniformity is proved in [1].

Then $(\tilde{X}, f)$ is a covering space of X if and only if and only if

(3) There exist a symmetric open neighbourhood $\tilde{C}$ of the diagonal in $\tilde{X} \times \tilde{X}$ and an open element C of $\mathscr{A}$ such that:

(3a) For every $\tilde{x} \in \tilde{X}$, $f(\tilde{C}[\tilde{x}]) = C[f(\tilde{x})]$.

(3b) If $\tilde{\mathscr{B}}$ is the family consisting of all subsets of $\tilde{X} \times \tilde{X}$ of the form $\tilde{C} \cap F^{-1}[A]$ for all $A \in \mathscr{A}$, where $F: \tilde{X} \times \tilde{X} \to X \times X$ is defined by $F(\tilde{x}, \tilde{y}) = (f(\tilde{x}), f(\tilde{y}))$, then $\tilde{\mathscr{B}}$ is a basis for a uniformity $\tilde{\mathscr{A}}$ in $\tilde{X}$ inducing the given topology of $\tilde{X}$.

Proof of necessity. Suppose that $(\tilde{X}, f)$ is a covering space of X. By Lemma 1, we can choose a covering of X satisfying (1) and (2) and which by the paracompactness of X is an even covering. (See [2] p. 156). Then clearly we can find a symmetric open neighbourhood C of the diagonal in $X \times X$ such that the covering $\mathscr{V} = \{C[x] \mid x \in X\}$ satisfies (1) and (2).

Define $\tilde{C}$ to be the set of all pairs $(\tilde{x}, \tilde{y})$ in $\tilde{X} \times \tilde{X}$ such that $F(\tilde{x}, \tilde{y}) \in C$ and such that $\tilde{x}$ and $\tilde{y}$ are contained in the same element $\tilde{W}$ of $\mathscr{F}(C[f(\tilde{x})])$. Hence, for each $\tilde{x} \in \tilde{X}$ $\tilde{C}[\tilde{x}]$ is the element of $\mathscr{F}(C[f(\tilde{x})])$ containing $\tilde{x}$.

We shall now show that $\tilde{C}$ is the required neighbourhood of the diagonal in $\tilde{X} \times \tilde{X}$. For this purpose we prove the following propositions:

(a) $\tilde{C} = \tilde{C}^{-1} = (\tilde{C} \circ \tilde{C}) \cap F^{-1}[C]$

(b) $\tilde{C}$ is an open neighbourhood of the diagonal in $\tilde{X} \times \tilde{X}$ satisfying (3a).

(c) The family $\tilde{\mathscr{C}}$, consisting of sets of the form $\tilde{C} \cap F^{-1}[A]$ for all $A \in \mathscr{A}$ and $A \subset C$, is a basis for a uniformity $\tilde{\mathscr{A}}$ of $\tilde{X}$.

(d) The uniformity $\tilde{\mathscr{A}}$ of $\tilde{X}$ defined in (c) induces the topology of $\tilde{X}$.

Proof of (a). Let $(\tilde{x}, \tilde{y})$ be an element of $\tilde{C}$. Thus, $\tilde{y}$ and $\tilde{x}$ are contained in the same element $\tilde{W}$ of $\mathscr{F}(C[f(\tilde{x})])$. Let $\tilde{V}_1$ and $\tilde{V}_2$ be the two elements of $\mathscr{F}(C[f(\tilde{y})])$ containing $\tilde{x}$ and $\tilde{y}$ respectively. Hence $\tilde{V}_1 \cap \tilde{W}$ and $\tilde{V}_2 \cap \tilde{W}$ are not empty. By property (2) we must have $\tilde{V}_1 = \tilde{V}_2$. Thus, $\tilde{x}$ and $\tilde{y}$ are contained in the same element of $\mathscr{F}(C[f(\tilde{y})])$ which means that $(\tilde{y}, \tilde{x}) \in \tilde{C}$. Hence $\tilde{C} = \tilde{C}^{-1}$. For proof of the second equality note that by $\tilde{C} \subset F^{-1}[C]$, we have $\tilde{C} \subset (\tilde{C} \circ \tilde{C}) \cap F^{-1}[C]$. On the other hand, $(\tilde{x}, \tilde{y}) \in (\tilde{C} \circ \tilde{C}) \cap F^{-1}[C]$ implies that $f(\tilde{y}) \in C[f(\tilde{x})]$ and that there exists $\tilde{z}, \tilde{z} \in \tilde{X}$ such that $(\tilde{x}, \tilde{z})$ and $(\tilde{z}, \tilde{y})$ are in $\tilde{C}$. Thus, $\tilde{x}, \tilde{y}$ and $\tilde{z}$ are contained in the same element $\tilde{W}$ of $\mathscr{F}(C[f(\tilde{z})])$. Now, if $\tilde{V}_1$ and $\tilde{V}_2$ are the elements of $\mathscr{F}(C[f(\tilde{x})])$ containing $\tilde{x}$ and $\tilde{y}$ respectively, it follows by (2) that $\tilde{V}_1 = \tilde{V}_2$. Hence $(\tilde{x}, \tilde{y}) \in \tilde{C}$. It follows that $(\tilde{C} \circ \tilde{C}) \cap F^{-1}[C] \subset \tilde{C}$ which completes the proof of (a).

Proof of (b). Let $(\tilde{x}, \tilde{y})$ be any element of $\tilde{C}$. There exist then open subsets V and W of X, such that $f(\tilde{x}) \in V$, $f(\tilde{y}) \in W$ and such that $V \times W \subset C$. Let $\tilde{U}$ be the element of $\mathscr{F}(C[f(\tilde{x})])$ containing $\tilde{x}$ and $\tilde{y}$. We put $\tilde{V} = \tilde{U} \cap f^{-1}[V]$ and $\tilde{W} = \tilde{U} \cap f^{-1}[W]$. It suffices to show that $\tilde{V} \times \tilde{W} \subset \tilde{C}$. But by (a) this last inclusion follows from the fact that

$$\begin{aligned} \tilde{V} \times \tilde{W} &= (\tilde{U} \times \tilde{U}) \cap [f^{-1}(V) \times f^{-1}(W)] \\ &= \tilde{U} \times \tilde{U} \cap F^{-1}(V \times W) \subset (\tilde{C} \circ \tilde{C}) \cap F^{-1}[C] = \tilde{C} \end{aligned}$$

Thus, $\tilde{C}$ is an open neighbourhood of the diagonal in $\tilde{X} \times \tilde{X}$, and by the definition of $\tilde{C}$ (3a) holds.

