NETS WITH CRITICAL DEFICIENCY

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This paper is a sequel to a paper of Bruck [2]. With only minor variations, we shall use Bruck's notation and terminology. All references to Bruck should be understood as references to [2].

For a net N of order n and deficiency d, the deficiency is said to be critical if $n = (d-1)^2$. If d is less than the critical deficiency, Bruck shows that

(1) N can be extended to an affine plane of order n in at most one way.

(2) The number of distinct transversals is less than or equal to dn.

(3) N can be embedded in an affine plane of order n if equality holds in (2).

In this paper, we show that if the deficiency is critical, then

(1′) N can be extended to an affine plane of order n in at most two ways. If two planes are obtained, they are related to each other by a construction due to the author.

(2′) The number of distinct transversals is less than or equal to $2dn$.

(3′) N can be extended to a plane in two different ways if equality holds in (2′).

We also show that N can be extended to a plane in at most one way if the critical value is exceeded only slightly.

We are concerned with the possibility that N may be extended to an affine plane in more than one way. Suppose that N can be extended to a plane π by adjoining the lines of a complementary net N_1. Then, if T is a transversal which is not a line of N_1, we shall say that T is an *extra transversal* (with respect to N_1).

THEOREM 1. *If T is an extra transversal with respect to N_1 and $n > (d-1)^2 - \frac{1}{2}(d-1)$ then*

(1) $n = (d-1)^2$

(2) *T is a subplane of N_1 of order $d-1$.*

Proof. Let p be a point belonging to T. We assert that every line of N_1 which goes through p must contain more than $\frac{1}{2}(d-1)$ points of T: The lines of N_2 are transversals (of N); by Bruck's Lemma 3.2, no line of N_1 can contain more than $d-1$ points of T. In the extreme case, suppose that $d-1$ lines of N_1 through p were to each

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contain $d-1$ points of T. This would account for $(d-1)(d-2)$ points of T distinct from p. The remaining line of N_1 which goes through p would then have to account for more than $(d-1)^2 - \frac{1}{2}(d-1) - (d-1)(d-2) = \frac{1}{2}(d-1)$ points of T. This establishes our assertion.

Let L_1 and L_2 be two lines of N_1 which intersect in the point s. Suppore that L_1 and L_2 each contain points of T. Now s is not joined (by lines of N) to any point of $L_1 \cap T$ or $L_2 \cap T$. By Bruck's Lemma 3.1, a point not in T is joined to all but $d-1$ points of T. Since $|L_1 \cap T| + |L_2 \cap T| > d-1$, we conculude that s must belong to T.

It is now readily verified that the points of T and the lines of N_1 which intersect T satisfy the postulates for an affine plane. This affine plane contains n points and d parallel classes; hence we must have $n = (d-1)^2$.

COROLLARY. *If $(d-1)^2 > n > (d-1)^2 - \frac{1}{2}(d-1)$ and N has more than a dn transversals, N cannot be embedded in an affine plane of order n.*

THEOREM 2. *If $n = (d-1)^2$ and N can be embedded in two affine planes π_1 and π_2 by adjoining complementary nets N_1 and N_2 respectively, then each line of N_2 is a subplane of N_1.*

Proof. If π_1 and π_2 are distinct, some line S of N_2 is not a line of π_1. By Theorem 1, S is a subplane of N_1. It was shown in the first part of the proof of Theorem 1 that a line of N_1 connot intersect S in just one point. Since the lines of N_2 not parallel to S do intersect S in one point, they must also be subplanes of N_1. It readily follows that the lines of N_2 parallel to S must also be subplanes of N_1.

REMARK. The relation between the two planes is precisely that given in [4].

COROLLARY. *A net with critical deficiency can be embedded in an affine plane in at most two ways.*

Proof. Given π_1 and N_1, it follows from Lemma 9 of [4] that, for any pair of points p and q, there is at most one subplane of order $d-1$ which contains p and q. Hence the lines of π_2 if it exists, are uniquely determined.

