MEASURABLE SETS OF MEASURES

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1. Introduction. Let M be the set of all countably additive, finite, signed measures on a σ-field Σ of subsets of a set X. There is a natural definition of measurability in M, namely, a subset of M is *measurable* if it is an element of Σ^*, the smallest σ-field of subsets of M such that: for each $A \in \Sigma$ the function $\mu \to \mu(A)$ is measurable from M to the Borel line. The purpose of this note, motivated by questions arising from (Dubins and Freedman, 1963) is to investigate the measurability and category of interesting subsets of M, under the assumption that Σ is countably generated.

Here are some results. If X is compact metric, and Σ is the σ-field of Borel subsets of X, then any subset of X with the Baire property is measurable for a residual set of probability measures (3.17). If also X is uncountable, there are weakly open, but not Σ^*-measurable, subsets of M; see (3.2). There is a G_δ in the three-dimensional unit cube whose convex hull is not Borel (3.22). If F is a continuous, strictly monotone, purely singular distribution function on the unit interval, then F is differentiable only on a set of the first category (4.8).

2. The abstract case. Let X be a nonempty set, $\mathscr{F}$ a countable field of subsets of X, and Σ the smallest σ-field including $\mathscr{F}$.

2.1. *Let $\mathscr{A}$ be a σ-field of subsets of a set Ω, and let φ map Ω into M. Then φ is measurable from $(\Omega, \mathscr{A})$ to (M, Σ^*) if and only if the function $\omega \to \varphi(\omega)(A)$ is measurable from $(\Omega, \mathscr{A})$ to the Borel line for each $A \in \mathscr{F}$.*

Proof. Routine.

2.2. *If φ is a measurable map from $(\Omega, \mathscr{A})$ to (M, Σ^*), and f is a bounded, measurable function from $(\Omega \times X, \mathscr{A} \times \Sigma)$ to the Borel line, then $\omega \to \int_X f(\omega, x)\varphi(\omega)(dx)$ is a measurable function from $(\Omega, \mathscr{A})$ to the Borel line.*

Proof. Extend from indicators of measurable rectangles.

2.3. *The σ-field Σ^* is countably generated.*

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Proof. Use (2.1).

2.4. *For each $\mu \in M$, the set $\{\mu\}$ is measurable.*

Proof. $\{\mu\} = \bigcap_{A \in \mathscr{F}} \{\nu \mid \nu \in M,\ \nu(A) = \mu(A)\}$.

2.5. *If φ_1 and φ_2 are measurable maps from $(\Omega, \mathscr{A})$ to (M, Σ^*), then so are $\varphi_1 + \varphi_2$ and $c\varphi_1$ for any real number c, and $\{\omega \mid \omega \in \Omega,\ \varphi_1(\omega) = \varphi_2(\omega)\} \in \mathscr{A}$.*

Proof. Use (2.1) for the first two assertions, and (2.4) for the third.

2.6. *If $\mu \in M$, and $A \in \Sigma$, for any $\delta > 0$ there is a set $A(\mu, \delta) \in \mathscr{F}$ whose symmetric difference with A has μ-measure less than δ.*

Proof. (Halmos 1958, Theorem D, page 56.)

Let M^+ be the set of nonnegative measures on (X, Σ).

2.7. *The set of nonnegative measures is measurable; so is the set of probability measures.*

Proof. $M^+ = \bigcap_{A \in \mathscr{F}} \{\mu \mid \mu \in M,\ \mu(A) \geqq 0\}$.

If $\mathscr{A}$ is a σ-field of subsets of the set Ω and $W \subset \Omega$, then $W\mathscr{A}$ is the σ-field of subsets of W having the form $W \cap A$, $A \in \mathscr{A}$. Abbreviate $M^+\Sigma^*$ to Σ^+. Recall that $\mu \in M$ is the difference of two unique, nonnegative, mutually singular measures μ^+ and μ^-. Let $|\mu| = \mu^+ + \mu^-$ and $\|\mu\| = |\mu|(X)$.

