ON A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN FOUR VARIABLES

R. P. Gilbert

1. Introduction. Bergman [1]–[6], [11] has considered the elliptic partial differential equation,

$$T_3[\Psi] = \frac{\partial^2 \Psi}{\partial x_\mu \partial x_\mu} + A(r^2) x_\mu \frac{\partial \Psi}{\partial x_\mu} + C(r^2)\Psi = 0 \,, \quad (\mu = 1, 2, 3) \tag{1.1}$$

where $A(r^2)$, $C(r^2)$ are analytic functions of the real variable $r^2 = x_\mu x_\mu$ $[\mu = 1, 2, 3]$ (Repeated indices mean the summation convention is used.) In this paper we shall investigate the four variable analogue of this equation, $T_4[\Psi] = 0$, and show that many of Bergman's results carry over to this case. Here, we need in many instances, the methods of several complex variables in order to find the natural generalizations.

In Bergman's theory,[1] the integral operator $B_3\ [f]$ plays an important role in studying the solutions of (1.1). In our case, there is an analogous operator [7]–[10], which is a four-variable analogue to $B_3\ [f]$:

$$H[X] = B_4[f] \equiv -\frac{1}{4\pi^2} \iint_D \frac{d\eta}{\eta} \frac{d\xi}{\xi} f(u; \eta, \xi) \,, \quad X \equiv (x_1, x_2, x_3, x_4) \,,$$

$$u \equiv x_1\Big(1 + \frac{1}{\eta\xi}\Big) + ix_2\Big(1 - \frac{1}{\eta\xi}\Big) + x_3\Big(\frac{1}{\xi} - \frac{1}{\eta}\Big) + ix_4\Big(\frac{1}{\xi} + \frac{1}{\eta}\Big) \tag{1.2}$$

and $D = \{|\xi| = 1\} \times \{|\eta| = 1\}$. The operator $B_4\ [f]$ maps analytic functions of three-complex variable onto harmonic, function-elements of four-variables. One may realize how analytic functions are transformed into harmonic functions, by considering the powers of u, which act as generating functions for the homogeneous, harmonic polynomials, $H_n^{kl}[X]$,

$$u^n = \sum_{k,l=0}^{n} H_n^{kl}[X] \xi^{-k} \eta^{-l} \tag{1.3}$$

The polynomials $H_n^{kl}(X)$ $[k, l = 0, 1, \cdots, n;\ n = 0, 1, 2, \cdots]$ form a complete, linearly independent system [13] [7] [8]. From the Cauchy formula for two complex variable we have an integral representation for these polynominals given by,

Received November 22, 1963. This research supported in part by the United States Air Force through the Air Force Office of Scientific Research Grant 400-64.

[1] See Bergman [1], Chapter II.

$$H_n^{kl}[X] = -\frac{1}{4\pi^2}\int_{|\xi|=1}\int_{|\eta|=1} u^n \xi^{k-1}\eta^{l-1} d\eta d\xi \; . \tag{1.4}$$

It is clear then, that the analytic function,

$$f(u;\eta,\xi) = \sum_{n=0}^{\infty}\sum_{m,p=0}^{n} a_{nmp}u^n\eta^m\xi^p \; , \tag{1.5}$$

is mapped onto a harmonic function,

$$H[X] = \sum_{n=0}^{\infty}\sum_{m,p=0}^{\infty} a_{nmp}H_n^{mp}[X] \; ,$$

defined in the small in the neighborhood of the origin, $[X] = (0)$. We shall refer to (1.6) as the normalized associate of $H[X]$ with respect *to* B_4 (see Bergman [1] pg. 62, Kreyszig [14] and Gilbert [8]).

2. The derivation of a Bergman integral operator generating solutions to $T_4\,[\Psi] = 0$. We shall establish the following theorem, which is four-variable analogue of Bergman's [5].

THEOREM 2.1. *Let $H(r,\tau)$, $|\tau| \leqq 1$, be a solution of*

$$(1-\tau^2)H_{r\tau} - \tau^{-1}(\tau^2+1)H_r + r\tau\Big(H_{rr} + \frac{3}{r}H_r + BH\Big) = 0 \; , \tag{2.1}$$

where

$$B = -\frac{r}{2}A_r - 2A - \frac{r^2}{4}A^2 + C \; , \quad r^2 = x_\mu x_\mu \; , \tag{2.2}$$

and $H_r/r\tau$ is continuous at $r = \tau = 0$. Then,

$$\begin{aligned} \Psi(X) &= \Omega[f] \\ &\equiv -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{\tau=-1}^{+1} E(r,\tau)f(u(1-\tau^2);\eta,\xi)d\tau \; , \end{aligned} \tag{2.3}$$

where

$$E(r,\tau) \equiv \exp\Big(-\frac{1}{2}\int_0^r A r dr\Big)H(r,\tau) \; , \tag{2.4}$$

is a solution of (1.1).

The proof of Theorem 2.1 follows closely the proof given by Bergman for the three-variable case. First, we list several formal relationships which exist between the variables:

$$\begin{aligned} \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x_\mu} &= \Big(1+\frac{1}{\eta\xi}\Big)^2 + (i)^2\Big(1-\frac{1}{\eta\xi}\Big)^2 + \Big(\frac{1}{\xi}-\frac{1}{\eta}\Big)^2 \\ &\quad + (i)^2\Big(\frac{1}{\xi}+\frac{1}{\eta}\Big)^2 = 0 \; , \end{aligned} \tag{2.5}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial E}{\partial x_\mu} = \frac{\partial u}{\partial x_\mu}\frac{\partial E}{\partial r}\frac{\partial r}{\partial x_\mu} = \frac{\partial u}{\partial r}\frac{\partial E}{\partial r} \tag{2.6}$$

$$x_\mu \frac{\partial E}{\partial x_\mu} = x_\mu \frac{\partial r}{\partial x_\mu}\frac{\partial E}{\partial r} = r\frac{\partial E}{\partial r}, \tag{2.7}$$

$$x_\mu \frac{\partial u}{\partial x_\mu} = u, \tag{2.8}$$

and

$$\frac{\partial}{\partial x_\mu} f[(1-\tau^2)u; \eta, \xi] = \frac{1}{2u}\frac{\partial u}{\partial x_\mu}\left(\tau - \frac{1}{\tau}\right) f_\tau \tag{2.9}$$