Proof of (c). Let $\tilde{A}$ be any element of $\tilde{C}$; then $\tilde{A} = \tilde{C} \cap F^{-1}[A]$ for some A of $\mathscr{A}$, $A \subset C$. Since $\mathscr{A}$ is a uniformity there exists an element B of $\mathscr{A}$, $B \subset C$ such that $B \circ B \subset A$. Let $\tilde{B} = \tilde{C} \cap F^{-1}[B]$. We have $\tilde{B} \circ \tilde{B} \subset (\tilde{C} \circ \tilde{C}) \cap F^{-1}[B \circ B] \subset (\tilde{C} \circ \tilde{C}) \cap F^{-1}[A]$. Since $A \subset C$, we have by (a) that

$$\tilde{B} \circ \tilde{B} \subset \tilde{C} \cap F^{-1}[A] = \tilde{A}$$

Thus, for every $\tilde{A} \in \tilde{\mathscr{C}}$ there exists $\tilde{B} \in \tilde{\mathscr{C}}$ such that $\tilde{B} \circ \tilde{B} \subset \tilde{A}$, Simple calculations show that $\tilde{\mathscr{C}}$ has all other properties of uniformity and (c) holds.

Proof of (d). Let $\tilde{S}$ be any open neighbourhood of an arbitrary point $\tilde{x}$ of $\tilde{X}$, and $\tilde{T} = \tilde{S} \cap \tilde{C}[\tilde{x}]$. Being an open subset of $X, f(\tilde{T})$ contains a set $A[f(\tilde{x})]$ for some $A \in \mathscr{A}$, $A \subset C$. Putting $\tilde{A} = \tilde{C} \cap F^{-1}[A]$ we now show that $\tilde{A}[\tilde{x}] \subset \tilde{T} \subset \tilde{S}$. In fact, let $g = f \mid \tilde{C}[\tilde{x}]$. We have $g(\tilde{A}[\tilde{x}]) = f(\tilde{A}[\tilde{x}]) \subset F(\tilde{A})[f(\tilde{x})] \subset A[f(\tilde{x})] \subset f(\tilde{T}) = g(\tilde{T})$. Since g is a homeomorphism, it follows that $\tilde{A}[\tilde{x}] \subset \tilde{T}$.

Finally, it is clear that for every $\tilde{A} \in \tilde{\mathscr{C}}$ and every $\tilde{x} \in \tilde{X}$, $\tilde{A}[\tilde{x}]$ is a neighbourhood of $\tilde{x}$, thus the uniformity $\tilde{\mathscr{A}}$ with basis $\tilde{\mathscr{C}}$ induces the topology of $\tilde{X}$ and (d) is proved.

To complete the proof of necessity of condition (3) we only have to note that the uniformity $\tilde{\mathscr{A}}$ defined in (c) satisfies condition (3b).

Proof of sufficiency of condition (3). Let $\tilde{\mathscr{A}}$ be the given uniformity of $\tilde{X}$, $\tilde{C}$ the given neighbourhood of the diagonal of $\tilde{X} \times \tilde{X}$ and C the element of $\mathscr{A}$ given in (3).

We have to show the existence of a covering $\mathscr{V}$ of X having property (1). For this purpose we prove the following two properties:

(e) If $(\tilde{x}, \tilde{y}) \in \tilde{C}$, then $f(\tilde{x}) = f(\tilde{y})$ implies $\tilde{x} = \tilde{y}$.

and

(f) For every $A \in \mathscr{A}$, $A \subset C$, let $\tilde{A} = \tilde{C} \cap F^{-1}[A]$.

Then, for every $\tilde{x} \in \tilde{X}$ $f(\tilde{A}[\tilde{x}]) = A[f(\tilde{x})]$.

Proof of (e). Suppose $f(\tilde{x}) = f(\tilde{y})$. Hence $F(\tilde{x}, \tilde{y})$ is in the diagonal of $X \times X$ and therefore, $(\tilde{x}, \tilde{y}) \in F^{-1}[A]$ for every $A \in \mathscr{A}$. If $(\tilde{x}, \tilde{y}) \in \tilde{C}$ then $(\tilde{x}, \tilde{y})$ is contained in every element of a basis for a uniformity of $\tilde{X}$, and since $\tilde{X}$ is a Hausdorff space we have $\tilde{x} = \tilde{y}$.

Proof of (f). By (3a) we have $F(\tilde{C}) = C$. Hence, $F(\tilde{A}) \subset F(\tilde{C}) \cap A = C \cap A = A$. It follows that $f(\tilde{A}[\tilde{x}]) \subset F(\tilde{A})[f(\tilde{x})] \subset A[f(\tilde{x})]$. On the other hand, if $y \in A[f(\tilde{x})]$ then $y \in C[f(\tilde{x})]$. Now by (3a) we have that $f(\tilde{C}[\tilde{x}]) = C[f(\tilde{x})]$; hence there exists $\tilde{y}, \tilde{y} \in \tilde{C}[x] \cap f^{-1}[y]$. Thus $(\tilde{x}, \tilde{y}) \in \tilde{C} \cap F^{-1}[A] = \tilde{A}$ and $\tilde{y} \in \tilde{A}[\tilde{x}]$. Hence $y \in f(\tilde{A}[\tilde{x}])$ and $f(A[\tilde{x}]) \supset A[f(\tilde{x})]$, which completes the proof of (f).

Now, let $\tilde{B}$ be a symmetric element of $\tilde{\mathscr{B}}$ (defined in 3b) satisfying $\tilde{B} \circ \tilde{B} \subset \tilde{C}$. Thus, $\tilde{B} = \tilde{C} \cap F^{-1}[B]$ for some $B \subset C$, $B \in \mathscr{A}$ and we can assume that $\tilde{B}$ and B are open. By (e) $f \mid \tilde{B}[\tilde{x}]$ is one-to-one for each $\tilde{x} \in \tilde{X}$. By (f) f is an open mapping. From (e) it also follows that $\{\tilde{B}[\tilde{x}] \mid \tilde{x} \in \tilde{X}\}$ are disjoint. Thus, $f \mid \tilde{B}[\tilde{x}]$ is a homeomorphism. To complete the proof we still have to show that

(f′) For every symmetric neighbourhood A of $\mathscr{A}$ and for every $x \in X$, $\tilde{A}[f^{-1}[x]] = f^{-1}(A[x])$ where $\tilde{A} = \tilde{C} \cap F^{-1}[A]$.