2.8. THEOREM. *The maps $\mu \to \mu^+$ and $\mu \to \mu^-$ are measurable from (M, Σ^*) to (M^+, Σ^+).*

Proof. By (2.5), it is enough to check the first assertion. By (2.6), if $A \in \Sigma$, then $\mu^+(A) = \sup\{\mu(A \cap B) : B \in \mathscr{F}\}$. Hence $\mu \to \mu^+(A)$ is measurable and (2.1) applies.

2.9. *The map $\mu \to |\mu|$ is measurable from (M, Σ^*) to (M^+, Σ^+). The function $\mu \to \|\mu\|$ is measurable from (M, Σ^*) to the Borel line.*

Proof. (2.5) and (2.8) imply the first assertion, and it implies the second.

Recall that for μ and ν in M there are two unique elements $S(\mu, \nu)$ and $A(\mu, \nu)$ of M with:

(i) $\mu = S(\mu, \nu) + A(\mu, \nu)$;

(ii) $S(\mu, \nu)$ and ν are mutually singular;
(iii) $A(\mu, \nu)$ is absolutely continuous with respect to ν.

2.10. THEOREM. *The maps* $S\colon (\mu, \nu) \to S(\mu, \nu)$ *and* $A\colon (\mu, \nu) \to A(\mu, \nu)$ *are measurable from* $(M \times M, \Sigma^* \times \Sigma^*)$ *to* (M, Σ^*).

Proof. If $\mu, \nu \in M^+$ and $A \in \Sigma$, then $S(\mu, \nu)(A) = \lim_{n\to\infty} \sup\{\mu(A \cap B) : B \in \mathscr{F}, \nu(B) < n^{-1}\}$, by (2.6) and (Halmos, 1958, Theorem *B*, page 125). By (2.1), S restricted to $M^+ \times M^+$ is $\Sigma^+ \times \Sigma^+$-measurable; apply (2.8) and (2.5).

2.11. *The set* of (μ, ν) *in* $M \times M$ *with* μ *absolutely continuous with respect to* ν *is in* $\Sigma^* \times \Sigma^*$, *as are the set of pairs* (μ, ν) *with* μ *equivalent to* ν, *and the set with* μ *and* ν *mutually singular.*

Proof. Use (2.10) and (2.5).

Recall that an *atom* of Σ is a nonempty Σ-measurable set with no proper nonempty Σ-measurable subset. Write $\alpha(\Sigma)$ for the collection of atoms of Σ. It $\mu \in M$, then $\{A\colon A \in \alpha(\Sigma)$ and $\mu(A) > 0\}$ is countable. If this set of atoms is empty, μ is *continuous*; if $\|\mu\| = \Sigma\{|\mu(A)| : A \in \alpha(\Sigma)\}$, then μ is *atomic*. Any $\mu \in M$ is the sum of a unique atomic $\mu_a \in M$ and a unique continuous $\mu_c \in M$.

2.12. THEOREM. *The maps* $\mu \to \mu_a$ *and* $\mu \to \mu_c$ *are measurable from* (M, Σ^*) *to* (M, Σ^*).

Proof. As usual, it suffices to verify that, for a fixed $A \in \Sigma$, the function $\mu \to \mu_c(A)$ is Σ^+-measurable on M^+. For this purpose, let $\{\Pi_n\colon n = 1, 2, \cdots\}$ have the following properties:

(i) each Π_n is a partition of A into a finite number of elements of Σ;

(ii) Π_{n+1} is a refinement of Π_n;