Now, using relations (2.5)–(2.9) we compute formally, that

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x_\mu \partial x_\mu} = & -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{\tau=-1}^{+1} f[u(1-\tau^2); \eta, \xi] \\ & \times \left\{ \frac{\partial^2 E}{\partial x_\mu \partial x_\mu} - \frac{\partial\left(\tau - \frac{1}{\tau}\right)\frac{1}{r}E_r}{\partial \tau} \right\} d\tau, \end{aligned} \tag{2.10}$$

and consequently

$$\begin{aligned} T_4(\Psi) \equiv & -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{\tau=-1}^{-+} f\left\{ \frac{\partial^2 E}{\partial x_\mu \partial x_\mu} - \frac{\partial\left(\tau - \frac{1}{\tau}\right)\frac{1}{r}E_r}{\partial \tau} \right. \\ & \left. + A\left(x_\mu \frac{\partial E}{\partial x_\mu} - \frac{x_\mu}{u}\frac{\partial u}{\partial x_\mu} \frac{\partial \frac{1}{2}\left(\tau - \frac{1}{\tau}\right)E}{\partial \tau} \right) + CE \right\} d\tau \\ = & -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{\tau=-1}^{+1} f\left\{ \frac{\partial^2 E}{\partial x_\mu \partial x_\mu} - \frac{\partial\left(\tau - \frac{1}{\tau}\right)\frac{1}{r}E_r}{\partial \tau} \right. \\ & \left. + A_r E_r - A \frac{\partial \frac{1}{2}\left(\tau - \frac{1}{\tau}\right)}{\partial \tau} + CE \right\} d\tau \end{aligned} \tag{2.11}$$

In order to verify that is a solution of $T_4[\Psi] = 0$, we show that the terms in the brace of the integrand (2.11) vanish identically. By setting $E(r, \tau) = H(r, \tau)\exp\left(-1/2\int_0^r A r dr\right)$, and computing the terms involved we obtain

$$\begin{aligned} & \frac{1}{r\tau}\left\{ (1-\tau^2)H_{r\tau} - \tau^{-1}(\tau^2+1)H_r + r\tau\left(H_{rr} + \frac{3}{r}H_r\right) \right. \\ & \left. + r\tau\left(-\frac{r}{2}A - 2A + \frac{r^2}{4}A^2 + C\right)H \right\} \exp\left(-\frac{1}{2}\int_0^r A r dr\right) \equiv 0\,; \end{aligned} \tag{2.12}$$

consequently, if a solution $H(r, \tau)$ exists to (2.1), then (2.3) represents a solution of $T_4(\Psi) = 0$.

One can show the existance of an absolutely, and uniformly convergent solution,

$$H(r, \tau) = 1 + \sum_{n=1}^{\infty} \tau^{2n} b^{(n)}(r) , \tag{2.13}$$

where the $\{b^{(n)}(r)\}$ are sequences of functions defined by the recurrence formulas

$$\begin{cases} b_r^{(1)} + rB = 0 \\ (2n-1)b_r^{(n)} + rb_{rr}^{(n-1)} - (2n-5)b_r^{(n-1)} + rBb^{(n-1)} = 0 , \quad (n=2, 3, \cdots) \\ b^{(n)}(0) = 0 , \quad (n = 1, 2, \cdots) \end{cases} \tag{2.14}$$

and where $|\tau| \leqq 1$, and $r \leqq p < P_0/\sqrt{2}$ (distance of the first singularity of B from the origin). The recurrence formulas are obtained by substituting the series (2.13) into (2.1), and the convergence may be proved by the method of dominants as Bergman has done in the case of three-variables [5].

As an illustration of the above method we consider the Klein-Gordon equation

$$\Box \Psi - m^2\Psi = 0 , \tag{2.15}$$

which describes the motion of massive, free elementary- particles, of spin zero. Here our recursive formulae become

$$\begin{aligned} &(2n-1)b_r^{(n)} + rb_{rr}^{(n-1)} - (2n-5)b_r^{(n-1)} - rm^2b^{(n-1)} = 0 , \qquad (n \geqq 2) \\ &b_r^{(1)} + rm^2 , \qquad b^{(n)}(0) = 0 , \end{aligned}$$

which have the solution,

$$b^{(n)} = -2\frac{(n-1)(mr)^{2n-2}}{(2n-1)!} + \frac{(mr)^{2n}}{(2n)!} ; \tag{2.16}$$

Consequently,

$$\begin{aligned} E(r, \tau) &= H(r, \tau) \equiv 1 + \sum_{1}^{\infty} \tau^{2n} b^{(n)}(r) \\ &= (1-\tau^2)\cosh(m\tau r) + \frac{\tau}{mr}\sinh(m\tau r) . \end{aligned} \tag{2.17}$$

3. A Class of vector solutions. In an earlier paper [8] we introduced a class of harmonic vectors $\overrightarrow{H}[X]$, whose components were given by

$$H_\mu[X] = \frac{-1}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1}\frac{d\eta}{\eta}\frac{d\xi}{\xi}f(u, \eta, \xi)N_\mu(\eta, \xi)\ , \tag{3.1}$$

where N_μ is a component of the vector,

$$\vec{N}(\eta, \xi) \equiv \left(1 + \frac{1}{\eta\xi}, i\left[1 - \frac{1}{\eta\xi}\right], \left[\frac{1}{\xi} - \frac{1}{\eta}\right], i\left[\frac{1}{\xi} + \frac{1}{\eta}\right]\right) \tag{3.2}$$

and $u = \vec{N}\cdot\vec{X}$. In the same way we introduce a class of solutions to $T_4[\Psi] = 0$ by setting

$$\begin{aligned}\Psi_\mu(X) &= \Omega_\mu[f] \\ &\equiv -\frac{1}{4\pi^2}\int_{|\eta|=1}\frac{d\eta}{\eta}\int_{|\xi|=1}\frac{d\xi}{\xi}N_\mu(\eta, \xi)\int_{\tau=-1}^{+1}E(r, \tau)f[u(1 - \tau^2); \eta, \xi]d\tau\ .\end{aligned} \tag{3.3}$$

