ON THE NORMAL BUNDLE OF A MANIFOLD

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In the Michigan lecture notes of 1940 [8] Whitney proved that any manifold in the cobordism class of P_2 cannot be embedded in R^4 with a normal field while non-orientable manifolds in the trivial cobordism class may or may not have a normal field. We will give a new proof of this result using some of the recent notions of differential topology. As one would expect, Whitney's theorem is a special case of a more general theorem and for the statement of this theorem we introduce some notation.

Let M^n be a compact smooth n-manifold. Let $\bar{w}_i$ be the dual Stiefel Whitney classes of M^n.

DEFINITION. Let $\sigma(M^n) = 0$ if $\bar{w}_1 \cdot \bar{w}_{n-1} = 0$ and $\sigma(M^n) = 1$ if $\bar{w}_1 \cdot \bar{w}_{n-1} \neq 0$.

Clearly $\sigma(M^n)$ is just a Stiefel Whitney number [6]. Note also that by a result of Massey [5], $\sigma(M^n) = 0$ unless $n = 2^j$.

THEOREM 1. *For any embedding of M^n in R^{2n} the (twisted) Euler class is congruent to* $2\sigma \bmod 4$.

This result is a slight sharpening of the theorem of Massey [4]; the proof is given in § 4 after some preliminary results in §§ 2 and 3.

Let χ be the Euler characteristic of M^2. In Whitney's theorem the role of σ in Theorem 1 is played by χ. It is not hard to verify that for 2-dimension manifolds $\sigma = \chi \bmod 2$. In addition, for 2-dimensional manifolds we can prove (section 6)

THEOREM 2. *For each k and each value of σ there is a manifold M^2 and an embedding of M^2 in R^4 with twisted Euler class $2\sigma + 4k$.*

We have not been able to show that a single manifold has an embedding for each k. Whitney exhibited two embeddings of the Klein bottle, one with a trivial Euler class and one with a non-trivial one.

We also have this weaker result (section 7) for other values of n.

THEOREM 3. *For every even n there exists a manifold M^n and an embedding of M^n in R^{2n} with no normal field.*

It is known that if $n \neq 2^j$ and $n > 3$, then every n-manifold embeds

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in R^{2n-1}. Hence this result asserts in addition that some n-manifolds have inequivalent embeddings in R^{2n}.

It is interesting to note that the principal lemma yielding Theorem 1 also gives a new proof of the following slightly strengthened version of a result of Levine [2] and Mahowald [3].

THEOREM 4. *Suppose M^n is orientable in addition. If there exists a class d of dimension $(n-k-1)/2$ such that $d \cup Sq^1 d \cup \bar{w}_k \neq 0$, then M^n does not embed in R^{n+k+1}.*

In [3] only the application of this result to give—P_n does not embed in R^{2n-2} if $n = 2^j + 1$—is given.

2. Some lemmas. In this section we will derive some information about a particular secondary cohomology operation. Let K be a semi-simplicial complex and let $u \in C^{2k}(K; Z)$ such that $\delta u = 2v$. If w is an integer (a mod j) cocycle we write $[w]$ ($[w]_j$) for the cohomology class containing w. We have the following results, some of which are well known.

2.1. $Sq^1[u]_2 = [v]_2$ and $\beta_2[u]_2 = [v]$ where β_j is the Bockstein coboundary connected with the sequence $0 \to Z \to Z \to Z_j \to 0$.

2.2. If $\mathfrak{p}$ is the Pontriagin square operation $\mathfrak{p}: H^{2k}(K; Z_2) \to H^{4k}(K; Z_4)$ then $\mathfrak{p}([u]_2) = [u \cup u + u \cup_1 \delta u]_4$.