(iii) $A\Sigma$ is the smallest σ-field of subsets of A which includes $\bigcup_n \Pi_n$. For $\delta > 0$, let $\varphi_{n,\delta}(\mu) = \Sigma\{\mu(B) : B \in \Pi_n, \mu(B) < \delta\}$. Clearly, $\varphi_{n,\delta}$ is Σ^+-measurable on M^+, and increases to a Σ^+-measurable function φ_δ on M^+ as n increases to ∞. As δ decreases to 0 through a fixed sequence, φ_δ decreases to a Σ^+-measurable function φ on M^+. The argument will be completed by showing that $\varphi(\mu) = \mu_c(A)$ for $\mu \in M^+$. If $A_n \in \Pi_n$ and $A_n \supset A_{n+1}$ for $1 \leqq n < \infty$, then $\bigcap_{n=1}^\infty A_n$ is empty or an atom of Σ, and in either case has μ_c-measure 0. The famous lemma of König (1936, Theorem 6, page 81) then implies $\lim_{n\to\infty} \max\{\mu_c(B) : B \in \Pi_n\} = 0$; so $\varphi_\delta(\mu) \geqq \mu_c(A)$. For the reverse inequality, if $\varepsilon > 0$ there is a positive δ so small that $\Sigma\{\mu_a(B) : B \in \alpha(\Sigma), B \subset A, \mu_a(B) \leqq \delta\} < \varepsilon$, which implies $\varphi_\delta(\mu) \leqq \mu_c(A) + \varepsilon$.

2.13 *Both the set of atomic measures and the set of continuous measures are measurable.*

Proof. (2.12) and (2.5).

2.14. *The set* G *of probability measures with* $\max\{\mu(A) : A \in \alpha(\Sigma)\} > 9/10$ *is measurable. Let* g *be any function from* G *to* X *such that, for all* $\mu \in G$, *the* μ*-measure of the* Σ*-atom containing* $g(\mu)$ *is greater than* $9/10$. *Then* g *is* $G\Sigma^*$*-measurable.*

Proof. Adapt the argument for (2.12).

3. The compact metric case. If Ω is a topological space, $\sigma(\Omega)$ means the smallest σ-field of subsets of Ω which includes the topology. In this section, X is a nonempty compact metric space, and Σ is $\sigma(X)$, the σ-field of Borel subsets of X. According to a famous theorem of Riesz, M can be identified with the dual of $C(X)$, the Banach space of all continuous real-valued functions on X with the sup norm: $\|f\| = \max\{|f(x)| : x \in X\}$. Unless otherwise noted, M has the weak * topology; and subsets of M have the relative weak * topology.

3.1. *The smallest* σ*-field including the weak * topology of* M *is* Σ^*; *that is,* $\Sigma^* = \sigma(M)$.

Proof. Easy.

Let P be the set of probability measures on (X, Σ). Then P is a compact metrizable subset of M^+; and M^+ is a closed subset of M. It is less widely known that M^+ is metrizable; it is complete and separable in this metric:

$$\rho^*(\mu, \nu) = \sum_{j=1}^{\infty} 2^{-j} \|f_j\|^{-1} \left| \int f_j d\mu - \int f_j d\nu \right|,$$

where $\{f_j : 1 \leqq j < \infty\}$ is dense in $C(X)$. Thus $\Sigma^+ = \sigma(M^+)$ is the Borel σ-field of M^+, and $P\Sigma^* = \sigma(P)$ is the Borel σ-field of P.

3.2. THEOREM. *If* X *is uncountable, there is a weakly open subset of* M *which is not* Σ^**-measurable.*

Proof. Let N be a nonanalytic subset of X, and $E = \{\mu : \mu \in M, \mu\{x\} > 9/10$ for some $x \in N\}$. Then E is weakly open. If EP were an analytic subset of P, then—using the notation and result of (2.14) $EP \subset G$ and $N = g(EP)$ would be analytic, a contradiction.

Recall that the *support* $C(\mu)$ of $\mu \in M$ is the smallest closed subset K of X with $|\mu|(X - K) = 0$. It is familiar that a closed subset E of X includes $C(\mu)$ if and only if $\int f d\mu = 0$ for each $f \in C(X)$ vanishing on E.

3.3. *If $\mu_n \in M$, $1 \leqq n < \infty$ and $\mu_n \to \mu \in M$, then $C(\mu)$ is a subset of the closure of $\bigcup_{n=1}^{\infty} C(\mu_n)$.*

Proof. Easy.