If X is chosen so that the integrand is absolutely integrable we may exchange orders of integration and write (3.3) as

$$\begin{aligned}\Psi_\mu(X) &= -\frac{1}{4\pi^2}\int_{\tau=-1}^{+1}E(r, \tau)\int_{|\eta|=1}\int_{|\xi|=1}\frac{d\eta}{\eta}\frac{d\xi}{\xi}N_\mu(\eta, \xi)f[u(1 - \tau^2); \eta, \xi]d\tau \\ &= 2\int_0^{+1}E(r, \tau)H_\mu[X(1 - \tau^2)]d\tau\ .\end{aligned} \tag{3.4}$$

We next consider the line integral,

$$\int_{\mathfrak{T}}\Psi_\mu[X]dx_\mu \equiv 2\int_{\mathfrak{T}}dx_\mu\int_0^1 E(r, \tau)H_\mu[X(1 - \tau^2)]d\tau\ , \tag{3.5}$$

where $\mathfrak{T}$ lies on the hypersphere, $\|X\| = R$.

Here we may interchange orders of integration and write

$$\begin{aligned}\int_{\mathfrak{T}}\Psi_\mu(X)dx_\mu &\equiv 2\int_0^1 E(R, \tau)\left\{\int_{\mathfrak{T}}H_\mu[X(1 - \tau^2)]dx_\mu\right\}d\tau \\ &= 2\int_0^1\frac{E(R, \tau)}{1 - \tau^2}\left\{\int_{\mathfrak{F}}H_\mu[Y]dy_\mu\right\}d\tau\ ,\end{aligned} \tag{3.6}$$

where

$$\mathfrak{F} \equiv \{Y \mid Y_\mu = (1 - \tau^2)x_\mu;\ \mu = 1, 2, 3, 4;\ X \in \mathfrak{T}\}\ . \tag{3.7}$$

Now as Bergman [1] [5], and later Mitchell [16] have done in the case of the equation $T_3[\Psi] = 0$, we assume the associate of $H_\mu(X)$ is rational,

$$f(u, \eta, \xi) = \frac{p_1(u, \eta, \xi)}{p_2(u, \eta, \xi)} = \eta^2\xi^2\frac{P_1[X; \eta, \xi]}{P_2[X; \eta, \xi]}\ , \tag{3.8}$$

and then

$$H_\mu(X[1-\tau^2]) \tag{3.9}$$
$$\equiv -\frac{1}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1}\frac{d\eta}{\eta}\frac{d\xi}{\xi}\frac{P_1[X;(1-\tau^2);\eta,\xi]}{P_2[X;(1-\tau^2);\eta,\xi]}N_\mu(\eta,\xi)\eta\xi\,.$$

It has been shown in Gilbert [9], that the integral (3.9) may be evaluated in terms of *Weierstrass integrals* of the first, second and third kind [18]. (see also [1], [16], [10]).

With this in mind, let us define the families of sets,

$$\mathfrak{S}_1^2(\tau) \equiv \{X \mid p_0[X(1-\tau^2);\eta] \equiv 0;\ \forall\eta\}$$
$$\mathfrak{S}_2^2(\tau) \equiv \left\{X \,\middle|\, R\left(P_2, \frac{\partial P_2}{\partial\xi}\right) \equiv (-1)\frac{n(n-1)}{2}\left(p_0^{2n-1}V[X(1-\tau^2)];\eta\right) = 0\,\forall\eta\right\}$$
$$\mathfrak{S}_3^2(\tau) \equiv \left\{X \,\middle|\, \forall\eta,\ R\left(V, \frac{\partial V}{\partial\eta}\right) = 0\right\} \cup \left\{X \,\middle|\, R\left(p_0, \frac{\partial p_0}{\partial\eta}\right) = 0,\ \forall\eta\right\},$$
$$\mathfrak{S}^2(\tau) \equiv \mathfrak{S}_1^2(\tau) \cup \mathfrak{S}_2^2(\tau) \cup \mathfrak{S}_3^2(\tau)\,, \tag{3.10}$$

where

$$V(X;\eta) \equiv \prod_{0\le\alpha<\beta\le\eta}[A_\beta(X;\eta) - A_\alpha(X,\eta)]$$

is the Vandermonde determinant associated with the equation,

$$P_2[X;\eta,\xi] \equiv \sum_{\nu=0}^{n} p_\nu[X;\eta]\xi^{n-\nu} = 0\,, \tag{3.11}$$

and $R(f, g)$ is the *resultant* of f and g.

The genus $\rho[X(1-\tau^2)]$ of the Riemann surface $\mathfrak{R}[X(1-\tau^2)]$ which lies over the η plane and is associated with the algebraic equation $P_2[X(1-\tau^2);\eta,\xi] = 0$ (τ-fixed) is constant for

$$\forall X \in (E^4 - \mathfrak{S}^2(\tau)\}\,.$$

Furthermore, let

$$H_\alpha[X(1-\tau^2);\eta,\xi],\ \tilde{H}_\alpha[X(1-\tau^2);\eta,\xi],\ H[X(1-\tau^2);\eta_\nu,\xi_\nu,\eta,\xi] \tag{3.12}$$

be Weierstrass integrands of the first, second, and third kind respectively, associated with the Riemann surface $\mathfrak{R}[X(1-\tau^2)]$, and let their periods taken over $\rho[X(1-\tau^2)]$ cycles $K_\beta[X(1-\tau^2)]$ be given by

$$2\omega_{\alpha\beta}[X(1-\tau^2)],\ 2\eta_{\alpha\beta}[X(1-\tau^2)],\ \Omega_\beta[X(1-\tau^2)] \tag{3.13}$$
$$(\beta = 1, 2, \cdots, p),\ (\alpha = 1, 2, \cdots, p)\,,$$

and their periods over the conjugate cycles $\tilde{K}_\beta[X(1-\tau^2)]$ be given by

$$2\tilde{\omega}_{\alpha\beta}[X(1-\tau^2)],\ 2\tilde{\eta}_{\alpha\beta}[X(1-\tau^2)],\ \tilde{\Omega}_\beta[X(1-\tau^2)]\,. \tag{3.14}$$