LEMMA 1. [Bruck's Lemma 3.3]. *Let N be a finite net of order $n = m^2$, deficiency $d = m+1$. Assume $m > 2$, so that N is nontrivial and nondegenerate.*

(i) *If S, T are distinct transversals with more than one common* *point, then they have exactly* $d - 1 = m$ *common points. Moreover* (a) *each point of* $S - T$ *is joined to each point of* $T - S$ *and* (b) *if* p, q *are any two distinct points in* $S \cap T$ *then every point not in* $S \cup T$ *is joined to at least one of* p, q.

(ii) *If* S, T, V *are three distinct transversals such that* S *has* m *points in common with each of* T, V *then* T, V *have at most one common point.*

LEMMA 2. *Under the hypotheses of Lemma* 1, *no point is on more than* $2(m + 1)$ *transversals.*

Proof. Suppose that the point p is contained in $m + 1 + i$ transversals and that h pairs of these transversals intersect in m points. By part (ii) of Lemma 1, no point distinct from p can be in more than two of these transversals. The number of points distinct from p in the union of all the transversals through d will be $(m + 1 + i)(m^2 - 1) - h(m - 1)$. But the number of points not joined to p is $d(n - 1) = (m + 1)(m^2 - 1)$. [See Bruck's Lemma 3.1].

Hence $(m + 1)(m^2 - 1) \geqq (m + 1 + i)(m^2 - 1) - h(m - 1)$, or $h \geqq i(m + 1)$. But the "overlaps" account for $h(m - 1)$ points, each belonging to two transvervals through p, so we must also have $(m + 1)(m^2 - 1) \geqq h(m - 1)$ or $(m + 1)^2 \geqq h$.

Hence $(m + 1)^2 \geqq i(m + 1)$, $(m + 1) \geqq i$. Lemma 2 then follows from the definition of i.

LEMMA 3. *Under the hypotheses of Lemma* 1, *suppose that there is a transversal* T *with the property that every pair of points in* T *belongs to another transversal. Then the number of transversals intersecting* T *in* m *points is equal to* $m^2 + m$.

Proof. Consider the subsets of T formed by intersecting T with those other transversals which intersect T in m points. These subsets, and the points of T, define an affine plane of order m. (See Lemma 8 in [4].) The number of transversals intersecting T in m points is then equal to the number of lines in an affine plane of order m.

THEOREM 3. *If* $n = m^2$, $d = m + 1$, *and* N *has* $2dn = 2(m + 1)m^2$ *transversals, and* $m > 3$, *then* N *can be extended to two different affine planes of order* n.

Proof. Since there are $2(m + 1)m^2$ transversals, each containing m^2 points, the number of point-transversal incidences is $2(m + 1)m^4$. There are m^4 points; if $t(p)$ is the number of transversals through p, we have $2(m + 1)m^4 = \Sigma t(p)$, where p ranges over all m^4 points.

Since $t(p) \leqq 2(m+1)$ for each p, we must have $t(p) = 2(m+1)$ for each point.

We may now apply a similar argument to the set of transversals through a fixed point p and the $(m+1)(m^2-1)$ points to which p is not joined. From the fact that no two points can be on more than two transversals (Lemma 1) we arrive at the conclusion that each pair of points which are not joined will be on exactly two transversals. Now let p be any point and let S be some transversal which contains p. For each point $q \neq p$ in S, there is another transversal T which contains p and q. Thus T contains $m-1$ points of S different from p. Applying Lemma 1 and using the fact that S contains m^2 points, we obtain $m+1$ transversals $T_1, T_2, \cdots, T_{m+1}$, where

$$|T_i \cap S| = m, i = 1, \cdots, m+1$$
$$T_i \cap T_j = p \text{ if } i \neq j .$$

Each point of S distinct from p is on exactly one of the T_i. Furthermore,

$$|T_1 \cup T_2 \cup \cdots \cup T_{m+1} - p| = (m+1)(m^2-1) .$$

No transversal contains two points which are joined; $(m+1)(m^2-1)$ is the number of points not joined to p. Hence each point not joined to p is on exactly one of the T_i.