Let 2^X be the space of nonempty closed subsets of X, endowed with the usual compact metric topology (Hausdorff, 1927, Section 28).

3.4. *If M_s is a metrizable subset of M and does not contain the zero measure, the restriction of C to M_s is lower semi-continuous in the sense of (Kuratowski, 1932, page 148).*

Proof. Use (3.3).

Let M_0 be the set of nonzero elements of M.

3.5. *The map C is measurable from $(M_0, \sigma(M_0))$ to $(2^X, \sigma(2^X))$.*

Proof. M_0 is the countable union of metrizable sets. Then use (3.4) and (Kuratowski, 1932, page 152).

3.6. *For each $K \in 2^X$, the set of probability measures whose support is K is a G_δ in P.*

Proof. Use (3.4) and (Kuratowski, 1932, page 151).

3.7. *The set of nonnegative measures whose supports have no isolated points is an $F_{\sigma\delta}$ in M^+, as is the set of nonnegative measures whose supports have no interior.*

Proof. Since the set of perfect, nonempty subsets of X is a G_δ in 2^X, as is the set of closed, nowhere dense, nonempty subsets, (3.4) and (Kuratowski, 1932, page 152) apply.

3.8 *The real-valued function $(\mu, K) \to \mu(K)$ is upper semi-continuous on $M^+ \times 2^X$ with the product topology.*

Proof. Endow $C(X)$ with the norm topology. There is a natural embedding of 2^X into $C(X)$: assign to $K \in 2^X$ the function $\hat{K} \in C(X)$ whose value at $x \in X$ is

$$1 - [(\text{distance from } x \text{ to } K)/(\text{diameter of } X)] .$$

As is easily verified, $K \to \hat{K}$ is continuous (and 1-1, although this will not be used); moreover, $\hat{K}^m$ decreases pointwise to the indicator of K as n increases to ∞. Since the function $(\mu, f) \to \int f d\mu$ is continuous on $M^+ \times C(X)$, the functions $(\mu, K) \to \int \hat{K}^n d\mu$ are continuous on $M^+ \times 2^X$. This sequence decreases pointwise to the function $(\mu, K) \to \mu(K)$ as n increases to ∞.

3.9. *The function* $(\mu, K) \to \mu(K)$ *is measurable from* $(M \times 2^X, \sigma(M) \times \sigma(2^X))$ *to the Borel line.*

Proof. Use (2.8) and (3.8).

3.10. *The function* $(\mu, \nu) \to \mu[C(\nu)]$ *is measurable from* $[M \times M, \sigma(M) \times \sigma(M)]$ *to the Borel line.*

Proof. Use (3.5) and (3.9).

3.11. *The set of* (μ, K) *in* $M^+ \times 2^X$ *with* $\mu(K) = 0$ *is a* G_δ.

Proof. Use (3.8).

3.12. *For each dense subset* G *of* X *the set of* μ *in* P *with* $\mu(G) = 1$ *is dense in* P.

Proof. Approximate $\mu \in P$ by a finite linear combination of point masses.

3.13. *The set* P_+ *of* μ *in* P *assigning positive probability to all nonempty open subsets of* X *is a dense* G_δ *in* P.

Proof. For each open subset V of X, $\{\mu\colon \mu \in P, \mu(V) = 0\}$ is closed. Let $\{V_n\colon 1 \leqq n < \infty\}$ be a basis for the topology of X. Then $P - P_+$ is $\bigcup_{n=1}^{\infty}\{\mu\colon \mu \in P, \mu(V_n) = 0\}$, an F_σ. Plainly, P_+ is dense.

3.14. *The set of continuous* μ *in* P *is a* G_δ. *It is dense in* P *if and only if* X *has no isolated points.*

Proof. For the first assertion, if $\delta > 0$, then $\{\mu\colon \mu \in P$, and $\mu\{x\} \geqq \delta$ for some $x \in X\}$ is closed. For the second, if X has no isolated points, then each open subset of X has cardinality c and supports a continuous $\mu \in P$. The converse is easy.