For each fixed value of X, and τ we may evaluate the integral

(3.9) by first integrating with respect to ξ,

$$H_\mu[X(1-\tau^2)] = \frac{1}{2\pi i}\sum_{\sigma=1}^{n}\int_{\mathscr{L}_\sigma(\tau)} \frac{\eta\xi_\sigma P_1[X(1-\tau^2);\eta,\xi_\sigma]}{\frac{\partial}{\partial\xi}P_2[X(1-\tau^2);\eta,\xi_\sigma]} N_\mu(\eta,\xi_\sigma)d\eta \tag{3.15}$$

(where $\mathscr{L}_\sigma(\tau)$ is that subset of $\{|\eta|=1\}$ for which the root $\xi_\sigma \equiv A_\sigma[X(1-\tau^2);\eta]$ lies inside $\{|\xi|=1\}$, and then by using the Weierstrass decomposition theorem [18], [1], [10] to write,

$$\begin{aligned} &\frac{\eta\xi P_1[X(1-\tau^2);\eta,\xi]N_\mu(\eta,\xi)}{\frac{\partial}{\partial\xi}P_2[X(1-\tau^2);\eta,\xi]} \\ &\quad = \sum_{\nu=1}^{r} C_{\mu\nu}[X(1-\tau^2)]H_\mu[X(1-\tau^2);\eta_\nu,\xi_\nu;\eta,\xi] \\ &\quad - \sum_{\alpha=1}^{\rho}\{\tilde{g}_{\mu\alpha}[X(1-\tau^2)]H_{\mu\alpha}[X(1-\tau^2);\eta,\xi] - g_{\mu\alpha}[X(1-\tau^2)] \\ &\qquad\qquad \times \tilde{H}_{\mu\alpha}[X(1-\tau^2);\eta,\xi]\} \\ &\quad + \frac{\partial}{\partial\eta}\left\{\sum_{\nu=1}^{r} F_{\mu\nu}[X(1-\tau^2);\eta,\xi]\right\},\ \sum_{\nu=1}^{r} C_{\mu\nu}[X(1-\tau^2)] = 0\,, \end{aligned} \tag{3.16}$$

where $\rho \equiv \rho[X(1-\tau^2)]$, $r \equiv r[X(1-\tau^2)]$ is the number of infinity points of the integrand (4.15), and the $F_\nu[X(1-\tau^2);\eta,\xi]$ are rational functions of η,ξ. Using Theorem 1 from Gilbert [9] we then may express $H_\mu[X(1-\tau^2)]$ in the form,

$$\begin{aligned} H_\mu[X1-\tau^2)] &= \sum_{\sigma=1}^{\eta}\sum_{\lambda=1}^{k_\sigma}\Bigg\{\sum_{\nu=1}^{r} C_{\mu\nu}[X(1-\tau^2)] \\ &\qquad \times \log\frac{E_\mu[X(1-\tau^2);\eta_\sigma^{(2\lambda)},\xi_\sigma^{(2\lambda)};\eta_\nu,\xi_\nu;\eta_0,\xi_0]}{E_\mu[X(1-\tau^2);\eta_\sigma^{(2\lambda)},\xi_\sigma^{(2\lambda)};\eta_\nu,\xi_\nu;\eta_0,\xi_0]} \\ &+ \sum_{\alpha=1}^{\rho}\Bigg(\tilde{C}_{\mu\alpha}[X(1-\tau^2)]\log\frac{E_{\mu\alpha}[X(1-\tau^2);\eta_\mu^{(2\lambda)},\xi_\mu^{(2\lambda)}]}{E_{\mu\alpha}[X(1-\tau^2);\eta_\mu^{(2\lambda-1)},\xi_\mu^{(2\lambda-1)}]} \\ &\quad - C_{\mu\alpha}[X(1-\tau^2)]\log\frac{\tilde{E}_{\mu\alpha}[X(1-\tau^2);\eta_\mu^{(2\lambda)},\xi_\mu^{(2\lambda)}]}{\tilde{E}_{\mu\alpha}[X(1-\tau^2);\eta_\mu^{(2\lambda-1)},\xi_\mu^{(2\lambda-1)}]}\Bigg) \\ &+ \sum_{\nu=1}^{r}(F_{\mu\nu}[X(1-\tau^2);\eta_\mu^{(2\lambda)},\xi_\mu^{(2\lambda)}] - F_{\mu\nu}[X(1-\tau^2);\eta_\mu^{(2\lambda-1)},\xi_\mu^{(2\lambda-1)}])\Bigg\} \end{aligned} \tag{3.17}$$

Here we have assumed that $X \notin \mathfrak{S}^2(\tau)$ for $\forall\tau \in [0,1]$.

We next introduce the families of sets,

$$\mathscr{M}^3(\tau) \equiv \{X \mid V[X(1-\tau^2);\eta]=0;\ \forall\eta\in|\eta|=1\}\,,$$
$$\mathscr{N}^3(\tau) \equiv \{X \mid P_0[X(1-\tau^2);\eta]=0;\ \forall\eta\in|\eta|=1\}\,;$$

for each fixed $X \notin \mathfrak{S}^2(\tau)\exists$ only a finite set $\{\tau_k\}$

$$\subset(0,1)\ni X\in(\mathscr{M}^3(\tau_k)\cup\mathscr{N}^3(\tau_k))\,, \qquad (k=1,2,\cdots,N[X])\,.$$

Let us define the intervals on (0, 1),

$$\varDelta_k[X] \equiv \{\tau \mid \tau_{k-1} < \tau < \tau_k\} ,$$

where we take $\tau_0 \equiv 0$, and $\tau_n \equiv 1$; for $\forall\tau\exists\ \tau \in \varDelta_k[X]_k$ (where X is fixed and $\notin \mathfrak{S}^2(\tau)$) there can be no poles of $\eta\xi P_1 N_\mu/\partial P_2/\partial\xi$ (defined on the Riemann surface associated with $P_2 = 0$), which coincide with the path of integration $|\eta| = 1$. Another way of saying this is that,

$$X \notin \bigcup_{\tau\in\varDelta_k} \mathscr{M}^3(\tau) \qquad (\text{for } k = 1, 2, \cdots, N[X]) ,$$

hence for each k, the number of poles, $r_k[X]$ inside $\{|\eta| = 1\}$ remains constant for all $\tau \in \varDelta_k[X]$. We designate these values of η as the set of points

$$N_k[K] = \left\{\overset{(k)}{\eta_\nu}[X]\right\}_{\nu=1}^{r} , \qquad r = r_k[X] . \tag{3.18}$$