We are now ready to define the lines of a complementary net N_1 so that the lines of N_1 together with the lines of N form an affine plane π_1. The point p is on $2(m+1)$ transversals. Let $S = S_1, S_2, \cdots, S_{m+1}$ be the transversals through p which are distinct from the T_i.

DEFINITION. L is a line of N_1 if and only if L is a transversal of N which intersects one of the S_i in m points.

Now each T_i intersects $S = S_1$ in m points. Thus the T_i are lines through p. Consider the intersections of the T_i with the other S_i, say S_2. Since no point $(\neq p)$ of S_2 is joined to p by a line of N, each point of S_2 is on one of the T_i. (Recall that every point not joined to p is on one of the T_i). It readily follows that the m^2-1 points of S_2 distinct from p occur in $m+1$ sets of $m-1$ each, one set of $m-1$ for each T_i. That is, $|T_i \cap S_j| = m$ for each i and j.

Now any transversal which goes through p is either one of the S_i or one of the T_i. Thus we have exactly $m+1$ lines of N_1 which go through p: namely, the T_i. (Note that S_i and S_j both intersect T_1 in m points implies that S_1 cannot intersect S_j in m points.)

We wish to show that two lines of N_1 can have at most one point in common. Let L_1 and L_2 be two lines of N_1. If L_1 and L_2 both intersect the same S_i in m points, then $|L_1 \cap L_2| = 1$ or 0 by Lemma 1.

If either L_1 or L_2 goes through p, there is some S_1 which is intersected in m points by both L_1 and L_2. Hence assume that neither L_1 nor L_2 goes through p, that L_1 intersects S_i in m points while L_2 intersects S_j in m points, where $i \neq j$. Moreover, we may assume that L_1 does not intersect S_j in m points and L_2 does not intersect S_i in m points.

Now S_i is included in the union of $T_1, \cdots, T_{m+1}$. Since L_1 can intersect each of these in at most one point, the m points of $L_1 \cap S_i$ must be located, one each, on m of the transversals $T_1, \cdots, T_{m+1}$. The points of L_2 must be similarly distributed among $T_1, \cdots, T_{m+1}$.

Since L_2 does not contain two points of S_i, at most one point of $L_1 \cap L_2$ can belong to S_i. Similarly, at most one point of $L_1 \cap L_2$ can belong to S_j. Thus we have at least $m-1$ points of $L_1 - L_2$ which are in S_i and are distributed, one each, to $m-1$ of $T_1, \cdots, T_{m+1}$. A similar remark holds for the points of $(L_2 - L_1) \cap S_j$. If $m > 3$, there must exist T_k which contains a point of $L_1 - L_2$ and a point of $L_2 - L_1$. That is, some point of $L_1 - L_2$ is not joined to some point of $L_2 - L_1$.

By Lemma 1, this cannot happen if $|L_1 \cap L_2| = m$. We conclude that $|L_1 \cap L_2| = 1$ or 0. Thus, in general, distinct lines of N_1 have 1 or 0 points in common.

Now let us see how many lines we have defined for N_1. By Lemma 3, there are $m^2 + m$ lines intersecting each of $S_1, \cdots, S_{m+1}$ in m points. However, $T_1, \cdots, T_{m+1}$ intersect *each* of the S_i in m points. Assuming for the moment that there are no further duplications, we get a total of $(m+1)[(m^2+m)-(m+1)]+(m+1) = (m+1)m^2 = dn$ lines in N_1. By further counting arguments of a standard nature and making use of the fact that lines of N_1 do not intersect in more than one point, it readily follows that N_1 is a complementary net in terms of which we can define an affine plane π_1.

