ARENS MULTIPLICATION AND CONVOLUTION

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1. Introduction. Let L denote the group algebra of a locally compact Abelian (LCA) group $\mathscr{G}$. For elements x and y in L the product of x and y is given by

$$xy(\beta) = \int x(\beta - \alpha)y(\alpha)d\alpha \qquad \beta \in \mathscr{G} ,$$

where the integral is taken over the entire group and with respect to Haar measure.

Let L^* and L^{**} denote the first and second conjugate spaces of L, respectively. As a result of [1], a multiplication can be introduced in L^{**} in the following manner. Let $x, y \in L$; $f, g \in L^*$; and $F, G \in L^{**}$. The elements xf and $F \circ f$ in L^* and $G \circ F$ in L^{**} are defined by:

$$xf(y) = f(xy) \qquad y \in L , \tag{1.1}$$

$$F \circ f(x) = F(xf) \qquad x \in L , \tag{1.2}$$

$$G \circ F(f) = G(F \circ f) \qquad f \in L^* . \tag{1.3}$$

The multiplication in L^{**} given by (1.3) will be referred to as the Arens product. Some of the properties of the Arens product in L^{**} have been developed in [2].

It is well-known that the spaces L^* and L^{**} have realizations in terms of functions on $\mathscr{G}$ [5, p. 148] and finitely additive measures on $\mathscr{G}$ [6], respectively. One difficulty which arises with the Arens product is that there seems to be no means of obtaining the functions and measures which correspond to elements of the form $F \circ f$ and $G \circ F$, respectively. To avoid excessive notation we will use $f, g, \cdots$ to denote elements of L^* and their corresponding realizations as functions. Any statement involving $f, g, \cdots$ as functions will be interpreted as a locally almost everywhere statement (see [5, p. 141]) even though a reference to locally almost everywhere (l.a.e.) may not appear. Similarly, $F, G, \cdots$ will denote elements of L^{**} and their corresponding realizations as finitely additive measures.

In the case of xf, an obvious application of the Fubini theorem yields

$$xf(\beta) = \int f(\beta + \alpha)x(\alpha)d\alpha \qquad \beta \in \mathscr{G} , \tag{1.4}$$

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which provides a realization of xf as a function. Proceeding formally, one obtains the following "equations":

$$(1.5) \qquad F\circ f(x) = \iint f(\beta+\alpha)x(\alpha)d\alpha dF(\beta) = \iint f(\beta+\alpha)dF(\beta)x(\alpha)d\alpha \ ,$$

$$(1.6) \qquad G\circ F(f) = \iint f(\beta+\alpha)dF(\beta)dG(\alpha) \ .$$

If the equations in (1.5) were valid, then the function $h(\alpha) = \int f(\beta+\alpha)dF(\beta)$ would be a realization of $F\circ f$; however, as a general statement, (1.5) is invalid on two counts. In 4.3 it is shown that the function $h(\alpha)$ need not even be measurable (measurability will always be with respect to Haar measure) and in 3.5 it is shown that even if $h(\alpha)$ is measurable, the second "equality" in (1.5) may not be valid.

The formal equations in (1.5) and (1.6) suggest a second pair of operations analogous to the operations defined in (1.2) and (1.3).

For each F in L^{**}, f in L^* and β in $\mathscr{G}$, $T_\beta f$ and $F*f$ are defined as follows:

$$(1.7) \qquad T_\beta f(\alpha) = f(\alpha+\beta) \qquad \alpha\in\mathscr{G} \ ,$$

$$(1.8) \qquad F*f(\beta) = F(T_\beta f) \qquad \beta\in\mathscr{G} \ .$$

Thus, for each β in $\mathscr{G}$, T_β is an operator (bounded linear transformation) on L^*. Let $\mathscr{T} = \{T_\beta: \beta\in\mathscr{G}\}$. Also, for each f in L^* and each F in L^{**}, $F*f$ is a well-defined functon on $\mathscr{G}$, though it may not be measurable. For simplicity, the expression $F(T_\beta f)$ is used instead of $\int f(\beta+\alpha)dF(\alpha)$.

Let $\mathscr{B}$ denote $\{F\in L^{**}$: $F*f$ is a measurable function for each f in $L^*\}$. Again, to avoid excessive notation, the function $F*f$ for each f in L^* and F in $\mathscr{B}$ will be identified with the element of L^* of which $F*f$ is a realization. Let π denote the natural map of L into L^{**}. Clearly $\pi x*f = xf$ for each x in L and f in L^*. Also, an easy computation shows that $\pi x*f$ is a continuous function, so $\pi L\subset\mathscr{B}$.

For each F in $\mathscr{B}$ and G in L^{**}, $G*F$, the *convolution* of G and F, is defined by

$$(1.9) \qquad G*F(f) = G(F*f) \qquad f\in L^* \ .$$

It is clear that $G*F$ is an element of L^{**} and that $G*F(f) = \iint f(\beta+\alpha)dF(\alpha)dG(\beta)$ for each f in L^*.

Formulas (1.8) and (1.9) define the operations which are suggested by (1.5) and (1.6) and which are analogous to (1.2) and (1.3).

The two main objectives of this paper are: (i) to compare the operations introduced above and (ii) to compare various algebras obtained from these operations.

In § 2 it is noted that $\{F \in \mathscr{B}: F * \pi x = \pi x * F$ for each x in $L\}$, which will be denoted by $\mathscr{A}$, is the largest set which contains πL and in which the Arens product agrees with convolution. It is also noted that $\mathscr{A} = L^{**}$ in case $\mathscr{G}$ is a discrete group. In §§ 3 and 4 examples are given to show that $\mathscr{A}$ may be different from $\mathscr{B}$ and $\mathscr{B}$ may be different from L^{**}, respectively. In § 6 it is established that for all non-compact groups and for certain compact groups πL is a proper subset of $\mathscr{A}$.

In § 2 it is also observed that convolution and the Arens product can be used to make various subspaces of L^{**} Banach algebras and that L^* is a module over these algebras when the module operation is chosen as in (1.2) or (1.8), depending on the multiplication in the algebra. The fact that L^* is a module over these algebras is then used in § 5 to identify various quotients of these algebras with algebras of operators on certain subspaces of L^*. These identifications are used in the latter part of 5 to characterize the measure algebra of $\mathscr{G}$ as a certain operator algebra and to relate the measure algebra to the various quotient algebras mentioned above.

The following notation, as well as all notation introduced above, will be used throughout this paper. If X is a normed linear space, then X^* will denote the conjugate space of X, $O(X)$ will denote the Banach algebra of operators on X and for each sebset X_1 of X, ${}^\circ X_1$ will denote $\{f \in X^*: f(x) = 0, x \in X_1\}$. For a subset ξ_1 of a Banach algebra ξ, $C(\xi_1, \xi)$ will denote $\{A \in \xi: AB = BA, B \in \xi_1\}$. For each subset E of a given set S, C_E will denote the characteristic function of E and $S \backslash E$ will denote the complement of E in S.

2. Properties of the Arens product and convolution.

This section contains a list, in the form of lemmas and theorems, of some of the properties of the operations introduced in § 1. In particular, Theorems 2.4, 2.8, and 2.9 summarize the information needed in 5. In the remaining theorems, the Arens product and convolution are compared.

The following lemma is an immediate consequence of the definitions.

2.1. LEMMA. *For each F in L^{**} the following conditions are satisfied.*

(i) $F \circ \pi x = \pi x \circ F \qquad x \in L$,

(ii) $F \circ T_\beta f = T_\beta(F \circ f) \qquad \beta \in \mathscr{G}, f \in L^*$,

(iii) $\|F \circ f\| \leqq \|F\| \|f\| \qquad f \in L^*$.

Let C_u denote the subspace of L^* consisting of the elements which can be realized as uniformly continuous functions. Whenever an element f in C_u is identified with a function, it will be assumed that the function is the unique realization of f as a uniformly continuous function.

2.2. LEMMA. *The set* $\{xf: x \in L, f \in L^*\}$ *is a dense subset of* C_u.

Proof. An easy computation shows that $xf \in C_u$ for each x in L and f in L^*. Let m denote Haar measure and for each compact neighborhood V of the identity in $\mathscr{G}$, let $e_V = (m(V))^{-1}C_V$. If $f \in C_u$, then $\| e_V f - f \| \to 0$ as $V \to 0$. Therefore, the closure of $\{xf: x \in L, f \in L^*\} = C_u$.

2.3. LEMMA. *The following statements are equivalent*:

(i) $F \in {}^0C_u$.

(ii) $F \circ f = 0 \quad f \in L^*$.

(iii) $G \circ F = 0 \quad G \in L^{**}$.

Proof. (i) $\Rightarrow$ (ii). If $F \in {}^0C_u$, then $F \circ f(x) = F(xf) = 0$ for each x in L and f in L^*. Therefore, $F \circ f = 0$ for each f in L^*. (ii) $\Rightarrow$ (iii). If $G \in L^{**}$, then $G \circ F(f) = G(F \circ f) = 0$ for each f in L^*. (iii) $\Rightarrow$ (i). For x in L and f in L^*, $F(xf) = F \circ \pi x(f) = \pi x \circ F(f) = 0$. Since $\{xf: x \in L, f \in L^*\}$ is dense in C_u, $F \in {}^0C_u$, which completes the proof.

For each F in L^{**} and f in L^*, let $A_F f = F \circ f$. Let $\mathscr{L} = \{A_{\pi x}: x \in L\}$. For convenience A_x will be used instead of $A_{\pi x}$ when $x \in L$.