From the definition of the set $\{\tau_\kappa\}$ it also follows that

$$X \notin \bigcup_{\tau\in\varDelta_k} \mathscr{N}^3(\tau) \qquad (k = 1, 2, \cdots, N[X]) ,$$

hence the Riemann surface

$$\mathbf{R}[X(1-\tau^2)] \quad \text{associated with} \quad P_2[X(1-\tau^2); \eta, \xi] = 0$$

[equation (3.11)] has constant genus for all $\tau \in \varDelta_k[X]$. Consequently, our evaluation of $\varPsi_\mu[X]$ is made by computing a term (3.17) for each τ-interval $\varDelta_k[X]$, where

$$\rho \equiv \rho[X(1-\tau^2)] = \rho_k \qquad (\text{a constant}) \text{ for } \qquad \tau \in \varDelta_k[X] ,$$
$$r \equiv r[X(1-\tau^2)] = r_k \qquad (\text{a constant}) \text{ for } \qquad \tau \in \varDelta_k[X] ,$$

and summing the integrals

$$\varPsi_\mu[X] = 2 \sum_{k=1}^{N(X)} \int_{\tau_{k-1}}^{\tau_k} E(r, \tau) H_\mu[X(1-\tau^2)] d\tau \tag{3.19}$$

We now return to evaluate the integral (3.7), where $\mathfrak{X}'$ is a closed curve lying on the hypersphere $\|X\| = R$,

$$\mathfrak{X}' \equiv \{X \mid x_\mu = x_\mu(s); 0 \leqq s \leqq 1\} ,$$

and $\mathfrak{Y}'$ is the image of $\mathfrak{X}'$ under the mapping, $y_\mu = (1-\tau^2)x_\mu$, for τ fixed, and $0 \leqq \tau \leqq 1$. The integration involved on the right-hand-side of (4.7) is then over the two-dimensional region,

$$\mathfrak{Y}^2 \equiv \{Y \mid y_\mu = (1-\tau^2)x_\mu(s);\ 0 \leqq s \leqq 1,\ 0 \leqq \tau \leqq 1\} \tag{3.20}$$

We assume further, that $\mathfrak{X}'$ is chosen such that for τ-fixed

$$\mathfrak{T}' \cap \{\mathscr{M}^3(\tau) \cup \mathscr{N}^3(\tau)\}$$

is a discreet set of points $\{X_k(\tau)\}_{k=1}^m$; consequently, as τ varies the points $X_k(\tau)$ trace out curves Γ'_k. This construction (see Mitchell [16]) suggests a subdivision of the s-interval such that for

$$s \in (s_\mu, s_{\mu+1}),\ [0, 1] = \sum_\mu [s_{\mu-1}, s_\mu]$$

there are a constant number of intersections of the curves Γ'_k ($k = 2, \cdots, M$) with the $ray\{\tilde{X} \mid \tilde{X} = (1-\tau^2)X(s); 0 \leqq \tau \leqq 1,\ s \text{ fixed}\}$. $\mathfrak{T}'_\mu$ is that subset of $\mathfrak{T}'$ for which $s \in (s_{\mu-1}, s_\mu)$, and N_μ is the number of τ-sub-interals $[0, 1]$ is broken into such that for $\tau \in \Delta_k[X]$, $X \in \mathfrak{T}'_\mu$, the number of poles r_{μ_k} and the genus ρ_{μ_k} will remain constant. This leads us to consider the line interval broken up into the sum,

$$\sum_{\alpha=1}^{m} \sum_{k=1}^{N_\alpha} \int_{X_\alpha}^{X_{\alpha+1}} \left\{ \int_{\tau_{k-1}}^{\tau_K} E(r, \tau) H_\mu^{(\alpha)}[X(1-\tau^2)] d\tau \right\} dx_\mu \,; \tag{3.21}$$

the superscript α in the integrand indicates, that for each $(s_{\alpha-1}, s_\alpha)$ interval we must perform a different Weierstrass decomposition (3.17).

The end points s_μ are found to be those s-values for which Γ'_k coincides with $\mathfrak{F}^2 \cap \{s = s_\mu\}$ for a finite number of τ-subintervals.

4. Considerations from the theory of algebraic surfaces. Using (3.1) the line integral (3.5) may be written as

$$\begin{aligned}
\int_{\mathfrak{T}'} \Psi_\mu[X] dx_\mu &= 2\int_{\mathfrak{T}'} dx_\mu \int_0^1 E(r, \tau) H_\mu[X(1-\tau^2)] d\tau \\
&= -\frac{1}{2\pi^2} \int_{\mathfrak{T}'} dx_\mu \int_0^1 E(r, \tau) \int_{|\eta|=1} \int_{|\xi|=1} \frac{d\eta}{\eta} \frac{d\xi}{\xi} f\big(u(1-\tau^2), \eta, \xi\big) \\
&\qquad \times N_\mu(\eta, \xi) d\tau \\
&= -\frac{1}{2\pi^2} \int_0^1 \frac{E(r, \tau)}{1-\tau^2} \int_{|\eta|=1} \int_{|\xi|=1} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \int_{\mathfrak{C}^1(\eta, \xi, \tau)} f(v, \eta, \xi) dv \,;
\end{aligned} \tag{4.1}$$

$\mathfrak{C}^1(\eta, \xi, \tau)$ is the image of $\mathfrak{T}'$ under the map $v = u[X; \eta, \xi](1-\tau^2)$ where η, ξ, and τ are held fixed, and the integrand is assumed absolutely integrable.