2.3. If $a \in H^i(X; Z)$ then let $\bar{a}$ be its mod 2 restriction. Then

$$\beta_4 \mathfrak{p}([u]_2) = [v \cup_1 v + u \cup v]$$

and

$$\overline{\beta_4 \mathfrak{p}([u]_2)} = Sq^{2k} Sq^1 [u]_2 + [u]_2 \cup [v]_2 \,.$$

Proof. By the coboundary formula [7] which also holds in s.s.c. we have $\delta(u \cup u + u \cup_1 \delta u) = 4(v \cup_1 v + u \cup v)$. This gives the first statement and the second now follows by definition.

2.4. If $u \cup u + \delta p$ is an integer cocycle then $u \cup_1 v$ is a mod 2 cocycle and $Sq^1([u \cup_1 v]) = Sq^{2k} Sq^1 [u]_2 + [u]_2 \cup [v]$.

Proof. By the coboundary formula we have

$$\begin{aligned}\delta(u \cup_1 v) &= u \cup v - v \cup u + \delta u \cup_1 v \\ &= 2(u \cup v) + 2(v \cup_1 v)\end{aligned}$$

since $\delta(u \cup u) = 0$ implies $u \cup v + v \cup u = 0$. Now 2.1 completes the proof.

2.5. If $u \cup u = 2b + \delta c$, then $b + u \cup_1 v$ is a mod 2 cocycle and

$$Sq^1[b + u \cup_1 v]_2 = Sq^2Sq^1[u]_2 + [u]_2 \cup Sq^1[u]_2 .$$

Proof. Note that $\delta(u \cup u) = 2(v \cup u + u \cup v) = 2\delta b$. Hence

$$v \cup u + u \cup v = \delta b$$

and the result follows as in 2.4.

In 2.4 we require that $u \cup u + \delta p$ is an integer cocycle, that is, we require that $\beta_2[u \cup u] = 0$. The universal example for such a class u is obtained by considering a fibering $p\colon X \to K(A_2, 2k)$ with fiber $K(Z, 4k)$ and k-invariant $2\beta_4\mathfrak{p}(\alpha)$ where α is the fundamental class of $K(Z_2, 2k)$. Let $\alpha' = p^*(\alpha)$. Then by 2.4, $\alpha' \cup_1 Sq^1\alpha'$ is a cocycle and not a coboundary (since $\alpha' \cup Sq^1\alpha' \neq 0$). Let $\varepsilon = \alpha' \cup_1 Sq^1\alpha'$.

Let SA be the suspension of A and let $s\colon H^j(A) \to H^{j+1}(SA)$ be the suspension isomorphism. There is a natural map $f\colon SK(Z_2, 2k-1) \to X$ such that f^* is an isomorphism in dimension $2k$.

2.6. With the above notation there is a class $\beta \in p^*H^*(K(Z_2, 2k); Z_2)$ (that is a primary operation) such that $f^*(\beta + \varepsilon) = s(\alpha \cup Sq^1\alpha)$ where $s\colon H^j(K(Z_2, 2k-1)) \simeq H^{j+1}(SK(Z_2, 2k-1))$. If β satisfies the above equation then $\beta + Sq^{2k}$ will do so too.

Proof. As a vector space $H^{4k}(SK; Z_2)$ is generated by

$$f^*p^*H^{4k}(K(Z_2, 2k)) \quad \text{and} \quad s(\alpha \cup Sq^1\alpha) .$$

Hence $f^*(\varepsilon) = \lambda s(\alpha \cup Sq^1\alpha) + \beta$ where $\lambda = 0$ or 1 and β satisfies the theorem. By direct computation we see that

$$Sq^1 s(\alpha \cup Sq^1\alpha) = Sq^{2k}Sq^1 s\alpha \notin f^*p^*Sq^1H^{4k}(K(Z, 2k); Z_2) .$$

But by 2.4 $Sq^1f^*(\varepsilon) = Sq^{2k}Sq^1 s\alpha$. Since

$$Sq^1\lambda s(\alpha \cup Sq^1\alpha) + Sq^1\beta = Sq^{2k}Sq^1 s\alpha$$

if and only if $\lambda = 1$ and $Sq^1\beta = 0$ we are finished.