Proof of (f′). By (f) $\tilde{A}[\tilde{x}] \subset f^{-1}(A[x])$ for every $\tilde{x} \in f^{-1}(x)$. Hence, $\tilde{A}[f^{-1}(x)] \subset f^{-1}(A[x])$.

On the other hand, since A and $\tilde{A}$ are symmetric, we have that $\tilde{y} \in f^{-1}(A[x])$ implies $x \in A[f(\tilde{y})]$. Therefore by (f), there exists $\tilde{x}, \tilde{x} \in \tilde{X}$ such that $f(\tilde{x}) = x$ and $(\tilde{x}, \tilde{y}) \in \tilde{A}$. Hence, $\tilde{y} \in \tilde{A}[\tilde{x}] \subset \tilde{A}[f^{-1}(x)]$ and we thus have $f^{-1}(A[x]) \subset \tilde{A}[f^{-1}(x)]$, and (f') is proved. Since $\tilde{B}$ is symmetric we have by (f') that $f^{-1}(B[x]) = \bigcup_{\tilde{x} \in f^{-1}[x]} \tilde{B}[\tilde{x}]$. It follows now that $f^{-1}(B[x])$ is a union of a family $\mathscr{F}(B[x]) = \{\tilde{B}[\tilde{x}] \mid \tilde{x} \in f^{-1}(x)\}$ consisting of pairwise-disjoint open sets, each of which is mapped homeomorphically onto $B[x]$ by f. Thus, $\mathscr{V} = \{B[x] \mid x \in X\}$ has property (1) and Theorem 1 is proved.

REMARK 1. Note that the fact of the uniformity $\mathscr{A}$ of X in Theorem 1 being maximal has been used, in the proof of Theorem 1, only for stating that $\mathscr{A}$ contains the neighbourhood C of the diagonal in $X \times X$ defined in (3). Thus, if the maximality of $\mathscr{A}$ is replaced by the statement that the uniformity $\mathscr{A}$ contains an open element C such that $\mathscr{V} = \{C[x] \mid x \in X\}$ has properties (1) and (2) the proof of Theorem 1 is still valid.

REMARK 2. Note that the only reason for X being locally connected is to ensure the existence of a covering of X having both properties

(1) and (2); hence the restriction that X is locally connected may be replaced by the assumption that there exists a covering of X having properties (1) and (2).

We shall now use Theorem 1 to give a solution to the original problem concerning locally-connected metric spaces. For this purpose we require some additional lemmas.

LEMMA 2. *Let X be a paracompact space and C a neighbourhood of $X \times X$. There exists a uniformity C of X which contains C, has a countable basis and whose members are neighbourhoods of the diagonal in $X \times X$.*

Proof. It is known that in a paracompact space X, for every neighbourhood A of the diagonal in $X \times X$, there exists a symmetric neighbourhood of the diagonal B such that $B \circ B \subset A$ (see [2] p. 157). Now let $B_0 = C$ and, by induction, B_n the symmetric neighbourhood of the diagonal of $X \times X$ such that $B_n \circ B_n \subset B_{n-1}$ holds. The family $\{B_n\}$ $n = 1, 2, 3 \cdots$ is a basis for a uniformity C consisting of neighbourhoods of the diagonal, containing $C = B_0$ which has a countable basis and Lemma 2 is proved.

Further, one can easily prove the following

LEMMA 3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two uniformities in a set X. Then the family*

$$\mathscr{D} = \{D \mid D = A \cap B; A \in \mathscr{A}, B \in \mathscr{B}\}$$

is again a uniformity in X.

LEMMA 4. *Let X be a metrisable space, and C any neighbourhood of the diagonal in $X \times X$. There exists a uniformity $\mathscr{D}$ of X, containing C, having a countable basis and inducing the topology of X.*

Proof. Let $\mathscr{A}$ be a uniformity of X containing C, having a countable basis and consisting of neighbourhoods of the diagonal in $X \times X$ (see lemma 2). Let $\mathscr{B}$ be a uniformity of X having a countable basis and inducing the topology of X (any metric uniformity has such properties). By Lemma 3 $\mathscr{D} = \{D \mid D = A \cap B; A \in \mathscr{A}, B \in \mathscr{B}\}$ is again a uniformity of X. Evidently $\mathscr{D}$ has a countable basis and contains C. Finally, since the elements of $\mathscr{D}$ are neighbourhoods of the diagonal in $X \times X$, and since each element of $\mathscr{B}$ is one of $\mathscr{D}$, $\mathscr{D}$ induces the given topology of X.

COROLLARY 1. *Let X be a metrisable space and C a neighbourhood of the diagonal in $X \times X$. There exists a metric ρ of X inducing* *the topology of X and such that*

$$C \supset \{(x, y) \mid \rho(x, y) < 1\} .$$

Proof. By Lemma 4 there exists a uniformity $\mathscr{D}$ of X, having a countable basis, containing C and inducing the topology of X. By a well known theorem (see [2] p. 186) there exists a metric $\tilde{\rho}$ of X for which $\mathscr{D}$ is the metric uniformity. Thus, there exists $\varepsilon > 0$ such that $C \supset \{(x, y) \mid \tilde{\rho}(x, y) < \varepsilon\}$. The metric $\rho = \tilde{\rho}/\varepsilon$ is the required metric.

By Corollary 1, and since in a paracompact space each covering is even, we obtain:

COROLLARY 2. *Let $\mathscr{U}$ be a covering of a metrisable space X. There exists a metric ρ of X inducing the topology of X and such that the set of unit spherical regions in (X, ρ) refines $\mathscr{U}$.*

THEOREM 2. *Let $(\tilde{X}, f)$ be a covering space of a metrisable locally connected space X. There then exist metrics $\tilde{\rho}$ in $\tilde{X}$ and ρ in X, inducing the topologies of $\tilde{X}$ and X respectively and such that the family $\mathscr{S}$ of unit sperical regions in (X, ρ) has the following property:*

(A) *for every $S \in \mathscr{S}$, $f^{-1}[S]$ is a union of a family $\mathscr{T}(S)$ consisting of pairwise-disjoint unit spherical regions in $(\tilde{X}, \tilde{\rho})$ each of which is mapped isometrically onto S by f.*

Proof. By Lemma 1, the proof of Theorem 1 and Corollary 2 there exists a metric ρ of X inducing the topology of X and such that the covering $\mathscr{V}$ of X defined by $\mathscr{V} = \{C[x] \mid x \in X\}$ where $C = \{(x, y) \mid \rho(x, y) < 1\}$, has properties (1) and (2). We may assume that $\rho(x, y) \leqq 1$ for all x and y in X.