2.4. THEOREM. (i) *With the Arens multiplication* L^{**} *is a Banach algebra.* (ii) *With the operation defined as in* (1.2) L^* *is a left* $(L^{**}, \circ)$ *module.* (iii) *The map* $F \to A_F$ *for each* F *in* L^{**} *is a continuous algebraic homomorphism of* $(L^{**}, \circ)$ *into* $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$ *with kernel* ${}^0C_u$. (iv) ${}^0C_u$ *is a closed ideal in* $(L^{**}, \circ)$.

Proof. (i) and (ii) follow easily from the definitions given in (1.1)-(1.3). Statement (iii) follows from (ii), 2.1 and 2.3. Statement (iv) is a consequence of (iii).

Similar theorems will now be obtained for $\mathscr{A}$ and $\mathscr{B}$.

2.5. LEMMA. *For each* F *in* $\mathscr{B}$ *the following conditions are satisfied*:

(i) $F*f \in C_u \qquad f \in C_u$,

(ii) $F*T_\beta f = T_\beta(F*f) \qquad \beta \in \mathscr{G}, f \in L^*$,

(iii) $\| F*f \| \leqq \| F \| \| f \| \qquad f \in L^*$.

Proof. (i) If $\alpha, \beta \in \mathscr{G}$, then $|F*f(\alpha) - F*f(\beta)| = |F(T_\alpha f - T_\beta f)| \leqq \| F \| \| T_\alpha f - T_\beta f \|$ and $\| T_\alpha f - T_\beta f \| \to 0$ as $\alpha - \beta \to 0$. Conditions (ii) and (iii) are obvious.

Let $\mathscr{Q} = \{F \in \mathscr{B}: F*f = 0$ (l.a.e.) for each f in $L^*\}$.

2.6. LEMMA. *The following statements are equivalent.*
(i) $F \in \mathscr{Q}$.
(ii) $F \in \mathscr{B}$ *and* $H*F = 0$ *for each* H *in* $\mathscr{B}$.
(iii) $F \in \mathscr{B}$ *and* $\pi x * F = 0$ *for each* x *in* L.

Proof. (i) $\Rightarrow$ (ii). Clearly $\mathscr{Q} \subset \mathscr{B}$. If $F \in \mathscr{Q}$ and $f \in L^*$, then for H in $\mathscr{B}$, $H*F(f) = H(F*f) = 0$. (ii) $\Rightarrow$ (iii). $\pi L \subset \mathscr{B}$. (iii) $\Rightarrow$ (i). $\pi x(F*f) = \pi x * F(f) = 0$. Therefore, $F*f = 0$, which proves that $F \in \mathscr{Q}$.

2.7. LEMMA. $\mathscr{Q} = {}^0C_u \cap \mathscr{A}$.

Proof. Let $F \in \mathscr{A} \cap {}^0C_u$. For x in L and f in L^*, $\pi x * F(f) = F*\pi x(f) = F(xf) = 0$ since $xf \in C_u$. Therefore, $\pi x * F = 0$ for each x in L, so $F \in \mathscr{Q}$.

Now assume that $F \in \mathscr{Q}$. If $f \in C_u$, then $F*f \in C_u$ and $F*f = 0$ (l.a.e.). Therefore, $F*f(\beta) = 0$ for each β in $\mathscr{G}$. In particular, $F(f) = F*f(0) = 0$. Hence, $F \in {}^0C_u$. Since $F \in \mathscr{Q}$, $\pi x * F = 0$ for each x in L; on the other hand, if $f \in L^*$, then $F*\pi x(f) = F(xf) = 0$. Therefore, $\pi x * F = F*\pi x$ for each x in L, so $F \in \mathscr{A}$ by definition, which completes the proof.

For each F in $\mathscr{B}$ and f in L^* let $B_F f = F*f$. Note that for each x in L, $B_{\pi x} = A_x$.

2.8. THEOREM. (i) $(\mathscr{B}, *)$ *is a Banach algebra.* (ii) *With the operation defined as in* (1.8) L^* *is a left* $(\mathscr{B}, *)$ *module.* (iii) *The map* $F \rightarrow B_F$ *for each* F *in* $\mathscr{B}$ *is a continuous algebraic homomorphism of* $\mathscr{B}$ *into* $C(\mathscr{T}, 0(L^*))$ *with kernel* $\mathscr{Q}$. (iv) $\mathscr{Q}$ *is a closed ideal in* $\mathscr{B}$.

Proof. From the definitions it is easily verified that $(\mathscr{B}, *)$ is a normed algebra. For each $n = 1, 2, \cdots$, let $F_n \in \mathscr{B}$ such that $F_n \rightarrow F$, an element of L^{**}. If $f \in L^*$ and $\beta \in \mathscr{G}$, then $F_n(T_\beta f) \rightarrow F(T_\beta f)$. Therefore, $F*f$ is the pointwise limit of a sequence of measurable functions, so $F*f$ is measurable. Hence, $f \in \mathscr{B}$. Therefore, $\mathscr{B}$ is a closed subspace of L^{**} and since L^{**} is complete, $\mathscr{B}$ is also complete. (ii) follows easily from the definitions. (iii) is a consequence of (ii), 2.5 and the definition of $\mathscr{Q}$. (iv) follows from (iii).

2.9. THEOREM. (i) $(\mathscr{A}, *)$ *is a Banach algebra.* (ii) *With the operation defined as in* (1.8) L^* *is a left* $(\mathscr{A}, *)$ *module.* (iii) *The* *map* $F \to B_F$ *for each* F *in* $\mathscr{A}$ *is a continuous algebraic homomorphism of* $\mathscr{A}$ *into* $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$ *with kernel* $\mathscr{Q}$. (iv) $\mathscr{Q}$ *is a closed ideal in* $\mathscr{A}$.

Proof. (i) Since $\mathscr{A} = C(\pi L, \mathscr{B})$, $\mathscr{A}$ is a closed subalgebra of of $\mathscr{B}$ and therefore $\mathscr{A}$ is a Banach algebra.

From the definition of $\mathscr{A}$ it is clear that $B_F \in C(\mathscr{L}, 0(L^*))$ for each F in $\mathscr{A}$. Therefore, (ii), (iii) and (iv) follow from (ii), (iii) and (iv) of 2.8, respectively.

In the remaining theorems a comparison of convolution and the Arens product is made.

2.10. THEOREM. (i) *For each* F *and* G *in* L^{**} *and* f *in* L^*, $(G \circ F) * f = G * (F \circ f)$. (ii) *The sets* $\mathscr{A}$ *and* $\mathscr{B}$ *are right ideals in* $(L^{**}, \circ)$.

Proof. For $\beta \in \mathscr{G}$, $G \circ F(T_\beta f) = G(F \circ T_\beta f) = G(T_\beta F \circ f)$; however, $G \circ F(T_\beta f)$ and $G(T_\beta F \circ f)$ as functions of β are realizations of $(G \circ F) * f$ and $G * (F \circ f)$, respectively. (ii) Let $G \in \mathscr{B}$ and $F \in L^{**}$. Then $(G \circ F) * f = G * (F \circ f)$, a measurable function. Since $(G \circ F) * f$ is measurable for each f in L^*, $G \circ F \in \mathscr{B}$ by definition. To prove that $\mathscr{A}$ is a right ideal, let $G \in \mathscr{A}$, $F \in L^{**}$, $x \in L$ and $f \in L^*$. Then $(G \circ F) * \pi x(f) = (G \circ F)(xf) = G(F \circ (xf)) = G(x(F \circ f)) = G * \pi x(F \circ f) = \pi x * G(F \circ f) = \pi x(G * (F \circ f)) = \pi x((G \circ F) * f) = \pi x * (G \circ F)(f)$. Therefore, $(G \circ F) * \pi x = \pi x * (G \circ F)$ for each x in L, so $G \circ F \in \mathscr{A}$ by the definition of $\mathscr{A}$.

2.11. THEOREM. *The following statements are equivalent.*

(i) $F \in \mathscr{A}$.
(ii) $F \in \mathscr{B}$ *and* $F * f = F \circ f$ $\quad f \in L^*$.
(iii) $F \in \mathscr{B}$ *and* $G * F = G \circ F$ $\quad G \in L^{**}$.
(iv) $F \in \mathscr{B}$ *and* $\pi x * F = \pi x \circ F$ $\quad x \in L$.

Proof. (i) $\Rightarrow$ (ii). If $x \in L$ and $f \in L^*$, then $\pi x(F \circ f) = F \circ f(x) = F * \pi x(f) = \pi x * F(f) = \pi x(F * f)$. Therefore, $F \circ f = F * f$. (ii) $\Rightarrow$ (iii). If $f \in L^*$ and $G \in L^{**}$, then $G \circ F(f) = G(F \circ f) = G(F * f) = G * F(f)$. Clearly (iii) implies (iv). (iv) $\Rightarrow$ (i). If $x \in L$ and $f \in L^*$, then $F \circ \pi x(f) = F(xf) = F * \pi x(f)$. Therefore, $\pi x * F = \pi x \circ F = F \circ \pi x = F * \pi x$, which proves that $F \in \mathscr{A}$.

2.12. COROLLARY. $\mathscr{A}$ *is the maximum subalgebra of* $(\mathscr{B}, *)$ *which contains* πL *and in which the Arens product and convolution agree.*

Proof. Let $\mathscr{D}$ be a subalgebra of $(\mathscr{B}, *)$ which contains πL and in which the two products agree. If $F \in \mathscr{D}$, then $\pi x \circ F = \pi x * F$ for each x in L, and by 2.11, $F \in \mathscr{A}$. Therefore, $\mathscr{D} \subset \mathscr{A}$.