In 2.5 we required that $u \cup u \equiv 0 \bmod 2$. The universal example for such a class u is given by a fiber space $p_1\colon Y \to K(Z_2, 2k)$ with $K(Z_2, 4k-1)$ as the fiber and Sq^{2k} as the k-invariant. Since there is no homotopy in dimension $4k$ we have, letting $[u]_2 = p_1^*\alpha$:

2.7. The class $\mu = [b + u \cup_1 v] \in H^{4k}(Y; Z_2)$ is not spherical and hence is the universal example of a nontrivial natural cohomology operation which we write as μ too.

Let $g: SK(Z_2, 2k-1) \to Y$ be the natural map inducing an isomorphism g^* in dimension $2k$. By an argument identical to the proof of 2.6 we have 2.8. In the above notation $g^*(\mu + \beta') = s(\alpha \cup Sq^1\alpha)$ where $\beta' \in p_1^* H^*(K(Z_2, 2k), Z_2)$. If β' satisfies the above equation then $\beta' + Sq^{2k}$ will do so too.

3. Let γ_n be the universal n-plane bundle and let I be the trivial line bundle. The base space of I will usually be clear from the context. If ν is any n-plane bundle we let $T(\nu)$ be the Thom complex and $U \in H^n(T; Z_2)$ be the Thom class. Recall that in T, $U \cup U$ is equal to $U \cup \bar{w}_n$ which is the restriction mod 2 of an integer class $U \cup \chi$ where χ is the twisted Euler class (of order 2 if n is odd). Hence $\beta_2 Sq^n U = 0$. By usual obstruction theory, letting $n = 2k$, we see that there exists a map $g: T(\gamma_{2k}) \to X$ such that g^* is an isomorphism in dimension $2k$.

Lemma 3.1. *In the above notation we can find a β satisfying* 2.6 *such that* $g^*(\beta + \varepsilon) = U \cup \bar{w}_{n-1} \cup \bar{w}_1$, $n = 2k$.

Proof. Consider the diagram:

$$\begin{array}{ccc} ST(\gamma_{n-1}) \cong T(\gamma_{n-1} \oplus I) & \xrightarrow{g'} & SK(Z_2, n-1) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ T(\gamma_n) & \xrightarrow{g} & X \end{array}$$

where i is the map induced by the natural inclusion of $\gamma_{n-1} \oplus I$ in γ_n, and g' is defined by requiring $g'^*(s\alpha) = U'$, the Thom class of $T(\gamma_{n-1} \oplus I)$. Letting β be the class of 2.6, we have $g'^* f^*(\beta + \varepsilon) = s(U_{n-1} \cup U_{n-1} \cup \bar{w}_1) = U' \cup \bar{w}_{n-1} \cup \bar{w}_1$ where U_{n-1} is the Thom class of $T(\gamma_{n-1})$. Hence $g^*(\beta + \varepsilon) = U \cup \bar{w}_{n-1} \cup \bar{w}_1 + \alpha$ where $\alpha \in \ker i^*$. But $\ker i^*$ is generated by $Sq^n U = U \cup \bar{w}_n$. Therefore 2.6 completes the proof.

4. Proof of Theorem 1.

Notation. In the remaining sections it will be convenient to use a dot for the cup product.

Let M^n be embedded in R^{2n} and let $T(\eta)$ be the Thom complex of the normal bundle. By [6] M^n has a normal field if $n = 1 \bmod 2$ (it even embeds in R^{2n-1}) so we suppose n is even. The group $H^{2n}(T(n); Z) = Z$ and is generated by a class b such that $2jb = U \cdot \lambda$

($\bar{w}_n$ is zero, hence λ is zero mod 2). The cohomology operation μ is defined on U and by 2.7 and 3.1 we have $\mu(U) = [U \cdot \bar{w}_1 \cdot \bar{w}_{n-1} + jb]_2$. Since the top cohomology class of the Thom complex of a normal bundle to an embedding is spherical [6], $\mu(U) = 0$. Therefore $jb = U \cdot \bar{w}_1 \cdot \bar{w}_{n-1}$ (mod 2).