3.15. *If G is a dense G_δ in X, then the set G_1 of μ in P with $\mu(G) = 1$ is a dense G_δ in P.*

Proof. Let $\{U_n: 1 \leqq n < \infty\}$ be open sets whose intersection is G. Then $G_1 = \bigcap_{n=1}^\infty \bigcap_{j=1}^\infty \{\mu: \mu \in P, \mu(U_n) > 1 - j^{-1}\}$, and (3.12) applies.

Any superset of a dense G_δ is *residual*. The complement of a residual set is of the *first category*. A set not of the first category is of the *second category*.

3.16. *If F is of the first category in X, then F has outer measure* 0 *for a residual set of μ in P.*

Proof. (3.15).

Recall that $B \subset X$ has the *property of Baire* if there is an open subset of X whose symmetric difference with B is of the first category. For a discussion, see (Kuratowski, 1958, Section 11). If X is uncountable and $\mu \in P$, there are μ-measurable sets without the property of Baire; if μ is continuous, there are sets with the property of Baire whose inner μ-measure is 0, and whose outer μ-measure is 1. According to (Kuratowski, 1958, pages 421–423), there is a subset of X which is μ-measurable for no continuous $\mu \in P$. There is, however, a connection between measurability and the property of Baire:

3.17. THEOREM. *If B is of the second category in X and has the property of Baire, then B is μ-measurable and of positive μ-measure for a residual set of μ in P.*

Proof. B differs from a nonempty open set by a set of the first category. Apply (3.16) and (3.13).

3.18. *If $\mu \in P$ and either X has no isolated points or μ is continuous, then there is a dense G_δ in X of μ-measure* 0.

Proof. As in (Halmos, 1958, (4) on page 66).

3.19. *The set $P_\perp$ of pairs (μ, ν) with μ and ν mutually singular is a G_δ in $P \times P$. It is dense if and only if X has no isolated points.*

Proof. Let $\{f_n : 1 \leqq n < \infty\}$ be dense in the unit ball of $C(X)$, and let $F_n(\mu, \nu) = \max_{1 \leqq j \leqq n} \left| \int f_j d\mu - \int f_j d\nu \right|$. Then F_n is continuous on $P \times P$ for each n, and the sequence $\{F_n\}$ increases pointwise to $F: (\mu, \nu) \rightarrow \|\mu - \nu\|$. So F is lower semi-continuous, and $P_1 = F^{-1}\{2\}$ is a G_δ. For the second assertion, use (3.12) in one direction, and (3.13) in the other.

3.20. THEOREM. *For each μ in P, the set $\mu_\perp$ of ν in P singular with respect to μ is a G_δ. If X has no isolated points or μ is continuous, then $\mu_\perp$ is dense in P.*

Proof. $\mu_\perp$ is a G_δ by (3.19), and dense by (3.12), (3.18).

There are reasonable sets of probability measures which are not Borel. A first example.

3.21. *If X is uncountable, the set of probability measures with uncountable support is analytic but not Borel; the set of probability measures with countably infinite support is analytic but not Borel.*

Proof. As reported in (Kuratowski and Szpilrajn, 1932, pages 166–169), the set of uncountable closed subsets of X is analytic but not Borel in 2^X. To obtain the first assertion in (3.21), apply (Kuratowski and Szpilrajn, 1932, Proposition IV, page 163). The second follows from the first, because the set of probability measures whose support has k points or fewer is closed, for every natural number k.