2.13. THEOREM. *If $\mathscr{G}$ is discrete, then $\mathscr{A} = L^{**}$.*

Proof. Let $F \in L^{**}$, $f \in L^*$ and $x \in L$. Let $\{\alpha_1, \alpha_2, \cdots\}$ be the support of x. Then $xf(\beta) = \sum_{j=1}^{\infty} f(\beta + \alpha_j)x(\alpha_j)$ and $\sum_{j=1}^{n} f(\beta + \alpha_j)x(\alpha_j)$ converges uniformly to $xf(\beta)$ in the variable β since $|\sum_{j=n+1}^{\infty} f(\beta + \alpha_j)x(\alpha_j)| \leqq \| f \| \sum_{j=n+1}^{\infty} |x(\alpha_j)|$. Therefore, $F * \pi x(f) = F(xf) = \int \lim_{n\to\infty} \sum_{j=1}^{n} f(\beta + \alpha_j)dF(\beta) = \lim_{n\to\infty} \sum_{j=1}^{n} x(\alpha_j) \int f(\beta + \alpha_j)dF(\beta) = \pi x * F(f)$. Since $\pi x * F = F * \pi x$ for each x in L, $F \in \mathscr{A}$, which completes the proof.

3. Groups for which $\mathscr{A}$ differs from $\mathscr{B}$.

In 3.4 a sufficient condition for $\mathscr{A}$ to be different than $\mathscr{B}$ is given in terms of the existance of a certain type of measurable subset of the group. It is then shown in 3.13 that for second countable groups this certain type of measurable subset is very numerous. Theorem 3.5 summerizes the results in this section. Lemmas 3.10 and 3.11 contain the main ideas for the proof of 3.12. Throughout this section μ will denote Haar measure.

A proof for the following lemma can be obtained by slightly modifying the proofs of 2.1, 2.2 and 3.1 (a) in [7].

3.1. LEMMA. *Let V be a complex vector space with pseudonorm q. Let V_1 be a subspace of V. Let $\mathscr{S}$ be a commutative semigroup of linear transformations on V such that*

(i) $T(v) \in V_1 \qquad T \in \mathscr{S}, v \in V_1$,

(ii) $q(Tv) \leqq q(v) \qquad T \in \mathscr{S}, v \in V$.

If k is a liner functional on V_1 such that

(iii) $|k(v)| \leqq q(v) \qquad v \in V_1$,

(iv) $k(Tv) = k(v) \qquad T \in \mathscr{S}, v \in V_1$,

then k has a linear homogeneous extension k_1 to all of V such that

(v) $|k_1(v)| \leqq q(v) \qquad v \in V$,

(vi) $k_1(Tv) = k_1(v) \qquad T \in \mathscr{S}, v \in V$.

3.2. LEMMA. *If an LCA group $\mathscr{G}$ contains a measurable subset E such that for each finite subset $\{\beta_1, \cdots, \beta_m\}$ of $\mathscr{G}$ and each open set V of $\mathscr{G}$,*

(i) $\mu(\bigcap_{k=1}^{m} E + \beta_k \cap V) > 0$,

(ii) $\mu(\bigcap_{k=1}^{m} E' + \beta_k \cap V) > 0$,

then there exists a nonzero translation invariant element in ${}^0C_u$.

Proof. Let E be a subset of $\mathscr{G}$ satisfying the hypothesis. Let $f = C_E$. For each g in L^* let $N(g) = \inf\{2\|g - h\|: h \in C_u\}$. Let X denote the linear span of $\{T_\beta f: \beta \in \mathscr{G}\}$. For $0 \leqq k \leqq m$ let a_k be a complex number and let $\beta_k \in \mathscr{G}$. It will be shown that if $g = \sum_{k=1}^m a_k T_{\beta_k} f$, then

$$(1) \qquad |\textstyle\sum_{k=1}^m a_k| \leqq N(g).$$

Assume that (1) has already been established and for $g = \sum_{k=1}^m a_k T_{\beta_k} f$, an arbitrary element of X, let $I_1(g) = \sum_{k=1}^m a_k$. If $g = 0$, then by (1), $\sum_{k=1}^m a_k = 0$. This shows that I_1 is a well-defined on X. Clearly I_1 is a homogeneous linear function on X. Let $\mathscr{T}$ denote $\{T_\beta: \beta \in \mathscr{G}\}$, which is a commutative semi-group of linear transformations on L^*. In 3.1 let $V = L^*$, $V_1 = X$, $\mathscr{S} = \mathscr{T}$, $k = I_1$ and $q = N$. Conditions (i) and (iv) of 3.1 are clearly satisfied in this case. Condition (iii) follows from (1). Condition (ii) follows from the translation invariance of C_u.

Therefore, by 3.1 there exists a linear homogeneous extension I of I_1 to all of L^* such that

$$(2) \qquad |I(g)| \leqq N(g) \qquad g \in L^*,$$

$$(3) \qquad I(Tg) = I(g) \qquad T \in \mathscr{T}, g \in L^*.$$

Conditions (2) and (3) imply that I is a translation invariant element of ${}^0C_u$. Since $I(f) = I_1(f) = 1$, the proof will be complete when (1) is established.

Let g be as in (1) and note that $\sum_{k=1}^m a_k T_{\beta_k} f = \sum_{k=1}^m a_k C_{E-\beta_k}$. Let $h \in C_u$ and $\varepsilon > 0$. Choose an open set V in $\mathscr{G}$ such that $|h(\alpha) - h(\beta)| < \varepsilon$ for any α and β in V. Let

$$D_1 = \bigcap_{k=1}^m E - \beta_k \cap V \quad \text{and} \quad D_2 = \bigcap_{k=1}^m E' - \beta_k \cap V.$$

By the hypothesis, $\mu(D_1) > 0$ and $\mu(D_2) > 0$. Let

$$x = (2\mu(D_1))^{-1} C_{D_1} - (2\mu(D_2))^{-1} C_{D_2}.$$

Clearly $x \in L$ and $\|x\| = 1$. A simple computation shows that $2g(x) = \sum_{k=1}^m a_k$ and $|h(x)| \leqq \varepsilon$. Since ε was chosen arbitrarily, $2\|g - h\| \geqq |\sum_{k=1}^m a_k|$. Therefore, $N(g) \geqq |\sum_{k=1}^m a_k|$ since h was chosen arbitrarily in C_u.

3.3. LEMMA. *If I is a translation invariant element in ${}^0C_u \cap \mathscr{A}$, then $I = 0$.*

Proof. Since I is translation invariant, $I * f = I(f)e$ for each f in L^*. Therefore, $I(f)\pi x(e) = \pi x(I(f)e) = \pi x(I * f) = \pi x * I(f) =$

$I(xf) = 0$, since $xf \in C_u$ for each x in L and f in L^*. Since there exists an x in L such that $\pi x(e) \neq 0$, $I(f) = 0$ for each f in L^* and that completes the proof.

As a consequence of 3.2 and 3.3 we have the following theorem.

3.4. THEOREM. *If a group $\mathscr{G}$ contains a measurable set satisfying the conditions of* 3.2, *then the corresponding sets $\mathscr{A}$ and $\mathscr{B}$ are different.*

In the remainder of this section it will be assumed that $\mathscr{G}$ is a second countable group. An important result is Theorem 3.13, which shows that for second countable groups, the measurable subsets which satisfy the conditions of 3.2 are in some sense very numerous. As a consequence of this theorem and Theorem 3.4, we obtain the following result, which is the main theorem of this section.

3.5. THEOREM. *For every second countable group, the corresponding sets $\mathscr{A}$ and $\mathscr{B}$ are different.*

The following notation, in addition to the notation already introduced, will be used throughout the remainder of this section. If A and B are measurable subsets of $\mathscr{G}$, then "A is equivalent to B" will mean that $\mu(A \triangle B) = 0$. Let $\mathscr{V}$ denote the resulting equivalence classes. As usual, a measurable set E will be identified with its eqivalence class. For A and B in $\mathscr{V}$, let $\rho(A, B) = \arctan \mu(A \triangle B)$. It is shown in [4, p. 156] that ρ is a metric on $\mathscr{V}$ and that $(\mathscr{V}, \rho)$ is a complete metric space. Let $\mathscr{U}$ denote the sets of finite measure. Since $\mathscr{U}$ is a closed subset of $\mathscr{V}$, $(\mathscr{U}, \rho)$ is also a complete metric space. We will let $\mathscr{F}$ denote the sets of finite measure (equivalence classes of $\mathscr{U}$) which satisfy the conditions of 3.2. Let $\{V_j: j = 1, 2, \cdots\}$ be a basis of open sets of finite measure for the topology of $\mathscr{G}$. Let $\{D_j: j = 1, 2, \cdots\}$ be an increasing sequence of compact sets such that $\mathscr{G} = \bigcup \{D_j: j = 1, 2, \cdots\}$. The direct product of D_n, n times, with the product topology will be denoted by $(D_n)^n$. For each measurable set E, open set V and each $u = (\beta_1, \cdots, \beta_n)$ in $(D_n)^n$, let

$$H_n(E, u, V) = \mu(\bigcap \{E + \beta_j: j \leqq n\} \cap V) \quad \text{and}$$
$$K_n(E, V) = \inf \{H_n(E, u, V): u \in (D_n)^n\} .$$

Finally, for any sequence $\{A_j: j = 1, 2, \cdots\}$ of measurable sets, $\lim A_j$ will denote the pointwise limit (see [4, p. 126]) when this limit exists.

3.6. LEMMA. *Let V be an open set of finite measure, n a positive integer and E a measurable set. Then*

(i) $K_n(F, V) \to K_n(E, V)$ *as* $\mu(F \triangle E) \to 0$ *and*
(ii) $K_n(E_k, V) \to K_n(E, V)$ *as* $k \to \infty$ *if* $E = \lim E_k$.