Now, let $\mathscr{A}$ be the metric uniformity of ρ. By Theorem 1 and Remark 1, there exists a uniformity $\tilde{\mathscr{A}}$ of $\tilde{X}$ containing a symmetric open set $\tilde{C}$ such that $\tilde{\mathscr{A}}$, $\tilde{C}$, $\mathscr{A}$ and C satisfy (3). Exactly as in the proof of properties (a) and (e), in Theorem 1 it can be shown that C has the same properties:

(a) $\tilde{C} = \tilde{C}^{-1} = (\tilde{C} \circ \tilde{C}) \cap F^{-1}[C]$

and

(e) if $(\tilde{x}, \tilde{y}) \in \tilde{C}$, then $f(\tilde{x}) = f(\tilde{y})$ implies $\tilde{x} = \tilde{y}$.

Now let $\tilde{\rho}$ be defined on $\tilde{X} \times \tilde{X}$ as follows:

$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \begin{cases} \rho[f(\tilde{x}), f(\tilde{y})] & \text{if } (\tilde{x}, \tilde{y}) \in \tilde{C} \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{\rho}$ is symmetric and that $\tilde{\rho}(\tilde{x}, \tilde{x}) = 0$. Moreover, if $\tilde{\rho}(\tilde{x}, \tilde{y}) = 0$ then $(\tilde{x}, \tilde{y}) \in \tilde{C}$ and $f(\tilde{x}) = f(\tilde{y})$; hence by (e), $\tilde{x} = \tilde{y}$.

Thus, to prove that $\tilde{\rho}$ is a metric it remains to show that

(g) $\tilde{\rho}(\tilde{x}, \tilde{y}) + \tilde{\rho}(\tilde{y}, \tilde{z}) \geqq \tilde{\rho}(\tilde{x}, \tilde{z})$.

In the case of $(\tilde{x}, \tilde{z}) \in \tilde{C}$, or if $(\tilde{x}, \tilde{y})$ or $(\tilde{y}, \tilde{z})$ are not in $\tilde{C}$, (g) obviously holds. If $(\tilde{x}, \tilde{y})$ and $(\tilde{y}, \tilde{z})$ are both in $\tilde{C}$ while $(\tilde{x}, \tilde{z})$ is not, by (a) it follows that $(f(\tilde{x}), f(\tilde{z})) = F(\tilde{x}, \tilde{z}) \notin C$ and therefore

$$\tilde{\rho}(\tilde{x}, \tilde{y}) + \tilde{\rho}(\tilde{y}, \tilde{z}) = \rho[f(\tilde{x}), f(\tilde{y})] + \rho[f(\tilde{y}), f(\tilde{z})] \geqq \rho[f(\tilde{x}), f(\tilde{z})]$$
$$= 1 = \tilde{\rho}(\tilde{x}, \tilde{z}), \text{ and (g) holds.}$$

Hence, $\tilde{\rho}$ is a metric, and one can see that $\tilde{\mathscr{A}}$ is its metric uniformity. In fact, let $\mathscr{B}$ be the family of all sets of the form $\{(x, y) \mid \rho(x, y) < \delta\}$ for some $0 < \delta \leqq 1$. Since $\mathscr{B}$ is a basis for $\mathscr{A}$, it follows by (3b) that

$$\tilde{\mathscr{C}} = \{\tilde{B} \mid \tilde{B} = \tilde{C} \cap F^{-1}[B]; B \in \mathscr{B}\}$$

is a basis for $\tilde{\mathscr{A}}$. Now, for every $\tilde{B} \in \tilde{\mathscr{C}}$ there exists δ, $0 < \delta \leqq 1$ such that

(h) $\tilde{B} = \tilde{C} \cap F^{-1}\{(x, y) \mid \rho(x, y) < \delta\} = \{(\tilde{x}, \tilde{y}) \mid \tilde{\rho}(\tilde{x}, \tilde{y}) < \delta\}$

Hence, $\tilde{\mathscr{A}}$ is the metric uniformity of $\tilde{\rho}$.

Putting in (h) $\delta = 1$ it follows that $\{\tilde{C}[\tilde{x}] \mid \tilde{x} \in \tilde{X}\}$ is the set of unit spherical regions in $\tilde{X}$ and by the definition of $\tilde{C}$ (see Theorem 1) $\mathscr{F}(C[x]) = \{\tilde{C}[\tilde{x}] \mid \tilde{x} \in f^{-1}(x)\}$.

To complete the proof of Theorem 2, it remains to show that for every $\tilde{x} \in \tilde{X}$, $f \mid \tilde{C}[\tilde{x}]$ is an isometry. In fact, if $\tilde{y}$ and $\tilde{z} \in \tilde{C}[\tilde{x}]$ then $(\tilde{y}, \tilde{z}) \in \tilde{C} \circ \tilde{C}$. Now, if $(\tilde{y}, \tilde{z}) \in \tilde{C}$ it follows by the definition of $\tilde{\rho}$ that $\tilde{\rho}(\tilde{y}, \tilde{z}) = \rho[f(\tilde{y}), f(\tilde{z})]$. If $(\tilde{y}, \tilde{z}) \notin \tilde{C}$ we have by (a) $[f(\tilde{y}), f(\tilde{z})] \notin C$. Hence, $\tilde{\rho}(\tilde{y}, \tilde{z}) = 1 = \rho[f(\tilde{y}), f(\tilde{z})]$ and $f \mid \tilde{C}[\tilde{x}]$ is an isometry.

REMARK 3. Note that the essential property of the metric ρ of X used in the proof of Theorem 2 is that the family of unit spherical regions in (X, ρ) satisfies (1) and (2). Thus, if $(\tilde{X}, f)$ is a covering space of a metric space (X, ρ) such that the family $\mathscr{S}$ of unit spherical regions of (X, ρ) has properties (1) and (2) it follows that there exists a metric $\tilde{\rho}$ of $\tilde{X}$ such that $\mathscr{S}$ satisfies (A).

Note also that if $(\tilde{X}, f)$ is a cevering space of a compact locally-connected metric space (X, ρ^*) then, by Lebesque's covering lemma (see [2] p. 154), the metric ρ of X can be obtained by multiplying ρ^* by a constant.