5. Proof of Theorem 4. Suppose we have an embedding of the kind described. Let E and E_0 be the normal disk and sphere bundle respectively. Consider the sequence

$$T(\eta) = E/E_0 \xrightarrow{\tau} SE_0 \xrightarrow{Sf} SK(Z_2, j) \xrightarrow{g} Y$$

where g is defined in the paragraph just before 2.8 and Sf is the suspension of the map $f\colon E_0 \to K(Z_2, j)$ satisfying $f^*(\alpha) = a \cdot d$ where a is any class such that $\tau^*(sa) = U$. The map τ is the natural map.[1] Let $\lambda = fSf\tau$. Clearly λ is a defining map for μ. We have $g^*\mu = s(\alpha \cdot Sq^1\alpha)$ by 2.8. By direct computation $f^*(\alpha \cdot Sq^1\alpha) = a \cdot \bar{w}_k \cdot d \cdot Sq^1 d + b$ where b is in $\ker \tau^*$. Finally $\lambda^*(\mu) = U \cdot \bar{w}_k \cdot d \cdot Sq^1 d$ which is in the top cohomology class of $T(\eta)$ and hence must be zero. This contradiction proves the theorem.

6. Proof of Theorem 2. Let $f'\colon S^4 \to T(\gamma^2)$ be any map. By Theorem 36 [6] the map f' is homotopic to a map $f\colon S^4 \to T(\gamma^2)$ which is transverse regular on $G_{2,k}$ (the grassmann manifold of 2 planes in R^{2+k} which, if $k > 3$, is universal for classifying 2 plane bundles over 2-manifolds. Then $f^{-1}(G_{2,k}) = M^2$ is a sub-manifold of S^4 and $f/M^2\colon M^2 \to G_{2,k}$ is the classifying map of the normal bundle to an embedding of M^2 in $R^4 \subset S^4$. All that remains is to investigate the structure of $\pi_4(T(\gamma^2))$.

LEMMA 6.1. *The first few homotopy groups of $T(\gamma^2)$ are*

i	1	2	3	4
$\pi_i(T(\gamma^2))$	0	Z_2	0	Z.

The k-invariant with which the Z group is added is $2\beta_4\mathfrak{p}(\alpha)$ where α is the fundamental class of $K(Z_2, 2)$.

REMARK. It is interesting to note that this portion of the Postnikov tower for $T(\gamma^2)$ is the same as the corresponding portion for $\tilde{G}_n$, $n > 4$ where $\tilde{G}_n$ is the classifying space for oriented n-plane bundles. Indeed the k-invariants computed in [1] agree with these

[1] If we realize E/E_0 by adding a cone over E_0 to E, then E is naturally embedded in $E \cup {}_cE_0$ and $\tau\colon E \cup {}_cE_0 \to E \cup {}_cE_0/E$.

given here. The class $w_4 \in H^4(\tilde{G}_n; Z_2)$ is associated with $U \cdot w_1^2$ in $H^4(T(\gamma^2); Z_2)$ while w_2^2 and $U \cdot w_2$ are similarly associated.

Proof of the lemma. Since the Thom class of $T(\gamma^2)$ is also the fundamental class and since $Sq^1 U \neq 0$, the Hurewicz isomorphism theorem proves that $\pi_2(T(\gamma^2)) = Z_2$. Now $H^3(T(\gamma^2); J) = Z_2$ if $J = Z$ or Z_{2k} for any k and zero for other Z_p. Hence any homotopy group in dimension 3 must be attached with a nontrivial k-invtriant. But $H^4(K(Z_2, 2); Z_2)$ is generated by $Sq^2\alpha$ and $Sq^2 U = U \cdot w_2$ in $H^*(T(\gamma^2))$ and so $\pi^2(T(\gamma^2)) = 0$.