Proof. The following inequality will be established:

$$
\begin{aligned}
&| H_n(E, u, V) - H_n(F, u, V) | \\
(4) \qquad &\leqq n(\sup \{\mu(V - \beta \cap E \backslash F) : \beta \in D_n\} \\
&\quad + \sup \{\mu(V - \beta \cap F \backslash E) : \beta \in D_n\})
\end{aligned}
$$

where F is any measurable set and $u \in (D_n)^n$. First note that for any sets A and B of finite measure,

$$| \mu(A) - \mu(B) | \leqq \mu(A \cap B') + \mu(B \cap A') .$$

So for $u = (\beta_1 \cdots, \beta_n)$ we have that

$$
\begin{aligned}
&| H_n(E, u, V) - H_n(F, u, V) | \\
(5) \qquad &\leqq \mu\Big(\Big(\bigcap_{i=1}^{n} E + \beta_i \cap V\Big) \cap \Big(\bigcup_{j=1}^{n} (F + \beta_j \cap V)'\Big)\Big) \\
&\quad + \mu\Big(\Big(\bigcap_{j=1}^{n} F + \beta_j \cap V\Big) \cap \Big(\bigcup_{i=1}^{n} (E + \beta_i \cap V)'\Big)\Big) .
\end{aligned}
$$

Considering the first term of the right side of (5), we have that

$$
\begin{aligned}
&\mu\Big(\Big(\bigcap_{i=1}^{n} E + \beta_i \cap V\Big) \cap \Big(\bigcup_{j=1}^{n} (F + \beta_j \cap V)'\Big)\Big) \\
&= \mu\Big(\bigcup_{j=1}^{n} \Big(\Big(\bigcap_{i=1}^{n} E + \beta_i \cap V\Big) \cap (F + \beta_j \cap V)'\Big)\Big) \\
&\leqq \mu\Big(\bigcup_{j=1}^{n} ((E + \beta_j \cap V) \cap (F + \beta_j \cap V)')\Big) \\
&\leqq n\mu((E + \beta_j \cap V) \backslash (F + \beta_j \cap V)) \qquad \text{for some } j \leqq n .
\end{aligned}
$$

But,

$$n\mu((E + \beta_j \cap V) \backslash (F + \beta_j \cap V)) = n\mu(V - \beta_j \cap E \backslash F) .$$

Since $\beta_j \in D_n$, it follows that the first term of the right side of (5) is no greater than

$$n(\sup \{\mu(V - \beta \cap E \backslash F) : \beta \in D_n\} .$$

The inequality (4) now follows by applying the same argument to the second term of the right side of (5).

It follows from (4) that $H_n(F, u, V) \to H_n(E, u, V)$ as $\mu(F \triangle E) \to 0$ uniformly for $u \in (D_n)^n$, so (i) is established.

Let $\{E_k : k = 1, 2, \cdots\}$ be a sequence of measurable sets such that $E = \lim E_k$. For each β in D_n,

$$\mu(V - \beta \cap E \backslash E_k) \to 0 \quad \text{as } k \to \infty$$

since V has finite measure. The family of functions $\{\mu(V - (\cdot) \cap E \backslash E_k): k = 1, 2, \cdots\}$ form an equicontinuous family on D_n since V has finite measure and

$$|\mu(V - \beta' \cap E \backslash E_k) - \mu(V - \beta \cap E \backslash E_k)| \leqq \mu(V - \beta' \triangle V - \beta)$$

for any β' and β in D_n. Therefore,

$$\mu(V - \beta \cap E \backslash E_k) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \beta \in D_n .$$

The same argument applies to $\mu(V - \beta \cap E_k \backslash E)$. It follows from (4) that $H_n(E_k, u, V) \to H_n(E, u, V)$ as $k \to \infty$ uniformly for u in $(D_n)^n$. Therefore,

$$K_n(E_k, V) \to K_n(E, V) \quad \text{as } k \to \infty ,$$

which completes the proof.

3.7. LEMMA. *If E and D are measurable sets and D has finite measure, then*
(i) $\mu(D \cap E \backslash (E + \beta)) \to 0$ *as* $\beta \to 0$ *and*
(ii) $\mu(D \cap (E + \beta) \backslash E) \to 0$ *as* $\beta \to 0$.

Proof. Consider (i) and assume, first of all, that D is compact. For any β in $\mathscr{G}$, $D \cap E \backslash (E + \beta) =$

$$D \cap E \backslash (D \cap (E + \beta)) = D \cap E \backslash ((D - \beta \cap E) + \beta) .$$

If, in addition, β is in V, a compact symmetric neighborhood of 0, then

$$\begin{aligned} &D \cap E \backslash ((D - \beta \cap E) + \beta) \\ &\quad = D \cap (E \cap (D + V)) \backslash ((D - \beta \cap (E \cap (D + V))) + \beta) \\ &\quad = D \cap ((E \cap D + V) \backslash (E \cap (D + V) + \beta) . \end{aligned}$$

Hence, for $\beta \in V$

$$(6) \qquad \mu(D \cap E \backslash (E + \beta)) \leqq \mu(E \cap (D + V) \backslash (E \cap (D + V)) + \beta) .$$

The set $D + V$ is compact so $E \cap (D + V)$ has finite measure. Therefore, the characteristic function of $E \cap (D + V)$ is in L and since translation is continuous in L (see [9, p. 3]), the right hand member of (6) tends to 0 as β tends to 0. A proof for the case of an arbitrary set D of finite measure can now be obtained by approximating D from below by a compact set. A proof of (ii) can be obtained by slightly modifying the above proof.

3.8. LEMMA. *Let V be an open set of finite measure. If E is* *a measurable set and n is a positive integer such that*

$$H_n(E, u, V) > 0 \qquad u \in (D_n)^n ,$$

then $K_n(E, V) > 0$. In particular, if E is a dense open set, then $K_n(E, V) > 0$ for $n = 1, 2, \cdots$.

Proof. If E is a dense open set, then $H_n(E, u, V) > 0$ for each u in $(D_n)^n$. So the last assertion is a consequence of the first assertion, which will now be established.

Let $u = (\beta_1, \cdots, \beta_n)$ and $v = (\gamma_1, \cdots, \gamma_n)$ be elements of $(D_n)^n$. By an argument similar to the one used in the proof of 3.6 it can be shown that

$$\begin{aligned} | H_n(E, u, V) &- H_n(E, v, V) | \\ &\leqq \sum_{j=1}^{n} \mu(V - \beta_j \cap E \backslash (E + (\gamma_j - \beta_j))) \\ &\quad + \mu(V - \beta_j \cap (E + (\gamma_j - \beta_j)) \backslash E) . \end{aligned}$$

Since $V - \beta_j$ has finite measure for each $j = 1, 2, \cdots, n$ it follows from 3.7 that the jth term in the above sum can be made small by choosing γ_j close to β_j. Therefore, the sum can be made small by choosing v close to u in $(D_n)^n$. So, $H_n(E, u, V)$ is a continuous function of u as u ranges over the compact space $(D_n)^n$. Hence, $H_n(E, u, V)$ assumes its infimum $K_n(E, V)$ on $(D_n)^n$. It follows from the hypothesis that $K_n(E, V) > 0$, which completes the proof.

3.9. LEMMA. *For a measurable set E the following conditions are equivalent:*
(i) $K_n(E, V_n) > 0 \qquad n = 1, 2, \cdots$
(ii) $\mu(\bigcap \{E + \beta_j : j = 1, 2, \cdots, m\} \cap V) > 0$
whenever V is a nonvoid open set, m is a positive integer and $\{\beta_1, \cdots, \beta_m\} \subset \mathscr{G}$.

Proof. Assume (i) and let V be a non-void open set, m a positive integer and $\{\beta_1, \cdots, \beta_m\} \subset \mathscr{G}$. There exists an integer $n \geqq m$ such that $\{\beta_1, \cdots, \beta_m\} \subset D_n$ (recall that $\{D_n : n = 1, 2, \cdots\}$ is an increasing sequence) and $V_n \subset V$. Therefore,

$$\mu(\bigcap \{E + \beta_j : j = 1, 2, \cdots, m\} \cap V) \geqq K_n(E, V_n) > 0 .$$

It is immediate from 3.8 that (ii) implies (i).

3.10. LEMMA. *Let B be a measurable set of finite measure and V be an open set of finite measure. Given a positive integer n and a positive real number ε, there exists a closed nowhere dense set E*

such that:
(i) $\rho(E, B) < \varepsilon$ *and*
(ii) $K_n(E, V) > 0$.

Proof. Choose a dense open set U of finite measure such that $B \subset U$ and $\rho(B, U) < \varepsilon/3$. This can be done by forming the union of a tight open cover of B and a tight open cover of a countable dense set. By 3.8 $K_n(U, V) > 0$. From 3.6 and the regularity of μ it follows that there exists a compact set D contained in U such that $K_n(D, V) > 0$ and $\rho(U, D) < \varepsilon/3$. Again by using 3.6 we can choose a dense open set W of small enough measure so that $\rho(D\backslash W, D) < \varepsilon/3$ and $K_n(D\backslash W, V) > 0$. Setting $E = D\backslash W$ we see that E is a closed nowhere dense set, that $K_n(E, V) > 0$ and that $\rho(E, B) < \varepsilon$, which completes the proof.