If $f(u, \eta, \xi) = p_1(u, \eta, \xi)/p_2(u, \eta, \xi)$ is rational, then we may evaluate (3.22) by the theory of algebraic surfaces [19] [17]. For instance, let us consider the algebraic surface $\mathfrak{S}$ defined by

$$P_2(u, \eta, \xi) = \sum_{\nu=0}^{q} a_\nu(\eta, \xi) u^{q-\nu} \,, \tag{4.2}$$

where the $a_\nu(\eta, \xi)$ are polynomials in η, ξ and q is the degree of u in p_2. Furthermore, if $p_3 = (\partial/\partial u)P_2(u, \eta, \xi)$, then the resultant of p_2, p_3 with respect to u is [12],

$$
(4.3)\qquad R(p_2, q_3) = \begin{vmatrix}
a_0 & 0 & 0 & qa_0 & 0 \\
a_1 & a_0 & & (q-1)a_1 & 0 \\
 & a_1 & & & \\
 & & 0 & & \\
 & & a_0 & a_{q-1} & 0 \\
a_q & a_{q-1} & a_1 & 0 & qa_0 \\
0 & a_q & & & (q-1)a_1 \\
 & 0 & & & \\
 & & & 0 & \\
0 & 0 & a_q & 0 & a_{q-1}
\end{vmatrix}
$$

(which is also the discriminant of p_2 with respect to u); the u-roots of $p_2 = 0$, $u = u_\nu(\eta, \xi)$, may be found by computing

$$
(4.4)\qquad u = \frac{\partial R}{\partial a_0} \Big/ \frac{\partial R}{\partial a_1} = \frac{\partial R}{\partial a_1} \Big/ \frac{\partial R}{\partial a_2} = \text{etc.}
$$

$P_2(u, \eta, \xi)$ may be expressed as factors of its u-roots as

$$
p_2(u, \eta, \xi) = a_q \prod_{\nu=1}^{q} [u - u_\nu(\eta, \xi)] .
$$

The roots $u_\nu(\eta, \xi)$ $(\nu = 1, 2, \cdots, q)$ are distinct for all

$$
(\eta, \xi) \notin \{(\eta, \xi) \mid R(p_2, p_3) = 0\} .
$$

For each fixed value of τ we may determine a set of values (η, ξ) for which $u_\nu(\eta, \xi)$ lies inside $\mathfrak{C}^1(\eta, \xi, \tau)$ and the set for whicn the root lies outside. For instance, setting $\eta = e^{i\alpha}$, $\xi = e^{i\beta}$, and using the parametrization $x_\mu = x_\mu(s)$, $(0 \leqq s \leqq 1)$ we may determine when the root u_ν lies on $\mathfrak{C}^1$ by taking the real and imaginary parts of the equation,

$$
(4.5)\qquad u\{s; \alpha, \beta\} \equiv u[X(s); e^{i\alpha}, e^{i\beta}](1 - \tau^2) = u_\nu(e^{i\alpha}, e^{i\beta}) = u_\nu\{\alpha_j \beta\}
$$

and then eliminating s to obtain $\alpha = \Phi_j^{(\nu)}(\beta; \tau)$, $(k = 1, 2, \cdots, N)$. This gives us a subdivision of the α-interval, say,

$$
0 < \alpha_1^{(\nu)} < \cdots < \alpha_{2\mu}^{(\nu)} < \alpha_{2\mu+1}^{(\nu)} < \cdots < 1 \quad (\alpha_{2\mu} \equiv \Phi_{2\mu}^{(\nu)}(\beta, \tau))
$$

such that for $\alpha \in \Delta_\mu^{(\nu)}(\beta, \tau) \equiv (\alpha_{2\mu}^{(\nu)}, \alpha_{2\mu+1}^{(\nu)})$ the root u_ν lies inside $\mathfrak{C}^1$. A further subdivision may then be made such the same set of roots $\{u_\nu\}\nu \in I(\beta, \tau)$ lie inside $\mathfrak{C}^1$ for all $\alpha \in \Delta_\mu(\beta, \tau)$. Consequently we have for (4.1).

$$
(4.6)\qquad \frac{-1}{2\pi^2} \int_0^1 \frac{E(r, \tau)}{1 - \tau^2} \sum_{\nu \in I(\beta, \tau)} \int_{\mathscr{L}(\xi, \tau)} \int_{|\xi|=1} \frac{p_1(v_\nu, \eta, \xi)}{p_3(v_\nu, \eta, \xi)} \frac{d\xi}{\xi} \frac{d\eta}{\eta} d\tau
$$

Here $\mathscr{L}(\xi, \tau)$ is the union of certain closed subintervals on the unit η-circle,

$$\mathscr{L}(\xi, \tau) = \bigcup_{\nu} \{\eta = e^{i\alpha}; \alpha \in \Delta^{(\nu)}(\beta, \tau)\} \tag{4.7}$$

Next, we wish to obtain a subdivision of the s, and τ intervals similar to the one obtained in the previous section. Essentially, the problem is the same, to find the subintervals $(\tau_\mu, \tau_{\mu+1})$ and $\mathfrak{F}^1_\nu(\tau) \equiv (Y_\nu(\tau), Y_{\nu+1}(\tau))$, where $Y_\nu(\tau)$ are end points on the curve,

$$\mathfrak{F}^1(\tau) = \{Y \mid y_\mu = (1 - \tau^2)x_\mu(s);\ 0 \leqq s \leqq 1; \tau\text{–fixed}\}\ , \tag{4.8}$$

such that the Γ^1_k intersects $\mathfrak{F}^1_\nu$ in a constant number of points.[2] We consider a point $Y(\tau) \in \mathfrak{F}^1_\nu(\tau)$ for $\tau \in (\tau_\mu, \tau_{\mu+1})$. If $Y(\tau) \in \Gamma^1_k$, then $Y(\tau) \in \mathscr{M}^3(\tau) \cup \mathscr{N}^3(\tau)$; returning to our definition for $\mathscr{M}^3(\tau)$ and $\mathscr{N}^3(\tau)$, we realize that this implies the existence of an $\eta(\tau)$ with $|\eta(\tau)| = 1$, such that either $Y(\tau) \in \mathscr{M}^3[\tau; \eta(\tau)]$ or $Y(\tau) \in \mathscr{N}^3[\tau; \eta(\tau)]$.[3] Now, in order to find the τ-subdivision we note that $Y(\tau)$ either intersects $\mathscr{M}^3$ (or $\mathscr{N}^3$) in a finite number of points (as τ-varies) or in a finite number of τ-subintervals.[4] In the latter case, this gives us [one of the s-points of subdivision in the previous section] a $Y(\tau)$-point subdivision or as we have shown above a point of subdivision of the integration path $|\eta| = 1$. The τ-interval is now subdivided so that there are a constant number or intersections in each subinterval. One obtains then the following sum of integrals for (4.6).