A second example: it is natural to guess that the convex hull of a Borel set is Borel, especially since this happens to be true in two-dimensional Euclidean space. However,

3.22. THEOREM. *There is a G_δ of the unit cube in three-dimensional Euclidean space whose convex hull is not Borel.*

Proof. Let A be a G_δ of the unit square whose projection A^* on the x-axis is not Borel. Let $A_n = \{(x, y) : 0 \leqq x \leqq 1, -\infty < y < \infty, (x, y - n) \in A\}$, and $A_\infty = \bigcup_{n=-\infty}^{\infty} A_{2n}$. Let f be a homeomorphism of $(0, 1)$ onto $(-\infty, \infty)$, and let $B = \{(x, y)\colon 0 \leqq x \leqq 1, 0 < y < 1, (x, f(y)) \in A_\infty\}$. For any ε in $(0, 1)$, the projection of $B \cap \{(x, y)\colon 0 \leqq x \leqq 1, 0 \leqq y \leqq \varepsilon\}$ or of $B \cap \{(x, y)\colon 0 \leqq x \leqq 1, 1 - \varepsilon \leqq y \leqq 1\}$ onto the x-axis is A^*.

Let φ map the unit square into the unit cube by $\varphi(x, y) = \{x, y, 1/2 - [1/4 - (x - 1/2)]^{\frac{1}{2}}\}$. Thus φ maps the unit square homeomorphically onto a half-cylinder C, and $\varphi(B)$ is a G_δ. If its convex hull H were Borel, then $\varphi^{-1}(C \cap H)$ would be Borel. Also the section of $\varphi^{-1}(C \cap H)$ by the line $y = 1/2$, $0 \leqq x \leqq 1$, namely A^* translated upward by 1/2, would be Borel, a contradiction.

4. The unit interval. In this section, X is the closed unit interval.

4.1. *The set of μ in P with well-ordered support is complementary analytic but not Borel in P.*

Proof. The set of closed, nonempty, well-ordered subsets of X is complementary analytic but not Borel in 2^X (Kuratowski and Szpilrajn, 1932, page 166).

We conjecture that the set of probabilities whose support has a given order-type is Borel, but have verified this only for well-ordered order-types. More generally, for any compact metric space X, the collection of elements of 2^X homeomorphic to a fixed $K \in 2^X$ may be Borel. These conjectures have been confirmed in: Dana Scott, Invariant Borel sets, Fund. Math. **41** (1964) C. Ryll-Nardjewski, On Borel measurability of orbits, to appear ————————, On Freedman's problem, to appear.

Other questions arise from differentiation. For each x in $[0, 1)$ and real-valued function f on $[0, 1)$, the *upper and lower right derivatives* of f at x are

$$f^*(x) = \limsup_{y \to 0+} y^{-1}[f(x + y) - f(x)]$$

and

$$f_*(x) = \liminf_{y \to 0+} y^{-1}[f(x + y) - f(x)] .$$

The next main result is (4.5). For the preliminaries (4.2)–(4.4), let $A \in \Sigma^*$ and for $\mu \in A$ suppose the real-valued function f_μ on $[0, 1)$ is continuous, and for each $x \in [0, 1)$ the function $\mu \rightarrow f_\mu(x)$ is measurable from $(A, \sigma(A))$ to the Borel line.

4.2. *The function $(x, \mu) \rightarrow f_\mu^*(x)$ is measurable from $\{[0, 1) \times A, \sigma[0, 1) \times \sigma(A)\}$ to the extended Borel line.*

Proof. By a familiar argument, the function $(x, \mu) \rightarrow f_\mu(x)$ is measurable; and

$$f_\mu^*(x) = \lim_{n \to \infty} \sup \left\{ \frac{f_\mu(x + r) - f_\mu(x)}{r} : 0 < r < n^{-1},\ r \text{ rational} \right\} .$$

4.3. *If $0 \leqq a < b < 1$, then the functions $S_{[a,b)} : \mu \rightarrow \sup \{f_\mu^*(x)$: $a \leqq x < b\}$ and $I_{[a,b)} : \mu \rightarrow \inf \{f_\mu^*(x)$: $a \leqq x < b\}$ are measurable from*

$(A, \sigma(A))$ to the extended Borel line.

Proof. By a theorem of Dini (Saks, 1937, page 204),

$$S_{[a,b)}(\mu) = \sup\left\{\frac{f_\mu(y) - f_\mu(x)}{y - x}: a \leqq x < y < b\right\},$$

where x and y can be restricted to rational values. An identical argument shows that $I_{[a,b)}$ is measurable.