3.11. LEMMA. *Given a measurable set B of finite measure and a positive real number ε, there exists a sequence $\{E_j: j = 1, 2, \cdots\}$ of measurable sets such that for each positive integer k:*
(i) $\rho(B, \bigcup \{E_j: j \leqq k\}) < \varepsilon$,
(ii) E_k *is closed and nowhere dense,*
(iii) $K_k(E_k, V_k) > 0$,
(iv) $K_j(U_k, V_j) > a_j/2 \qquad j = 1, 2, \cdots, k$
where $U_k = \mathscr{G}\backslash\bigcup \{E_j: j \leqq k\}$ *and* $a_j = K_j(U_j, V_j)$ *for* $j = 1, 2, \cdots$.

Proof. Let $\mathscr{K}$ be the class of all finite sequences $\{E_j: j = 1, 2, \cdots, n\}$ of measurable sets satisfying conditions (i)–(iv). If $M = \{E_j: j = 1, 2, \cdots, m\}$ and $N = \{F_j: j = 1, 2, \cdots, n\}$ are elements of $\mathscr{K}$, then $M \leqq N$ means that $m \leqq n$ and $D_j = E_j$ for $j = 1, 2, \cdots, m$. With the ordering $\leqq$, $\mathscr{K}$ is a partially ordered set. It follows from 3.10 that there exists a measurable set E_1 such that $\{E_1\}$ satisfies conditions (i)–(iii). Since U_1 is a dense open set, $a_1 = K_1(U_1, V_1) > 0$ by 3.8. Therefore, $\{E_1\}$ is an element of $\mathscr{K}$.

Now let $\{E_j: j = 1, 2, \cdots, n\}$ be any element of $\mathscr{K}$. By 3.6 and the triangle inequality for the metric, there exists a $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$, then

$$\rho\left(B, \bigcup_{j=1}^{n} E_j \cup A\right) < \varepsilon \quad \text{and}$$
$$K_j(U_n\backslash A, V_j) > a_j/2 \qquad j = 1, 2, \cdots, n\ .$$

Using 3.10 with B taken as the empty set it follows that there exists a closed nowhere dense set E_{n+1} such that $\mu(E_{n+1}) < \delta$ and $K_{n+1}(E_{n+1}, V_{n+1}) > 0$. So, in particular,

$$\rho(B, \bigcup\{E_j: j = 1, 2, \cdots, n+1\}) < \varepsilon \quad \text{and}$$
$$K_j(U_{n+1}, V_j) > a_j/2 \qquad j = 1, 2, \cdots, n\ .$$

Since $U_{n+1} = U_n \backslash E_{n+1}$ is a dense open set, $a_{n+1} = K_{n+1}(U_{n+1}, V_{n+1}) > 0$ by 3.8, so

$$K_j(U_{n+1}, V_j) > a_j/2 \qquad j = n + 1\ .$$

Therefore, $\{E_j: j = 1, 2, \cdots, n+1\}$ is also an element of $\mathscr{K}$. This argument shows that any maximal chain in $\mathscr{K}$ is infinite. That $\mathscr{K}$ has a maximal chain follows from the axiom of choice. The desired sequence is obtained from a maximal chain in the obvious way, which completes the proof.

3.12. LEMMA. *The set* $\mathscr{F}$ *is a dense subset of* $\mathscr{U}$.

Proof. Let B be a measurable set of finite measure and let $0 < \varepsilon < \pi/2$. Let $\{E_j: j = 1, 2, \cdots\}$ be chosen as in 3.11 corresponding to B and ε. Let $E = \bigcup\{E_j: j = 1, 2, \cdots\}$. The set E is the limit of the increasing sequence $\{F_k: k = 1, 2, \cdots\}$ of measurable sets where $F_k = \bigcup\{E_j: j = 1, 2, \cdots, k\}$. By 3.1, $\rho(B, F_k) < \varepsilon$ for each $k = 1, 2, \cdots$, where $\rho(B, F_k) = \arctan(\mu(B\backslash F_k) + \mu(F_k\backslash B))$. If the measure of E were infinite, then $\mu(F_k\backslash B) \to \infty$ as $k \to \infty$. But, $\arctan \mu(F_k\backslash B) \leqq \rho(B, F_k) < \varepsilon < \pi/2$ for $k = 1, 2, \cdots$, which implies that $\{\mu(F_k\backslash B): k = 1, 2, \cdots\}$ is a bounded sequence. Hence, $\mu(E) < \infty$ and $\mu(F_k\backslash B) \to \mu(E\backslash B)$ as $k \to \infty$. Also, since B has finite measure, $\mu(B\backslash F_k) \to \mu(B\backslash E)$ as $k \to \infty$. Therefore, $\rho(B, F_k) \to \rho(B, E)$ as $k \to \infty$, so $\rho(B, E) \leqq \varepsilon$. To complete the proof it is sufficient to show that $E \in \mathscr{F}$ and since E has finite measure, showing that E belongs to $\mathscr{F}$ reduces to showing that E satisfies the conditions of 3.2.

For each $n = 1, \cdots$, it is clear that $K_n(E_n, V_n) \leqq K_n(E, V_n)$, so it follows from (iii) of 3.1 that

$$K_n(E, V_n) > 0 \qquad n = 1, 2, \cdots\ . \tag{6}$$

The sequence $\{U_n: n = 1, 2, \cdots\}$ is decreasing and $E' = \bigcap\{U_n: n = 1, 2, \cdots\}$. Therefore, by (ii) of 3.6

$$\lim_{n\to\infty} K_j(U_n, V_j) = K_j(E', V_j) \qquad j = 1, 2, \cdots\ .$$

It follows from (iv) of 3.11 that

$$K_j(E', V_j) \geqq a_j/2 > 0 \qquad j = 1, 2, \cdots\ . \tag{7}$$

From (6), (7) and 3.9 we conclude that E satisfies the conditions of 3.2, which completes the proof.

It was pointed out that $\mathscr{U}$ is a complete metric space, so $\mathscr{U}$ is a space of second category. The set $\mathscr{F}$ is compared with $\mathscr{U}$ in the following theorem.

3.13. THEOREM. *For second countable groups, the complement of $\mathscr{F}$ in $\mathscr{U}$ is a set of first catagory in $\mathscr{U}$.*

Proof. A set E of finite measure is in the complement of $\mathscr{F}$ if and only if condition (ii) of 3.9 fails for at least one of the sets E or E'. Therefore, it follows from 3.9 that the complement of $\mathscr{F}$ in $\mathscr{U}$ is the set

$$\bigcup_{n=1}^{\infty}\{E: \mu(E) < \infty, K_n(E, V_n) = 0\} \cup \bigcup_{n=1}^{\infty}\{E: \mu(E) < \infty, K_n(E', V_n) = 0\} .$$

The sets $\{E: \mu(E) < \infty, K_n(E, V_n) = 0\}$ and $\{E: \mu(E) < \infty, K_n(E', V_n) = 0\}$ are the zero sets in $\mathscr{U}$ of the maps $E \to K_n(E, V_n)$ and $E \to K_n(E', V_n)$, respectively. The first map is continuous on $\mathscr{V}$ by 3.6. The second map is a composition of the first map and the map $E \to E'$, which is clearly continuous. Therefore, the sets $\{E; \mu(E) < \infty, K_n(E, V_n) = 0\}$ and $\{E: \mu(E) < \infty, K_n(E', V_n) = 0\}$ are closed, and since the complement of each of these sets contains the set $\mathscr{F}$, which is dense in $\mathscr{U}$, the complement of $\mathscr{F}$ in $\mathscr{U}$ is a countable union of nowhere dense sets.

4. **A group for which $\mathscr{B}$ differs from L^{**}.** In this section (see 4.3) it is established that in the case of the real numbers, $\mathscr{B}$ and L^{**} are different.

The following notation will be used in this section in addition to notation already introduced. The sets $I_{k,n}$, $\mathscr{P}$, $E_{K,n}$ and U are defined as follows:

$$\begin{aligned} I_{k,n} &= \{\alpha: k2^{-n} < \alpha < (k+1)2^{-n}\} \qquad 0 \leqq k < 2^{n-1}; n = 1, 2, \cdots \\ \mathscr{P} &= \{(K, n): K \subset \{k: k \text{ an integer}, 0 \leqq k < 2^{n-1}\}, n = 1, 2, \cdots\} \\ E_{K,n} &= \bigcup \{(k2^{-n}, k2^{-n} + 2^{-n-1}); k \in K\} , \qquad (K, n) \in \mathscr{P} \\ U &= (0, 2^{-1})\backslash\{j2^{-n}: j \text{ an integer}, n = 1, 2, \cdots\} . \end{aligned}$$

For each pair (K, n) in $\mathscr{P}$ let $a(K, n)$ be an integer such that the map $(K, n) \to a(K, n)$ is one-to-one ($\mathscr{P}$ is countable). Let $E = \bigcup \{E_{K,n} + a(K, n): (K, n) \in \mathscr{P}\}$.