$$\frac{-1}{2\pi^2} \sum_{\lambda=0}^{N} \sum_{\mu=0}^{N_\lambda} \sum_{\nu \in I_{\lambda\mu}(\mathfrak{T})} \int_{\tau_{\lambda-1}}^{\tau_\lambda} \frac{E(r, \tau)}{1 - \tau^2} \int_{\mathscr{L}_\mu(\tau)} \int_{|\xi|=1} \frac{p_1(v_\nu, \eta, \xi)}{p_3(v_\nu, \eta, \xi)} \frac{d\xi}{\xi} \frac{d\eta}{\eta} d\tau \tag{4.9}$$

over $d_2(v, \eta, \xi) = 0$. $I_{\lambda\mu}(\mathfrak{T})$ is the index set of those roots, v, of $p_2 = 0$ which lie inside $\mathfrak{C}^1(\eta, \xi, \tau)$ where the others remain outside; $\bigcup_{\lambda=0}^{N}(\tau_{\lambda-1}, \tau_\lambda) = [0, 1]$ is the subdivision of the τ-interval and $\eta_\mu(\tau)$ $(\mu = 1, 2, \cdots, N_\lambda)$ are the points on $|\eta| = 1$ which correspond to the N_λ, Y-points on either $\mathscr{M}^3(\tau)$ or $\mathscr{N}^3(\tau)$ for $\tau \in (\tau_{\lambda-1}, \tau_\lambda)$.

The cartesian product $\{|\xi| = 1\} \times \mathscr{L}_\mu(\tau)$ (for fixed τ) is a *tubular chain* [16] on $\mathfrak{L}$. On $\mathfrak{L}$ $p_1(v, \eta, \xi)/p_3(v, \eta, \xi)$ may be represented as

$$\frac{p_1}{p_2} = \frac{p_1 C_2(v, \eta, \xi)}{R(p_2, p_3)}\ , \tag{4.10}$$

where $R(p_2, p_3) = C_1(v, \eta, \xi)p_2 + C_2(v, \eta, \xi)p_3$, C_1, C_2 are polynomials [19]

---

[2] This construction of subintervals is due to Mitchell **[16]** who employed it in the three-variable case.

[3] $\mathscr{M}^3[\tau; \eta(\tau)]$ is a restriction of $\mathscr{M}^3(\tau)$; see definition of $\mathscr{M}^3(\tau)$.

[4] In the definition of $\mathscr{M}^3(\tau)$, and $\mathscr{N}^3(\tau)$ the ξ-variable does not appear, hence it plays no part in our subdivision. Because of this we may make a similar subdivision as in the three-variable case **[16]**.

[10], and v is given by (5.4). If we designate the polar curves D_σ on $\mathfrak{L}$ by

$$D_\sigma \equiv \left\{g_\sigma(\eta, \xi) = 0;\ v = \frac{\partial R}{\partial a_{\sigma-1}}\Big/\frac{\partial R}{\partial a_\sigma}\right\}$$

then we may rewrite (5.9) as,

$$(4.11)\quad -\frac{1}{2\pi^2}\sum_{\lambda=0}^{N}\sum_{\mu=0}^{N_\lambda}\sum_{\nu\in I_{\lambda\mu}(\mathfrak{T})}\int_{\tau_{\lambda-1}}^{\tau_\lambda}\frac{E(r,\tau)}{1-\tau^2}\int_{\mathscr{L}_\mu(\tau)}\int_{|\xi|=1}\frac{p_1(v_\nu,\eta,\xi)C_2(v_\nu,\eta,\xi)}{\Phi(\xi)\prod_{\sigma=1}^{k} g_\sigma(\eta,\xi)^{\alpha_\sigma}}d\xi d\eta\,,$$

which may be reduced by partial fractions and subtraction of certain improper integrals [19] [10] to the form,

$$(4.12)\quad -\frac{1}{2\pi^2}\sum_{\lambda=0}^{N}\sum_{\mu=0}^{N_\lambda}\sum_{\nu\in I_{\lambda\mu}(\mathfrak{T})}\int_{\tau_{\lambda-1}}^{\tau_\lambda}\frac{E(r,\tau)}{1-\tau^2}\int_{\mathscr{L}_\mu(\tau)}\sum_{\sigma=1}^{k}\frac{V_\sigma(v_\nu,\eta,\xi)}{\frac{\partial g_\sigma}{\partial\xi}(\eta,\xi)}d\xi d\eta\,.$$

The functions,

$$W_\sigma(v,\eta,\xi) = \frac{V_\sigma(v,\eta,\xi)}{\frac{\partial g_\sigma(\eta,\xi)}{\partial\xi}}\,,$$

are algebraic functions defined on the Riemann surface $\mathfrak{R}_\sigma$ associated with the polar curve D_σ; consequently $W_\sigma(v, \eta, \xi)$ has a Weierstrass decomposition such as illustrated in (3.16)

$$(4.13)\quad \begin{aligned} W_\sigma(v,\eta,\xi) = &\sum_{\alpha=1}^{r_\sigma} d_{\sigma\alpha}H_\sigma(\eta_{\sigma\alpha},\xi_{\sigma\alpha};\eta,\xi) \\ &-\sum_{\beta=1}^{p_\sigma}\{\tilde{h}_{\sigma\beta}H_{\sigma\beta}(\eta,\xi) - h_{\sigma\beta}\tilde{H}_{\sigma\beta}(\eta,\xi)\} \\ &+\frac{d}{d\eta}\left\{\sum_{\alpha=1}^{r_\sigma}F_{\sigma\alpha}(\eta,\xi)\right\}\,, \qquad \sum_{\alpha=1}^{r_\sigma} d_{\sigma\alpha} = 0\,, \end{aligned}$$

where the terms have a similar meaning as before in (3.16). Each $\mathscr{L}_\mu(\tau)$ is the union of intervals on the unit η-circle, $\mathscr{L}_\mu(\tau) \equiv \bigcup_{i=1}^{m_\mu}(\eta_{2i}^{(\mu)}(\tau), \eta_{2i+1}^{(\mu)}(\tau))$, hence using a result from Theorem (2) of Gilbert [10] we have for (5.9)