Part 2. Covering spaces of metrisable spaces which are not locally connected. In this part, the original problem for not necessarily locally connected spaces is considered and the following result is obtained:

THEOREM 3. *Let $(\tilde{X}, f)$ be a covering space of a metrisable space X. There then exist metrics $\tilde{\rho}$ of $\tilde{X}$ and ρ of X inducing the topologies of $\tilde{X}$ and X respectively, such that the family $\mathscr{S}$ of unit spherical regions in (X, ρ) has the following property*:

(A_1) *for every $S \in \mathscr{S}$, $f^{-1}[S]$ is a union of a family $\mathscr{F}(S)$ consisting of pairwise-disjoint open sets in $(\tilde{X}, \tilde{\rho})$ each of which is mapped isometrically onto S.*

The proof of Theorem 3 will be given later, after some remarks and Example 2 which, as we hope, explains the need for this theorem.

Comparing Theorem 3 with Theorem 2, we see that in this case it is not claimed that the elements of $\mathscr{F}(S)$, are spherical regions. Indeed, in Example 2 a covering space $(\tilde{X}, f)$ of a metrisable non-locally connected space X is constructed in such a way that there do not exist metrics $\tilde{\rho}$ of $\tilde{X}$ and ρ of X for which the family of unit spherical regions $\mathscr{S}$ has property (A). For this purpose note that property (2) is not only a sufficient but a necessary condition for the validity of Theorem 2 (see Remark 2). In fact, if there exist metrics $\tilde{\rho}$ of $\tilde{X}$ and ρ of X for which the family of unit spheres in (X, ρ) has property (A) then the family of spherical regions of radius 1/2 in (X, ρ) has property (2).

Thus, it suffices to construct a covering space $(\tilde{X}, f)$ of a metrisable space X such that no covering of X has property (2). Such a covering space is constructed in the following

EXAMPLE 2. Let $\{g_n\}$ be the sequence defined by

$$g_0 = 1;\ g_1 = 2;\ g_n = 2\prod_{k=2}^{n}(2^k - 2) \qquad \text{for } n \geqq 2\ .$$

For each nonnegative integer n and for $m = 0, 1, \cdots g_n - 1$ let $I(m, n)$ be the segment in E^2 defined by

$$I(m, n) = \{(x, y) \mid 2m \leqq x \leqq 2m + 1, y = n\}\ .$$

Now let C denote the Cantor set in $[0, 1]$. We put

$$X = \{(x, y) \mid x \in C, 0 \leqq y \leqq 1\} \cup I(0, 0) \cup I(0, 1)\ .$$

To define $\tilde{X}$ note that for each integer n, C is contained in a union of 2^n disjoint segments of length $(1/3)^n$. We denote these segments by $D(n, k)$, $k = 0, 1 \cdots 2^n - 1$ and it is clear that $C(n, k) = C \cap D(n, k)$ is homeomorphic with C and that

$$C(n, k) = C(n + 1, 2k) \cup C(n + 1, 2k + 1)\ .$$

Now for each $\xi \in C$ and each two pairs (m_1, n_1) and (m_2, n_2), where $n_1 \neq n_2$, $0 \leqq m_i < g_{n_i}$, let $S(m_1, n_1; m_2, n_2; \xi)$ be the segment in E^2

having $\alpha_i = (2m_i + \xi, n_i)$ $i = 1, 2$ as end points. Finally, we put

$$P(m_1, n_1; m_2, n_2; n, k) = \bigcup \{S(m_1, n_1; m_2, n_2; \xi) \mid \xi \in C(n, k)\}$$

and call such a set a path of width $(1/3)^n$ connecting $I(m_1, n_1)$ and $I(m_2, n_2)$. We shall call the points $(2m_i + \xi, n_i)$, $\xi \in C(n, k)$, $i = 1, 2$, the end points of the path. Now let

$$\tilde{X}_0 = I(0, 0)$$
$$\tilde{X}_1 = I(0, 0) \cup I(0, 1) \cup I(1, 1) \cup P(0, 0; 0, 1; 1, 0) \cup P(0, 0; 1, 1; 1, 1) .$$

Suppose we have defined $\tilde{X}_q$, $q \geqq 1$, having the following properties (obviously holding for $q = 1$):

(i) $\tilde{X}_q \supset \tilde{X}_{q-1}$

(ii) $\tilde{X}_q \supset \bigcup \{I(m, n) \mid 0 \leqq m < g_n, 0 \leqq n \leqq q\}$

(iii) for each n, $0 < n \leqq q$, $0 \leqq m < g_n$, $\tilde{X}_q$ contains one and only one path of width $(1/3)^n$ having end points in $I(m, n)$. This path connects $I(m, n)$ and a segment $I(m_1, n-1)$ for some $0 \leqq m_1 < g_{n-1}$.

(vi) for each $0 \leqq n < q$, every point $\alpha = (2m + \xi, n)$, $0 \leqq m < g_n$, $\xi \in C$, is an end point of one and only one path in $\tilde{X}_q$. This path is either the path of width $(1/3)^n$ indicated in (iii) connecting $I(m, n)$ with $I(m_1, n-1)$ for some $0 \leqq m_1 < g_{n-1}$ or of width $(1/3)^{n+1}$ connecting $I(m_2, n)$ and $I(m_2, n+1)$ for some $0 \leqq m_2 < g_{n+1}$. Different paths connect $I(m, n)$ with different segments.

(v) All the paths contained in $\tilde{X}_q$ are disjoint.

Thus, it follows that for each m, $0 \leqq m < g_q$, $I(m, q)$ contains end points of one and only one path in $\tilde{X}_q$, this path is of width $(1/3)^q$. Hence, for each m, $0 \leqq m < g_q$, there exist $2^{q+1} - 2$ integers $0 \leqq k(0, m) < k(1, m) < \cdots k(2^{q+1} - 3, m) < 2^{q+1}$ such that if $\xi \in C(q + 1, k(r, m))$, $0 \leqq r < 2^{q+1} - 2$, then $(2m + \xi, q)$ is not an end point of a path contained in $\tilde{X}_q$. We put

$$\tilde{X}'_{q+1} = \bigcup \{P(m, q; (2^{q+1} - 2)m + r, q + 1; q + 1, k(r, m)) \mid 0 \leqq r < 2^{q+1} - 2, 0 \leqq m < g_q\}$$

and let

$$\tilde{X}_{q+1} = \tilde{X}_q \cup \tilde{X}'_{q+1} \cup [\bigcup \{I(m, q + 1) \mid 0 \leqq m < g_{q+1}\}] .$$

It can be seen that conditions (i) to (v) hold. Finally, let $\tilde{X} = \bigcup_{q=0}^{\infty} \tilde{X}_q$. Then $\tilde{X}$ satisfies

(i)′ $\tilde{X} \supset \bigcup \{I(m, n) \mid 0 \leqq m < g_n, n = 0, 1, 2 \cdots\}$

(ii)′ For each $0 < n$ and $0 \leqq m < g_n$, $\tilde{X}$ contains one and only one path of width $(1/3)^n$ which has end points in $I(m, n)$. This path connects $I(m, n)$ with a segment $I(m_1, n - 1)$ for some $0 \leqq m_1 < g_{n-1}$.