4.1. LEMMA. *Let H be a finite subset of U. Let n be a positive integer such that $I_{k,n} \cap H$ has at most one element for each k, $0 \leqq k < 2^{n-1}$. Let H_1 (H_2) be the elements of H which are contained in sets of the form $I_{2k,n+1}(I_{2k+1,n+1})$. If $M_i \subset H_i (i = 1, 2,)$ and $M_i \neq \phi$, then*

$$\mu(\bigcap \{E - \beta: \beta \in M_i\} \backslash \bigcup \{E - \gamma : \gamma \in H \backslash M_i\}) > 0 .$$

Proof. Consider the case of M_1. Let $K = \{k : I_{k,n} \cap M_1 \neq \phi\}$. If $\beta \in M_1$, then $\beta \in E_{K,n}$. Since $E_{K,n}$ is an open set, there exists a positive real number u depending on β such that $(-u, 0) + \beta \subset E_{K,n}$. If $\gamma \in H \backslash M_1$, then γ is not in the closure of $E_{K,n}$. So there exists a positive real number v depending on γ such that $(-v, 0) + \gamma \cap E_{K,n} = \phi$. Since H is a finite set, it follows that there exists a positive real number $t < 2^{-1}$ such that

$$(-t, 0) + \beta \subset E_{K,n} \qquad \beta \in M_1$$
$$(-t, 0) + \gamma \cap E_{K,n} = \phi \qquad \gamma \in H \backslash M_1 .$$

Therefore, it follows that

$$(-t, 0) + a(K, n) \subset E_{K,n} + a(K, n) - \beta \qquad \beta \in M_1 \tag{1}$$

$$(-t, 0) + a(K, n) \cap E_{K,n} + a(K, n) - \gamma = \phi \qquad \gamma \in H \backslash M_1 . \tag{2}$$

For any non-zero integer s,

$$(-2^{-1}, 2^{-1}) \cap (0, 2^{-1}) + s = \phi .$$

Since $(-t, 0) + \gamma \subset (-2^{-1}, 2^{-1})$ and $E_{L,m} \subset (0, 2^{-1})$ for each (L, m) in $\mathscr{P}$, it follows that

$$(-t, 0) + a(K, n) \cap E_{L,m} + a(L, m) - \gamma = \phi \quad \text{for } (L, m) \neq (K, n) \text{ and } \gamma \in H \backslash M_1 . \tag{3}$$

It follows from (1), (2) and (3) that

$$(-t, 0) + a(K, n) \subset \bigcap \{E - \beta: \beta \in M_1\} \backslash \bigcup \{E - \gamma: \gamma \in H \backslash M_1\} ,$$

which completes the proof in the case of M_1.

In the case of M_2, let $K = \{k: I_{k,n} \cap M_2 \neq \phi\}$. By noting that for $\beta \in M_2$ and $\gamma \in H \backslash M_2$, $\beta - 2^{-n-1}$ is in $E_{K,n}$ and $\gamma - 2^{-n-1}$ is in the complement of the closure of $E_{K,n}$, one can give a proof similar to the above proof in the case of M_2. This proof is omitted.

4.2. *Let N be a subset of U, H a finite subset of U and a_β a complex number for each β in H. Then*

$$| \Sigma_{\beta \in N \cap H} a_\beta | \leqq 2 \| \Sigma_{\beta \in H} a_\beta C_{E-\beta} \| .$$

In particular, $\{C_{E-\beta} : \beta \in U\}$ is a linearly independent set in L_∞.

Proof. Let n be a positive integer such that $H \cap I_{k,n}$ has at most one element for $0 \leqq k < 2^{n-1}$. Let H_1 and H_2 be as in 4.1 and let $M_i = H_i \cap N (i = 1, 2,)$. If $M_i \neq \phi$, then it follows from 4.1 that

$$(4) \qquad \mu(\bigcap \{E-\beta\colon \beta \in M_i\}\backslash\bigcup \{E-\gamma\colon \gamma \in H\backslash M_i) > 0 \;.$$

However, in any case,

$$\begin{aligned} |\Sigma_{\beta\in N\cap H}a_\beta| &\leqq |\Sigma_{\beta\in M_1}a_\beta| + |\Sigma_{\beta\in M_2}a_\beta| \\ &\leqq 2\,||\Sigma_{\beta\in H}a_\beta C_{E-\beta}|| \;, \end{aligned}$$

since if $M_i \neq \phi$, then (4) guarantees that the value $\Sigma_{\beta\in M_i}a_\beta$ is assumed on a set of positive measure.

To prove the last assertion assume that $\Sigma_{\beta\in H}a_\beta C_{E-\beta} = 0$. By letting $N = \{\beta\}$ with β an arbitrary element of H, we conclude from the inequality that $a_\beta = 0$ for each β in H, which completes the proof.

Let X denote the linear span in L_∞ of the set $\{C_{E-\beta}\colon \beta \in U\}$.

4.3. THEOREM. *For the group of real numbers there exists an element F in L^{**} and an element f in L^* such that $F*f$ is a non-measurable function.*

Proof. Let N be a non-measurable subset of U. For an arbitrary element g in X, $g = \Sigma_{\beta\in H}a_\beta C_{E-\beta}$, for some finite subset H of U and for some choice of complex numbers a_β corresponding to each β in H. For each g in X, let

$$F_1(g) = \Sigma_{\beta\in N\cap H}a_\beta \;.$$

It follows from 4.2 that F_1 is a well defined function on X and, moreover, that F_1 is a bounded linear functional on X. Therefore, there exists an F in L^{**} such that the restriction of F to X agrees with F_1. Let $f = C_E$ and note that $C_U(\beta)F*f(\beta) = C_U(\beta)F(C_{E-\beta}) = C_U(\beta)F_1(C_{E-\beta}) = C_N(\beta)$. Since C_U is a measurable function and C_N is not measurable, it follows that $F*f$ is a non-measurable function, which completes the proof.

5. Identification of some algebras.

In this section it will be assumed that $\mathscr{G}$ has the following property.

Property A. *There exists an element E in L^{**} such that $||E|| = 1$ and $E*f = f$ for each f in L^*.*

5.1. LEMMA. *If, for a group $\mathscr{G}$, there exists a sequence $\{x_n\colon n = 1, 2, \cdots\} \subset L$ such that*

(i) $||x_n|| \leqq 1 \qquad n = 1, \cdots$

(ii) $x_n f(\beta) \to f(\beta)$ l.a.e. *as $n \to \infty$ for each $f \in L^*$, then $\mathscr{G}$ has property A.*

Proof. Let E be a w^* limit point of the set $\{x_n: n = 1, 2, \cdots\}$ in the unit ball of L^{**}. Let V be an abitrary non-void open set of finite measure. There exists a set D of measure 0 (assuming a particular realization of f) such that $x_n f(\beta) \rightarrow f(\beta)$ for each $\beta \in V \backslash D$. Hence, given $\beta \in V \backslash D$ and $\varepsilon > 0$, there exists a positive integer n_1 such that

$$|x_n f(\beta) - f(\beta)| < \varepsilon \qquad n \geqq n_1 .$$

However, for some $n \geqq n_1$

$$|E(T_\beta f) - \pi x_n(T_\beta f)| < \varepsilon .$$

Therefore, since $x_n(T_\beta f) = x_n f(\beta)$, $|E(T_\beta f) - f(\beta)| < 2\varepsilon$. It follows that $E(T_\beta f) = f(\beta)$ for $\beta \in V \backslash D$. Since V was chosen arbitrarily we have shown that $E * f$ and f have realizations which agree l.a.e. Therefore, $E * f = f$, which completes the proof.

Let S_n be a cube in euclidean k-space which contains 0 and which has edges of length $1/n$. Let $x_n = (\mu(S_n))^{-1} C_{S_n}$. Then $\|x_n\| = 1$ and it follows from the classical differentiation theory (see [4, III. 12.6, p. 214]) that $\{x_n: n = 1, 2, \cdots\}$ satisfies (ii) of 5.1. Therefore, euclidean k-space has property A for each positive integer k. The question of whether or not all LCA groups have property A is unanswered.

Recall that whenever an element f in C_u is identified with a function, that function is assumed to be the unique uniformly continuous realization of f.

5.2. Lemma. *If E is an element of L^{**} such that $E * f = f$ for each f in L^*, then*

(i) $E(f) = f(0) \qquad f \in C_u$

(ii) $E \in \mathscr{A}$

(iii) $G \circ E = G \qquad G \in L^{**}$.

Proof. (i) If $f \in C_u$, then $E * f \in C_u$ by 2.5. Therefore, $E * f(\beta) = f(\beta)$ for each β in $\mathscr{G}$. In particular, $f(0) = E * f(0) = E(f)$. (ii) If $f \in L^*$ and $x \in L$, then $\pi x * E(f) = \pi x(E * f) = \pi x(f)$. Also, $E * \pi x(f) = E(xf) = xf(0) = \pi x(f)$ since $xf \in C_u$. Therefore, by the definition of $\mathscr{A}$, $E \in \mathscr{A}$. (iii) Since $E \in \mathscr{A}$, by 2.11, $E \circ f = E * f = f$ for each f in L^*. Therefore, $G \circ E(f) = G(E \circ f) = G(f)$ for each G in L^{**} and f in L^*, which completes the proof.

The following lemma and its proof are due to R. J. Lindahl.

5.3. Lemma. *If k is an element of C_u^*, then there exists an element F in $\mathscr{A}$ such that $\|F\| = \|k\|$ and F agrees with k on C_u.*

Proof. Let k be an element of C_u^*. By the Hahn-Banach theorem there exists an element F_1 in L^{**} such that $\| F_1 \| = \| k \|$ and F_1 agrees with k on C_u. Choose E as in the definition of property A and let $F = E \circ F_1$. It follows from the preceding lemma that $E \in \mathscr{A}$, so by 2.10, F is also in $\mathscr{A}$. If $G \in L^{**}$, then $G \circ (E \circ F_1 - F_1) = 0$ since E is a right identity. Therefore, by 2.3, $E \circ F_1 - F_1 \in {}^0C_u$. Hence, F agrees with k on C_u. Since F and k agree on C_u, $\| F \| \geqq \| k \|$; however, $\| F \| \leqq \| F_1 \| \, \| E \| = \| k \|$. Therefore, $\| F \| = \| k \|$, which completes the proof.

For the proofs of 5.4 and 5.8 recall that A_F and B_F are defined in the paragraphs preceding 2.4 and 2.8, respectively. As a result of 2.8 it is clear that $(\mathscr{B}/\mathscr{Q}, *)$ with the quotient norm is a Banach algebra.