$$(4.14)\quad \begin{aligned} -\frac{1}{2\pi^2}&\sum_{\lambda=0}^{N}\sum_{\mu=0}^{N_\lambda}\sum_{\nu\in I_{\lambda\mu}(\mathfrak{T})}\int_{\tau_{\lambda-1}}^{\tau_\nu}\frac{E(r,\tau)}{1-\tau^2} \\ &\times\left\{\sum_{\sigma=1}^{k}\sum_{i=1}^{m_\mu}\left[\sum_{\alpha=1}^{r_\sigma} d_{\sigma\alpha}\log\frac{E_\sigma(\eta_{2i+1}^{(\sigma)},\xi_{2i+1}^{(\sigma)};\eta_{\sigma\alpha},\xi_{\sigma\alpha};\eta_{\sigma0},\xi_{\sigma0})}{E_\sigma(_{2i}^{(\sigma)},\xi_{2i}^{(\sigma)};\eta_{\sigma\alpha},\xi_{\sigma\alpha};\eta_{\sigma0},\xi_{\sigma0})}\right.\right. \\ &+\sum_{\beta=1}^{p_0}\left(\tilde{C}_{\sigma\beta}\log\frac{E_{\sigma\beta}(\eta_{2i+1}^{(\sigma)},\xi_{2i+1}^{(\sigma)})}{E_{\sigma\beta}(\eta_{2i}^{(\sigma)},\xi_{2i}^{(\sigma)})} - C_{\sigma\beta}\log\frac{\tilde{E}_{\sigma\beta}(\eta_{2i+1}^{(\sigma)},\xi_{2i+1}^{(\sigma)})}{\tilde{E}_{\sigma\beta}(\eta_{2i}^{(\sigma)},\xi_{2i}^{(\sigma)})}\right) \\ &\left.\left.+\sum_{\alpha=1}^{r_\sigma}(F_{\sigma\alpha}(\eta_{2i+1}^{(\sigma)},\xi_{2i+1}^{(\sigma)}) - F_{\sigma\alpha}(\eta_{2i}^{(\sigma)},\xi_{2i}^{(\sigma)}))\right]\right\}\,. \end{aligned}$$

From the above discussion we have proved the following theorem, which is the four-variable analogue of results by Bergman [1] [5] and later Mitchell [16].

THEOREM 4.1. *Let $\Psi_\mu(X) = \Omega_\mu[f]$ be the μth component of a vector solution to $T_4[\Psi] = 0$, where $f(u, \eta, \xi)$ is a rational B_4-associate of a harmonic function $H[X]$. Let $\mathfrak{X}$ be a simple, closed curve on the hypersphere $\|X\| = r$ meeting the above discussed conditions. Then the integral of* (3.21),

$$\sum_{\alpha=1}^{M}\sum_{k=1}^{N_\alpha}\int_{X_\alpha}^{X_{\alpha+1}}\left\{\int_{\tau_{k=1}}^{\tau_k} E(r, \tau)H_\mu^{(\alpha,k)}[X(1-\tau^2)]\,d\tau\right\}dx_\mu$$
$$\equiv \sum_{\alpha=1}^{M}\sum_{k=1}^{N_\alpha}\int_{X_\alpha}^{X_{\alpha+1}} \Psi_\mu^{(\alpha,k)}(X)\,dx_\mu$$

may be evaluated in terms of an expression (4.14).

REFERENCES

1. S. Bergman. *Integral Operators in the Theory of Linear Partial Differential Equations*, Ergeb. Math. u. Grenzgeb, **23**, Springer, Berlin (1960).

2. ———, *Zur Theorie der ein-und mehrwertigen hamonischen Funktionen des dreidimensional Raumes*, Math. z. **24** (1926), 641-669.

3. ———, *Zur Theorie der algebraischen Potential Funktionen des dreidimensional Raumes*, Math. Ann., **99** (1929), 629-659; ibid., Vol. **101** (1929), 534-538.

4. ———, *Some properties of a harmonic function of three variables given by its series development*, Arch. Rat. Mech. Anal., **8** (1961), 207-222.

5. ———, *Multivalued harmonic functions in three variables*, Comm. Pure and Appl. Math., Vol. IX, No. 3, (1956), 327-338.

6. ———, *Integral operators in the study of an algebra and a coefficient problem in the theory of three-dimensional harmonic functions*, Duke Math. J. **30** No. 3, (1963) 447-460.

7. R. P. Gilbert, *Singularities of solutions to the wave equation in three dimensions.*, J. Reine u. Angew. Math. **205** (1960), 75-81.

8. ———, *On harmonic functions of four variables with rational P_4-associates*, Pacific J. Math., **13** (1963), 79-96.

9. ———, *Harmonic functions in four variables with algebraic and rational P_4-associates*, to appear in Annales Polonici Mathematici (1964).

10. ———, *Multivalued harmonic functions of four variables*, to appear in Journal d'Analyse Mathématique.

11. S. Bergman, *Operators generating solutions of certain differential equations in three variables and their properties*, Scripta Math., **26** (1961), 5-31.

12. W. S. Burside, and A. W. Panton, *Theory of Equations* II Longmans, Green and Co., London, (1904).

13. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functiod II*, McGraw-Hill, New York (1953).

14. E. Kreyszig, *Kanonische Integral Operatoren zur Erzeugung Harmonischer Funktionen von vier Vergenderlichen*, Archiv. der Math. XIV, (1963), 193-203.

15. V. Maric'. *On some properties of solutions of $\Delta_3\Psi + A(r^2)X\nabla\Psi + C(r^2)\Psi = 0$, to appear* in Pacific J. Math.

16. J. Mitchell, *Representation theorems for solutions of linear partial differential equations in three variable*, Arch. Rat. Mech. Anal., **3**, No. 5, (1959), 439–459.
17. E. Picard and G. Simart, *Theorie des Fonctions Algebrique de deux variables indépendentes*, Vol. I, II, Gauthiers-Villars et Fils, Paris, (1897).
18. K. Weierstrass, *Vorlleslngen über die Theorie der Abelschen Transcendenten*, Gasammelte Werke, IV, Mayer and Mayer, Berlin, (1902).
19. O. Zariski, *Algebraic Surfaces*, Chelsea, New York (1948).

UNIVERSITY OF MARYLAND