Now $H^4(T(\gamma^2); Z) = Z$, generated by $U \cdot \chi$ where χ is the twisted Euler class. Hence the rank of $\pi_4(T(\gamma^2))$ is 1. Since the restriction mod 2 of $U \cdot \chi$ is $Sq^2 U$, the Z component is attached with a nontrivial k-invariant. Finally $H^5(K(Z_2, 2); Z) = Z_4$ generated by $\beta_4 \mathfrak{p}(\alpha)$ and $\overline{(\beta_4 \mathfrak{p}(\alpha))} = Sq^2 Sq^1 \alpha + \alpha Sq^1 \alpha$ (see 2.3) and since $Sq^2 Sq^1 U + U \cdot U w_1 = U \cdot w_2 \cdot w_1 \neq 0$ the k-invariant for the Z component can not be $\beta_4 \mathfrak{p}(\alpha)$. Therefore it must be $2\beta_4 \mathfrak{p}(\alpha)$.

Let $p: X \rightarrow K(Z_2, 2)$ be the fiber map having $2\beta_4 \mathfrak{p}(\alpha)$ as k-invariant and $K(Z, 4)$ as fiber. By 2.4 we see that $H^4(X; Z_2) = Z_2 + Z_2$ generated by a new class $\alpha' \cup_1 Sq^1 \alpha'$ and by $Sq^2 \alpha'$ where $\alpha' = p^* \alpha$. Hence the natural map $f: T(\gamma^2) \rightarrow X$ induces an isomorphism $f^*: H^i(X) \rightarrow H^i(T(\gamma^2))$ for all coefficient groups if $i \leqq 4$. To complete the proof of the lemma we note that f^* is also an isomorphism in dimension 5.

Now we can complete the proof of Theorem 2. Since the order of the k-invariant is 2, $f'^*(U \cdot \chi) = 2j\aleph$ where $\aleph$ is a generator of $H^4(S^4; Z)$ and $j = [f']$, the homotopy class of f' in π_4 under some identification with the integers. Let η be the normal bundle for the embedding of M^2 in R^4 constructed above. Then the composite

$$S^4 \xrightarrow{\lambda_1} T(\eta) \xrightarrow{\lambda_2} T(\gamma^2)$$

(where λ_2 is the natural map and λ_1 is obtained by collapsing the complement of a normal neighborhood of M^2 to a point) is just f'. Since λ_1^* is an isomorphism in dimension 4, the twisted Euler class of the embedding is $2j$ times the twisted fundamental cohomology class.

7. **Proof of Theorem 3.** Let $T(\gamma^n)$ be the Thom complex of the universal n-plane bundle, n even. Then $H_n(T(\gamma^n); Z) = Z_2$ generated by the cycle dual to the Thom class U. Since $T(\gamma_n)$ is $(n-1)$-connected, we have $\pi_n(T(\gamma^n)) = Z_2$. Therefore by Serre's theorem, ([6], page 109) rank $H^{2n}(T(\gamma^n); Z) = \text{rank}\, \pi_{2n}(T(\gamma^n))$. In particular there is a map $f: S^{2n} \rightarrow T(\gamma^n)$ such that $f^*(U \cdot \chi) \neq 0$ where χ is the twisted Euler class. Now following the argument of § 6 we construct the desired manifold.

References

1. A. Dold and H. Whitney, *Classification of oriented sphere bundles over a 4-complex*, Ann. of Math., **69** (1959), 667-677.

2. J. Levine, Princeton Thesis.

3. M. Mahowald, *On the embeddability of the real projective spaces*, Proc. Am. Math. Soc., **13** (1962), 763-764.

4. W. S. Massey, *Normal vector fields on manifolds II*, Notices, Amer. Math. Soc., **10** (1963), p. 362.

5. ———, *On the Stiefel Whitney classes of a manifold*, Amer. J. of Math., **82** (1960), 92-102.

6. J. Milnor, *Lectures on Characteristic Classes*, Princeton mimeographed notes.

7. N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math., **48** (1947), 290-320.

8. H. Whitney, *On the topology of differentiable manifolds*, Lectures in Topology, Michigan Press, 1940.

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