5.4. THEOREM. *The algebra* $(\mathscr{B}/\mathscr{Q}, *)$ *is topologically isomorphic to* $C(\mathscr{T}, 0(L^*))$.

Proof. For each F in $\mathscr{B}$ let $\nu(F + \mathscr{Q}) = B_F$. By 2.8, ν is an isomorphism of $\mathscr{B}/\mathscr{Q}$ into $C(\mathscr{T}, 0(L^*))$. Let A be an arbitrary element in $C(\mathscr{T}, 0(L^*))$. Let E be as in the definition of property A and let F be the element of L^{**} whose value at f in $E(Af)$ for each f in L^*. Then $F*f(\beta) = F(T_\beta f) = E(AT_\beta f) = E(T_\beta Af) = E*Af(\beta)$ and $E*Af = Af$. Therefore, $F \in \mathscr{B}$ and $\nu(F + \mathscr{Q})f = F*f = Af$, so $\nu(F + \mathscr{Q}) = A$. Hence, ν maps $\mathscr{B}/\mathscr{Q}$ onto $C(\mathscr{T}, 0(L^*))$.

To see that ν is continuous let F be an element in $\mathscr{B}$ and ε a positive real number. Choose G in $\mathscr{Q}$ such that $\| F + \mathscr{Q} \| + \varepsilon \geqq \| F + G \|$. Now

$$\begin{aligned} \| F + G \| &= \sup \{ | (F + G)(T_\beta f) | : \| f \| \leqq 1, f \in L^*; \beta \in \mathscr{G} \} \\ &\geqq \sup \{ \operatorname{ess\,sup} \{ | (F + G)(T_\beta f) | : \beta \in \mathscr{G} \} : \| f \| \leqq 1, f \in L^* \} \\ &= \sup \{ \| F*f \| : \| f \| \leqq 1, f \in L^* \} = \| B_F \| = \| \nu(F + \mathscr{Q}) \| . \end{aligned}$$

We conclude that $\| F + \mathscr{Q} \| \geqq \| \nu(F + \mathscr{Q}) \|$ for each F in L^{**}, which implies that ν is continuous.

Since $C(\mathscr{T}, 0(L^*))$ is a Banach space, ν^{-1} is continuous by the interior mapping theorem.

5.5. COROLLARY. *Each element of* $C(\mathscr{T}, 0(L^*))$ *leaves* C_u *invariant.*

Proof. If $A \in C(\mathscr{T}, 0(L^*))$, then for some F in $\mathscr{B}$, $Af = F*f$ for each f in L^*. By 2.5, $F*f \in C_u$ whenever $f \in C_u$, which completes the proof.

In the remainder of this section $\mathscr{T}'$ will denote the translation operators in $0(C_u)$.

As a result of 2.9 it is clear that $(\mathscr{A}/\mathscr{Q}, *)$ with the quotient norm is a Banach algebra.

5.6. THEOREM. *The algebra $(\mathscr{A}/\mathscr{Q}, *)$ is topologically isomorphic to $C(\mathscr{T}', 0(C_u))$.*

Proof. For each F in $\mathscr{A}$ let $\nu_1(F + \mathscr{Q}) = B_F$. By 2.9, ν_1 is an isomorphism of $\mathscr{A}/\mathscr{Q}$ into $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$ and since ν_1 is the restriction to $\mathscr{A}/\mathscr{Q}$ of the map defined in the proof of 5.4, ν_1 is a bicontinuous map.

Let B be an arbitrary element of $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$. By 5.5, B leaves C_u invariant. Therefore, if $\nu_2(B)$ denotes the element of $0(C_u)$ obtained by restricting B to C_u, then ν_2 is clearly a homomorphism of $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$ into $C(\mathscr{T}', 0(C_u))$.

To complete the proof it suffices to show that ν_2 is an isometry and that $\nu = \nu_2\nu_1$ is an onto map.

To see that ν_2 is an isometry, first note that if $B \in 0(L^*)$, then

$$\|B\| = \sup\{|BT_\beta f(x)| : f \in L^*, \|f\| \leqq 1; x \in L, \|x\| \leqq 1; \beta \in \mathscr{G}\} .$$

For $B \in C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$, $BT_\beta f(x) = A_x BT_\beta f(0) = T_\beta BA_x f(0) = BA_x f(\beta)$ and $\sup\{|BA_x f(\beta)| : \beta \in \mathscr{G}\} = \|BA_x f\|$ since $BA_x f \in C_u$. Therefore,

$$\begin{aligned}\|B\| &= \sup\{\|BA_x f\| : f \in L^*, \|f\| \leqq 1; x \in L, \|x\| \leqq 1\} \\ &= \sup\{\|Bg\| : g \in C_u, \|g\| \leqq 1\} = \|\nu_2(B)\| .\end{aligned}$$

The equality preceding the last equality is a consequence of 2.2.

To see that ν is an onto map, let $A \in C(\mathscr{T}', 0(C_u))$ and let k denote that element of C_u^* whose value at f is $Af(0)$ for each f in C_u. By 5.3, there exists an element F in $\mathscr{A}$ such that F agrees with k on C_u. Since $F * f(\beta) = k(T_\beta f) = AT_\beta f(0) = T_\beta Af(0) = Af(\beta)$ for each β in $\mathscr{G}$ and f in C_u, $\nu(F + \mathscr{Q}) = A$. Since A was chosen arbitrarily in $C(\mathscr{T}', 0(C_u))$, we conclude that ν is an onto map, which completes the proof.

The following corollary is an immediate consequence of the preceding proof.

5.7 COROLLARY. *The restriction map is an isometric isomorphism of $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$ onto $C(\mathscr{T}', 0(C_u))$.*

It follows from 2.4 that $(L^{**}/C_u, \circ)$ with the quotient norm is a Banach algebra.

5.8. THEOREM. *The algebra $(L^{**}/C_u, \circ)$ is topologically isomorphic to $C(\mathscr{T}', 0(C_u))$.*

Proof. For each F in L^{**}, let $\nu_1(F + {}^0C_u) = A_F$. It follows from 2.4 that ν_1 is an isomorphism of $L^{**}/{}^0C_u$ into $C(\mathscr{T} \cup \mathscr{L}, 0(L^*))$. Let ν_2 be as in the proof of 5.6. Then $\nu = \nu_2\nu_1$ is an isomorphism of $L^{**}/{}^0C_u$ into $C(\mathscr{T}', 0(C_u))$.

Because of the interior mapping theorem, to complete the proof it suffices to show that ν is a continuous onto map.

Let $A \in C(\mathscr{T}', 0(C_u))$. By 5.6, there exists an F in $\mathscr{A}$ such that $F * f = Af$ for each f in C_u. Since $F \in \mathscr{A}$, by 2.11, $F * f = F \circ f$ for each f in L^*. Therefore, $\nu(F + {}^0C_u)f = F \circ f = F * f = Af$ for each f in C_u, which shows that ν is an onto map.

To prove that ν is continuous, let $F \in L^{**}$ and $\varepsilon > 0$. Choose $G \in {}^0C_u$ such that $\| F + {}^0C_u \| + \varepsilon \geqq \| F + G \|$. Now

$$\begin{aligned} \| F + G \| &= \text{such}\,\{| (F + G)(f) | : f \in L^*, \| f \| \leqq 1\} \\ &\geqq \sup\,\{| F(xf) | : x \in L, \| x \| \leqq 1; f \in C_u, \| f \| \leqq 1\} \\ &= \sup\,\{\| F \circ f \| : f \in C_u, \| f \| \leqq 1\} = \| \nu(F + {}^0C_u) \| . \end{aligned}$$

Therefore, $\| F + {}^0C_u \| \geqq \| \nu(F + {}^0C_u) \|$ for each F in L^{**}, which implies that ν is continuous. The proof is now complete.

The following result is a consequence of 5.6 and 5.8.

5.9. THEOREM. *The algebras* $(\mathscr{A}/\mathscr{Q}, *)$ *and* $(L^{**}/{}^0C_u, \circ)$ *are topologically isomorphic.*

In the remainder of this paper M will denote the measure algebra of the group $\mathscr{G}$. The convolution (product) of two elements μ_1 and μ_2 in M will be denoted by $\mu_1\mu_2$. It will be assumed that L is embedded in M in the natural way. For each μ in M, the operator A_μ on L^* is defined by

$$A_\mu f(x) = f(\mu x) \qquad f \in L^*, x \in L .$$

For f *in* L^*, $A_\mu f$ and μf will be used interchangeably. As usual, $\hat{\mu}$ will denote the Fourier-Stieljes transform of μ. Finally, for each A in $0(L^*)$ which leaves C_u invariant, A' will denote the element of $0(C_u)$ obtained by restricting A to C_u.

The previous theorems and the following lemmas will be used to obtain a characterization of M as an operator algebra on C_u.

5.10. LEMMA. *If X is a normed linear space and $H \in 0(X^*)$, then H is the adjoint of an element in $0(X)$ if and only if H is continuous in the X topology on X^*.*

Proof. If H is the adjoint of an element K in $0(X)$, then since $Hf(x) = f(Kx)$ for each f in X^* and x in X, H must be continuous in the X topology. Conversely, if H is continuous in the X topology, then for each x in X the function $f \to Hf(x)$ for each f in X^* is continuous in the X topology on X^*. Therefore, there exists [4, V. 3.9.] an element Kx in X such that $Hf(x) = f(Kx)$ for all f in X^*. Clearly K is a linear transformation on X. Furthermore,

$$\|K\| = \sup\{|f(Kx)| : f \in X^*, \|f\| \leqq 1; x \in X, \|x\| \leqq 1\} = \|H\| .$$

Therefore, $K \in 0(X)$ and $H = K^*$, the adjoint of K.