(iii)′ Every point $\alpha = (2m + \xi, n)$, $0 \leqq m < g_n$, $\xi \in C$, is an end point of one and only one path [in $\tilde{X}$. This path is either the path of width $(1/3)^n$ mentioned in (ii)′ connecting $I(m, n)$ with a segment $I(m_1, n-1)$ for some $0 \leqq m < g_{n-1}$ or a segment of width $(1/3)^{n+1}$ which connects $I(m, n)$ with some segment $I(m_2, n+1)$ for some $0 \leqq m_2 < g_{n+1}$. Different paths connects $I(m, n)$ with different segments.

(v)′ All the paths in $\tilde{X}$ are disjoint.

Figure 1 illustrates the set $\tilde{X}_2$.

In order to define f, $f: \tilde{X} \rightarrow X$, we first define it on $\bigcup \{I(m, n) \mid 0 \leqq m < g_n, n = 0, 1, 2, \cdots\}$. We put $f(2m + \xi, n) = (\xi, (1 + (-1)^n)/2)$ for $0 \leqq \xi \leqq 1$. Now we extend f linearly onto all of $\tilde{X}$, i.e: if α and β are end points of a segment in a path contained in $\tilde{X}$, we put $f(t\alpha + (1-t)\beta) = tf(\alpha) + (1-t)f(\beta)$, $0 \leqq t \leqq 1$. This mapping is continuous and maps $\tilde{X}$ onto X. Moreover, if we put

$$V_0 = X \cap \left\{(x, y) \mid y < \frac{3}{4}\right\}$$

$$V_1 = X \cap \left\{(x, y) \mid y > \frac{1}{4}\right\}$$

then $\mathscr{V}_0 = \{V_0, V_1\}$ satisfies (1). Indeed,

$$f^{-1}(V_0) = \tilde{X} \cap \left[\bigcup_{k=0}^{\infty} \left\{(x, y) \mid 2k - \frac{3}{4} < y < 2k + \frac{3}{4}\right\}\right]$$

and each component of $f^{-1}(V_0)$ contains one segment $I(m, 2k)$ and the intersections of all paths in $\tilde{X}$ which have end points in $I(m, 2k)$ with the set $\{(x, y) \mid 2k - (3/4) < y < 2k + (3/4)\}$, and this set is homeomorphic under f with V_0. Similar arguments hold for V_1. Moreover, it can be shown that a covering $\mathscr{V}$ of X satisfies (1) if and only if no element of $\mathscr{V}$ intersects both $I(0, 0)$ and $I(0, 1)$.

We shall show now that no covering $\mathscr{V}$ of X for which (1) holds satisfies (2). Indeed, let $\mathscr{V}$ be a covering of X for which (1) holds. Without loss of generality we may assume that $\mathscr{V}$ consists of intersections of open disks in E^2 with X. Let n_0 be such that $(1/3)^{n_0}$ is less than the Lebesque number of $\mathscr{V}$. Consider any segment $I(m_0, n_0)$, $0 \leqq m_0 < g_{n_0}$, and let ξ_1 and ξ_2 be the two end points of $D(n_0, k(m_0, 0))$, then the paths in $\tilde{X}$ of which $\tilde{\alpha}_1 = (2m_0 + \xi_1, n_0)$ and $\tilde{\alpha}_2 = (2m_0 + \xi_2, n_0)$ are end points connect $I(m_0, n_0)$ with two different segments-$I(m_1, n_0 + 1)$ and $I(m_2, n_0 + 1)$. For each $0 \leqq \eta \leqq 1$ the points (ξ_1, η) and (ξ_2, η), are both contained in some element of $\mathscr{V}$. Let V_0 be an element of $\mathscr{V}$ containing $\alpha_1 = (\xi_1, 0)$ and $\alpha_2 = (\xi_2, 0)$ and $V_1 \in \mathscr{V}$ containing $\beta_1 = (\xi_1, 1)$ and $\beta_2 = (\xi_2, 1)$. Assuming that n_0 is even we have that $\tilde{\alpha}_1 = (2m_0 + \xi_1, n_0)$ and $\tilde{\alpha}_2 = (2m_0 + \xi_2, n_0)$ as points of $\tilde{X}$ lie in the same element $\tilde{V}_0$ of $\mathscr{F}(V_0)$. Indeed, since V_0 is connected the elements of $\mathscr{F}(V_0)$ are connected, and since V_0 contains no points of $I(0, 1)$, the element $\tilde{V}_0$ of $\mathscr{F}(V_0)$ which contains $\tilde{\alpha}_1$ does not contain points of segments $I(m, n)$ other than $I(m_0, n_0)$. Thus, the point of $f^{-1}(\alpha_2)$ which $\tilde{V}_0$ contains must be $\tilde{\alpha}_2$. By the same arguments $\tilde{\beta}_1 = (2m_1 + \xi_1, n_0 + 1)$ and $\tilde{\beta}_2 = (2m_2 + \xi_2, n_0 + 1)$ lie in different elements $\tilde{V}_1^{(1)}$ and $\tilde{V}_1^{(2)}$ of $\mathscr{F}(v_1)$. Therefore, there must exist two elements V and U of $\mathscr{V}$, containing the points (ξ_1, η) and (ξ_2, η) for some $0 < \eta \leqq 1$, and two elements of $\mathscr{F}(U)$ intersect some element of $\mathscr{F}(V)$, (see Figure 2) and (2) does not hold.