We have yet to show that no transversal not through p can intersect S_i and S_j $(i \neq j)$ each in m points, i.e. that there are no further duplications in the previous paragraph. Suppose that there were such a transversal L. Since L intersects S_i in m points and since each of $T_1, \cdots, T_{m+1}$ intersects S_i in m points, we have (as before) that $L \cap S_i$ must have one point each on some m of $T_1, \cdots, T_{m+1}$. A similar remark holds for $L \cap S_j$. By Lemma 1, L can intersect each T_k in at most one point. We conclude that $|L \cap S_i \cap S_j| \geqq m-1$, contrary to the condition that $S_i \cap S_j = p$.

Applying Theorem 1 of this paper and Theorem 6 of [4], the extra transversals with respect to N_1 enable us to define a second complementary net N_2 and a second plane π_2.

COROLLARY. *If $n = (d-1)^2$, there are at most $2dn$ transversals.*

Proof. Applying Theorem 1, every transversal is either a line of N_1 or a line of N_2.

Now many of the known finite non-Desarguesian planes are constructed by a process which amounts to extending the same net in more than one way. The construction in [4] is an example; see [3] for other cases. Consider the following situation: Let π be an affine Desarguesian plane of order n and let T be a set of n points of π such that not all of the points of T lie on any one line. Let N be the (possibly degenerate) net consisting of all parallel classes which have no lines intersecting T in more than one point. If N contains three or more parallel classes, then T is a transversal of the non-degenerate net N. Henceforth, assume that this is the case—i.e., $d \leqq n - 2$.

Now every collineation of N carries transversals into transversals. Let G be the group of central collineations of π with axis L_∞. Let us say that a pair of points belongs to a parallel class if the line through them belongs to this parallel class. Let p_1, q_1 and p_2, q_2 be two point-pairs such that p_1, q_1 and p_2, q_2 both belong to the same parallel class. Then there is exactly one element of G which carries p_1 into p_2, q_1 into q_2—i.e., G is transitive on point-pairs in each parallel class of N_1.

Let $\mathfrak{T}$ be the set of transversals which are images of T under the collineations in G. For every parallel class in the complementary net N_1, there is at least one pair of points in T which belongs to this parallel class. It follows that every pair of points which belongs to a parallel class in N_1 belongs to at least one transversal in $\mathfrak{T}$.

Now G is of order $n^2(n-1)$. If the subgroup of G which leaves T invariant has order $n(n-1)/d$, then $\mathfrak{T}$ will contain exactly nd transversals. In this case, each pair not joined in N will belong to exactly one member of $\mathfrak{T}$; the members of $\mathfrak{T}$ can be taken as the lines of a new complementary net N_2 permitting N to be extended to a new affine plane π_2.

The above argument will go through for more general planes and more general groups of collineations.

THEOREM 4. *Let π be an affine plane of order n, consisting of two complementary nets N and N_1, where N is of deficiency d. Let T be a transversal of N which contains at least one point-pair in each parallel class of N_1. Suppose that either N or N_1 admits a group G of collineations such that*

(1) *G has order $n^2(n-1)g$ and is transitive on point-pairs in each parallel class of N_1*

(2) *The subgroup of G which leaves T invariant has order $n(n-1)g/d$. Then the images of T under G define a new net N_2 such*

that N and N_2 form an affine plane.

Proof. The argument is essentially the same as the one already given. G will carry transversals into transversals if it acts as a collineation group on either N or N_1, whether or not G acts as a collineation group on the full plane.

The discussion given in [3] shows, in effect, that the Andre planes are obtained by a process which one may interpret as a matter of successive applications of Theorem 4. Furthermore, the hypotheses of Theorem 3 are satisfied in all known cases for which the situation described in Theorem 2 occurs. This corresponds to the fact that in these cases an appropriate coordinate system for π_1 is a right vector space over a field of order m.

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