5.11. LEMMA. *An element A in $0(L^*)$ is an L continuous element of $C(\mathscr{T}, 0(L^*))$ if and only if $A = A_\mu$ for some μ in M.*

Proof. By 5.10, an element A in $0(L^*)$ is continuous in the L topology if and only if A is the adjoint of an element K in $0(L)$. The adjoint A of an element K in $0(L)$ is an element of $C(\mathscr{T}, 0(L^*))$ if and only if K commutes with the translation operators on L. Therefore, A is an L continuous element of $C(\mathscr{T}, 0(L^*))$ if and only if $A = A_\mu$ for some μ in M (see [10, 3.8.4]), which completes the proof.

Note that by 2.5, each F in $\mathscr{A}$, B_F leaves C_u invariant so B_F' is well-defined.

5.12. LEMMA. *If F is an element of $\mathscr{A}$, then B_F is continuous in the L topology on L^*, if and only if B_F' is continuous in the L topology on C_u.*

Proof. Suppose that $G \in \mathscr{A}$ and that B_F' is continuous in the L topology on C_u. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in L^* which converges in the L topology to an element f in L^*. Note that for each f in L^* and x in L, $xf = A_x f$ and A_x is the adjoint of an element of $0(L)$. Therefore, by 5.10, $\{xf_\lambda : \lambda \in \Lambda\}$ is a net in C_u which converges in the L topology to xf. Hence, $B_F'(xf_\lambda)$ converges in the L topology to $B_F'(xf)$, so $B_F'(xf_\lambda)(y)$ converges to $B_F'(xf)(y)$ for each y in L. However, for each g in L^*, $B_F'(xg)(y) = F * (xg)(y) = F \circ (xg)(y) = F(xyg) = F \circ g(xy) = F * g(xy) = B_F g(xy)$. Therefore, $B_F f_\lambda(xy)$ converges to $B_F(xy)$ for each x and y in L. Since each element of L is the product of two elements of L (see [3]), $B_F f_\lambda$ converges to $B_F f$ in the L topology on L^* and we conclude that B_F is continuous in the L topology on L^*.

5.13. THEOREM. *The mapping $\mu \to A_\mu'$ for each μ in M is an isometric isomorphism of M onto the L continuous elements of $C(\mathscr{T}', 0(C_u))$.*

Proof. Let $\mu \in M$. It is easily verified that for each f in C_u, $A_\mu f$ can be realized as the function whose value at β is $\int f(\beta + \alpha) d\mu(\alpha)$

for each β in $\mathscr{G}$. Therefore, it is clear that A_μ leaves C_u invariant and that $A'_\mu \in C(\mathscr{T}', 0(C_u))$. Since A_μ is an adjoint operator, from 5.11 we conclude that A_μ is L continuous. Therefore, A'_μ is continuous in the L topology on C_u. The mapping $\mu \rightarrow A'_\mu$ is clearly an isometry, so to complete the proof we must show that this mapping is onto.

Let A be an L continuous element of $C(\mathscr{T}', 0(C_u))$. By 5.6, there exists an F in $\mathscr{A}$ such that $A = B'_F$. By 5.12, B_F is L continuous and by 2.5, $B_F \in C(\mathscr{T}, 0(L^*))$. Therefore, from 5.11 we conclude that $B_F = A_\mu$ for some μ in M. Hence, $A = A'_\mu$, which completes the proof.

6. Groups for which πL differs from $\mathscr{A}$. The content of this section is Theorem 6.1, which is a summary of Theorems 6.2 and 6.5.

6.1. THEOREM. *For any noncompact group or for any group with property A, πL is a proper subset of $\mathscr{A}$.*

It was first pointed out by R. J. Lindahl that the following theorem is a consequence of 5.3. A proof can be gotten from the proof of 5.6.

6.2. THEOREM. *In any group $\mathscr{G}$ with property A, πL is a proper subset of $\mathscr{A}$.*

Proof. The map ν in the proof of 4.6 maps $\{\pi x + \mathscr{Q} : x \in L\}$ onto $\{A'_x : x \in L\}$ and $\{A'_x : x \in L\}$ is properly contained in $\{A'_\mu : \mu \in M\}$. However, $\{A'_\mu : \mu \in M\} \subset C(\mathscr{T}', 0(C_u))$, and since ν maps $\mathscr{A}/\mathscr{Q}$ onto $C(\mathscr{T}', 0(C_u))$, we conclude that for every group $\mathscr{G}$ with property A, πL is a proper subset of $\mathscr{A}$, which completes the proof.

The remainder of this section is devoted to establishing the existance of a special translation invariant element in L^{**} of a noncompact group. Notation introduced in the paragraph following 5.9 will be used in the following theorems.

6.3. THEOREM. *For a noncompact group $\mathscr{G}$, there exists an element I in L^{**} such that*

(i) $I(f) \geqq 0 \qquad f \in L^*, f \geqq 0$
(ii) $I(\mu f) = \hat{\mu}(0) I(f) \qquad \mu \in M, f \in L^*$
(iii) $\| I \| = 1$.

Proof. Let $X = \{f \in L^* : f$ has a real valued realization$\}$ and $X^+ = \{f \in X : f$ has a nonnegative realization$\}$. Let $\mathscr{W} = \{A \in 0(L^*) : A = A_\mu, \mu \geqq 0, \|\mu\| = 1\}$. Let $\mathscr{S} = \{D \in 0(X) : D = A \mid X, A \in \mathscr{W}\}$. The family $\mathscr{S}$ is a commutative semi-group of operators on X which leaves

X^+ invariant and e (the element of L^* having the identically 1 function as a realization) is an interior point of X^+ such that $D(e) = e$ for each D in $\mathscr{S}$. It follows from [8, 3.1, p. 33] that there exists an element ψ in X^* such that

$$\psi(f) \geqq 0 \qquad f \in X^+ \tag{1}$$

$$\psi(Af) = \psi(f) \qquad A \in \mathscr{S}, f \in X . \tag{2}$$

$$\| \psi \| = 1 . \tag{3}$$

For each f in L^*, let $I(f) = \psi(f_1) + i\psi(f_2)$ where f_1 and f_2 are the real and imaginary parts of f, respestively. Then I satisfies the following conditions:

$$I(f) \geqq 0 \qquad f \in X^+ \tag{4}$$

$$I(Af) = I(f) \qquad A \in \mathscr{W}, f \in L^* . \tag{5}$$

$$\| I \| = 1 . \tag{6}$$

Condition (5) follows from the fact that the real and imaginary parts of Af are Af_1 and Af_2, respectively. To establish (6) let $f \in L^*$ such that $\|f\| \leqq 1$. Then $|I(f)| = e^{-i\theta}I(f) = I(e^{-i\theta}f)$ for some θ and $I(e^{-i\theta}f) = \psi(Re^{-i\theta}f) \leqq \|\psi\| \|f\|$. Therefore, $\|I\| \leqq \|\psi\| = 1$, so $\|I\| = 1$. To complete the proof of 6.3 we need only verify that I satisfies (ii).

Note that (5) is equivalent to

$$I(\mu f) = \hat{\mu}(0)I(f) \qquad \hat{\mu}(0) = 1, \|\mu\| = 1, f \in L^* . \tag{7}$$

Now let μ be any real valued measure in M. Then there exist non-negative elements μ^+ and μ^- in M such that $\mu = \mu^+ - \mu^-$. Let $a = \|\mu^+\|$ and $b = \|\mu^-\|$. Without loss of generality, we may assume that $a \neq 0 \neq b$. Then for each f in L^*, $I(\mu f) = I(\mu^+ f - \mu^- f) = I(\mu^+ f) - I(\mu^- f) = aI(a^{-1}\mu^+ f) - bI(b^{-1}\mu^- f) = (a - b)I(f)$, since $a^{-1}\mu^+$ and $b^{-1}\mu^-$ are measures which correspond to operators in $\mathscr{W}$. However, $a - b = \hat{\mu}(0)$. Therefore, $I(\mu f) = \hat{\mu}(0)I(f)$ for each real valued measure μ in M and each f in L^*. Finally, if μ is an arbitrary element of M, then there exist real valued elements μ_1 and μ_2 in M such that $\mu = \mu_1 + i\mu_2$ and (ii) now follows from the linearity of I and the above remarks, which completes the proof.

6.4. THEOREM. *If I is an element of L^{**} and*
(i) $I(\mu f) = \hat{\mu}(0)I(f) \qquad \mu \in M, f \in L^*$,
then I is a translation invariant element in $\mathscr{A}$.

Proof. If μ is a unit point mass at β, then $\hat{\mu}(0) = 1$ and $\mu f = T_\beta f$ for each f in L^*. Therefore, (i) implies that I is translation invariant.

Since I is translation invariant, $I \in \mathscr{B}$. To show that $I \in \mathscr{A}$, let $f \in L^*$ and $x \in L$. Then $I * \pi x(f) = I(xf) = \hat{x}(0)I(f)$ and $\pi x * I(f) = \pi x(I * f) = \pi x(I(f)e) = \pi x(e)I(f) = \hat{x}(0)I(f)$. Therefore, by the definition of $\mathscr{A}$, $I \in \mathscr{A}$.

6.5. THEOREM. *For any noncompact group, πL is a proper subset of $\mathscr{A}$.*

Proof. In 6.3 and 6.4 it is established that in the case of a noncompact LCA group, $\mathscr{A}$ contains a nonzero translation invariant element; however, πL contains no such element.

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