Before proving Theorem 3 let us introduce the following notions: Let X be a topological space and $\mathscr{W}$ a convering of X. We say that a finite subset F of X, $F = \{x_0, x_1, x_2 \cdots x_n\}$ is a chain in $\mathscr{W}$ if, for every i, $i = 0, 1, 2 \cdots n - 1$, x_i and x_{i+1} are both contained in some

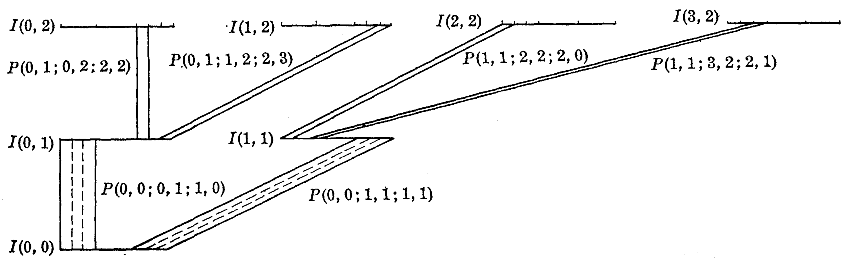

Fig. 1

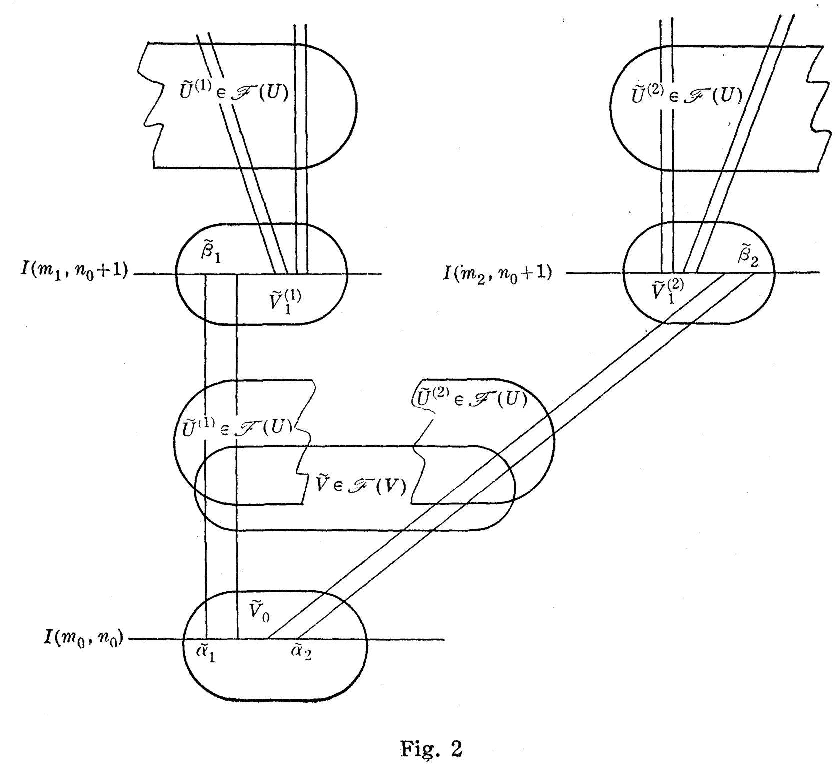

Fig. 2

element W of $\mathscr{W}$. The $\mathscr{W}$-component of x in X is the set of all points y of X contained, together with x, in some chain in $\mathscr{W}$. It is clear that the $\mathscr{W}$-components are disjoint open subsets of X.

Proof of theorem 3. Let $\mathscr{W}^*$ be a covering of X having property (1), and $\mathscr{W}$ a locally finite covering of X consisting of open sets whose closures refine $\mathscr{W}^*$. Furthermore, let W be any element of $\mathscr{W}$ and W^* an elements of $\mathscr{W}^*$ such that $\bar{W} \subset W^*$. We put

$$\mathscr{F}(W) = \{\tilde{W} \mid \tilde{W} = \tilde{W}^* \cap f^{-1}(W); \tilde{W}^* \in \mathscr{F}(W)\} .$$

It follows that $\mathscr{W}$ also satisfies (1), and since $\mathscr{W}$ is locally-finite and for every $W \in \mathscr{W}$, $\mathscr{F}(W)$ is discrete—the covering $\tilde{\mathscr{W}} = \bigcup_{W \in \mathscr{W}} \mathscr{F}(W)$ is locally finite.

Given any metric ρ in X, we shall show that ρ can be "lifted" into X, i.e: there exists a metric $\tilde{\rho}$ of $\tilde{X}$ such that for every $W \in \mathscr{W}$ and every $\tilde{W} \in \mathscr{F}(W)$, $f \mid \tilde{W}$ is an isometry between $\tilde{W}$ and W.

We now define $\tilde{\rho}$ as follows:

If $\tilde{x}$ and $\tilde{y}$ are elements of $\tilde{X}$ belonging to different $\tilde{\mathscr{W}}$-components, we put $\tilde{\rho}(\tilde{x}, \tilde{y}) = 1$. If $\tilde{x}$ and $\tilde{y}$ are in the same component, we put

$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \inf \sum_{i=0}^{n-1} \rho[f(\tilde{x}_i), f(\tilde{x}_{i+1})]$$

where the infinum is taken over all chains $\{\tilde{x} = \tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n = \tilde{y}\}$ in $\tilde{\mathscr{W}}$ connecting $\tilde{x}$ and $\tilde{y}$. It is clear that $\tilde{\rho}$ is pseudometric and that $\tilde{\rho}(\tilde{x}, \tilde{y}) \geqq \rho[f(\tilde{x}), f(\tilde{y})]$. We shall now show that $\tilde{\rho}$ has the following properties:

(k_1) For every $\tilde{x} \in \tilde{X}$ there exists an open neighbourhood $\tilde{T}(\tilde{x})$ such that $\tilde{x} \in \tilde{T}(\tilde{x}) \subset \tilde{W}$ for every $\tilde{W}$ of $\tilde{\mathscr{W}}$ which contains $\tilde{x}$.

(k_2) There exists a positive real number $\delta(\tilde{x})$ such that

$$\tilde{S}(\tilde{x}, \delta(\tilde{x})) = \{\tilde{y} \mid \tilde{\rho}(\tilde{x}, \tilde{y}) < \delta(\tilde{x})\} \subset \tilde{T}(\tilde{x}) .$$

Proof of (k_1). Since $\tilde{\mathscr{W}}$ is locally finite, each $\tilde{x}$ in $\tilde{X}$ is contained in an open set $\tilde{M}(\tilde{x})$ intersecting only a finite number of elements of $\tilde{\mathscr{W}}$.

Let $\tilde{T}(\tilde{x})$ be the intersection over the family $\tilde{\mathscr{M}}(\tilde{x})$ which consists of $\tilde{M}(\tilde{x})$, of the elements of $\tilde{\mathscr{W}}$ which contain $\tilde{x}$ and of the complements of the closures of elements of $\tilde{\mathscr{W}}$ which do not contain $\tilde{x}$ in their closure. Then $\tilde{T}(\tilde{x})$ is an open neighbourhood of $\tilde{x}$ satisfying (k_1). By the definition of $\tilde{T}(\tilde{x})$ it follows that

(1) if $\tilde{W} \in \tilde{\mathscr{W}}$ then $\tilde{W} \cap \tilde{T}(\tilde{x}) \neq \phi$ implies $\tilde{x} \in \bar{\tilde{W}}$.

Proof of k_2. Let $\delta(\tilde{x})$, $\delta(\tilde{x}) < 1$ be a positive number such that

$S[f(\tilde{x}), \delta(\tilde{x})] = \{y \mid y \in X; \rho[f(\tilde{x}), y] < \delta(\tilde{x})\} \subset f(\tilde{T}(\tilde{x}))$. We shall show that ($k_2$) holds.

In fact, let $\tilde{y}$ be any point such that $\tilde{y} \notin \tilde{T}(\tilde{x})$. If $\tilde{y}$ does not belong to the $\tilde{\mathscr{W}}$-component of $\tilde{x}$ then $\tilde{\rho}(\tilde{x}, \tilde{y}) = 1 > \delta(\tilde{x})$ and thus, $\tilde{y} \notin \tilde{S}(\tilde{x}, \delta(\tilde{x}))$. Therefore we may assume that $\tilde{y}$ belongs to the $\tilde{\mathscr{W}}$-component of $\tilde{x}$. Then for every $\varepsilon > 0$ there exists a chain $\{\tilde{x} = \tilde{x}_0, \tilde{x}_1 \cdots \tilde{x}_n = \tilde{y}\}$ in $\tilde{\mathscr{W}}$ such that

$$\tilde{\rho}(\tilde{x}, \tilde{y}) + \varepsilon \geqq \sum_{i=0}^{n-1} \rho[f(\tilde{x}_i), f(\tilde{x}_{i+1})] .$$

Let i_0 be the first subscript such that $\tilde{x}_{i_0+1} \notin \tilde{T}(\tilde{x})$. We shall show that $f(\tilde{x}_{i_0+1}) \notin f(\tilde{T}(\tilde{x}))$. Suppose to the contrary that $f(\tilde{x}_{i_0+1}) \in f(\tilde{T}(\tilde{x}))$. There then exists an element $\tilde{z}$ of $\tilde{T}(\tilde{x})$ (and therefore $\tilde{z} \neq \tilde{x}_{i_0+1}$) and such that $f(\tilde{z}) = f(\tilde{x}_{i_0+1})$. By definition of a chain it follows that there exists a $\tilde{W}$, $\tilde{W} \in \tilde{\mathscr{W}}$, such that $\tilde{x}_{i_0}$ and $\tilde{x}_{i_0+1} \in \tilde{W}$. Since $\tilde{x}_{i_0} \in \tilde{T}(\tilde{x})$, $\tilde{W} \cap \tilde{T}(\tilde{x})$ is not empty and therefore by (l) $\tilde{x} \in \bar{\tilde{W}}$. On the other hand, if $\tilde{W}_1$ is the element of $\mathscr{F}[f(\tilde{W})]$ containing $\tilde{z}$, we have by $\tilde{z} \in \tilde{T}(\tilde{x})$ and by (l) that $\tilde{x} \in \bar{\tilde{W}}_1$. Hence, $\bar{\tilde{W}}_1 \cap \bar{\tilde{W}} \neq \phi$ which contradicts the fact that $\mathscr{F}(f(\tilde{W}))$ is discrete. Thus, $f(\tilde{x}_{i_0+1}) \notin f(\tilde{T}(\tilde{x}))$ and therefore $\rho(f(\tilde{x}), f(\tilde{x}_{i_0+1})) \geqq \delta(\tilde{x})$. Thus we have:

$$\begin{aligned} \tilde{\rho}(\tilde{x}, \tilde{y}) + \varepsilon \geqq \sum_{i=0}^{n-1} \rho[f(\tilde{x}_i), f(\tilde{x}_{i+1})] &\geqq \sum_{i=0}^{i_0} \rho[f(\tilde{x}_i), f(\tilde{x}_{i+1})] \\ &\geqq \rho[f(\tilde{x}), f(\tilde{x}_{i_0+1})] \geqq \delta(\tilde{x}) . \end{aligned}$$

Hence, for every $\varepsilon > 0$ we have $\tilde{\rho}(\tilde{x}, \tilde{y}) \geqq \delta(\tilde{x}) - \varepsilon$, and thus $\tilde{\rho}(\tilde{x}, \tilde{y}) \geqq \delta(\tilde{x})$. It follows that $\tilde{S}[\tilde{x}, \delta(\tilde{x})] \subset \tilde{T}(\tilde{x})$ and (k_2) is proved.

Suppose now that $\tilde{\rho}(\tilde{x}, \tilde{y}) = 0$. Then by $\tilde{\rho}(\tilde{x}, \tilde{y}) \geqq \rho(f(\tilde{x}), f(\tilde{y}))$, we have $\rho[f(\tilde{x}), f(\tilde{y})] = 0$, hence $f(\tilde{x}) = f(\tilde{y})$ and by (k_2) $\tilde{y} \in \tilde{T}(\tilde{x})$. Since by ($k_1$) $f \mid \tilde{T}(\tilde{x})$ is a homeomorphism, it follows that $\tilde{x} = \tilde{y}$ and therefore $\tilde{\rho}$ is a metric. Moreover, by the definition of $\tilde{\rho}$ we have that for each $\tilde{W} \in \tilde{\mathscr{W}}$, $f \mid \tilde{W}$ is an an isometry. Therefore by (k_2) we have that for each $\eta < \delta(\tilde{x})$, $\tilde{x} \in \tilde{W} \in \tilde{\mathscr{W}}$

$$S(\tilde{x}, \eta) = \tilde{W} \cap f^{-1}[S(f(\tilde{x}), \eta)]$$

which implies that $\tilde{\rho}$ induces the topology of $\tilde{X}$.

Now if we take the metric ρ of X to be a metric having the property that the set $\mathscr{S}$ of unit spherical region in (X, ρ) refines $\mathscr{W}$ (see corollary 2) (A_1) holds and the proof of Theorem 3 is completed.

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