4.4. *The set A_1 of $\mu \in A$ with f_μ continuously differentiable on $[0, 1)$ is measurable.*

Proof. By the same result of Dini, f_μ is continuously differentiable on $[0, 1)$ if and only if f_μ^* is continuous there. Hence $A_1 = \bigcap_{j=1}^\infty B_j$, where B_j is the set of $\mu \in A$ with $-\infty < I_{[0,1-2^{-j})}(\mu) \leqq S_{[0,1-2^{-j})}(\mu) < \infty$ and

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq 2^n - 2^{n-j}} \{S_{[(k-1)/2^n, k/2^n)}(\mu) - I_{[(k-1)/2^n, k/2^n)}(\mu)\} = 0 .$$

If $\mu \in M$, its *distribution function* F_μ is defined as $F_\mu(x) = \mu[0, x]$ for $x \in X$.

4.5. THEOREM. *The set C_k of $\mu \in M$ whose distribution function F_μ has a kth continuous derivative $F_\mu^{(k)}$ on $[0, 1)$ is measurable; and the function $(x, F_\mu^{(k)}) \to F_\mu^{(k)}(x)$ is measurable from $\{[0, 1) \times C_k, \sigma[0, 1) \times \sigma(C_k)\}$ to the Borel line.*

Proof. For $k = 0$, use (2.12). Then apply (4.4) and (4.2) inductively.

If a real-valued function f on $[0, 1)$ is infinitely differentiable there, let $S_n(f, x_0, x) = \sum_{j=0}^n f^{(j)}(x_0)/j!\ (x - x_0)^j$. Then f is *analytic* on $[0, 1)$ if each $x_0 \in [0, 1)$ has a neighborhood $N(x_0)$ in $[0, 1)$ on which $S_n(f, x_0, \cdot)$ converges uniformly to f.

4.6. THEOREM. *The set of μ in M with F_μ analytic on $[0, 1)$ is measurable.*

Proof. The set $C_\infty = \bigcap_{k=1}^\infty C_k$ is measurable, and for each $x_0 \in X$, the function $(\mu, x) \to S_n(F_\mu, x_0, x)$ is measurable from $\{C_\infty \times [0, 1), \sigma(C_\infty) \times \sigma[0, 1)\}$ to the Borel line. If J is an interval, then $R_{n,J,x_0}: \mu \to \sup_{x \in J, 0 \leqq x < 1} | F_\mu(x) - S_n(F_\mu, x_0, x)|$ is measurable on $(C_\infty, \sigma(C_\infty))$, since x can be restricted to rational values. Therefore, the set $A(x_0, J)$ of $\mu \in C_\infty$ with $\lim_{n\to\infty} R_{n,J,x_0}(\mu) = 0$ is in $\sigma(C_\infty)$. The set of μ with F_μ analytic on $[0, 1)$ is

$$\bigcap_{h=1}^{\infty} \bigcap_{k=h}^{\infty} \bigcap_{j=0}^{2^k-2^{k-h}-1} A(j/2^k, [(j-1)/2^k, (j+1)/2^k]) .$$

Let Pr be the set of μ in P which are continuous, singular with respect to Lebesgue measure, and assign positive measure to all nonempty open sets.

4.7. *The set Pr is a dense G_δ in P.*

Proof. (3.14), (3.20), (3.13).

According to (Saks, 1937, Chapter IV), if $\mu \in Pr$, then F_μ is differentiable with derivative 0 on a set of Lebesgue measure 1, and differentiable with derivative ∞ on a set of μ-measure 1. Topologically speaking, however, F_μ is differentiable essentially nowhere:

4.8 THEOREM. *The set of pairs (x, μ) with $F_\mu^*(x) = \infty$ and $F_{\mu*}(x) = 0$ is a G_δ in $[0, 1) \times Pr$. Each of its section is dense.*

Proof. Let W be the set of pairs (x, μ) in $[0, 1) \times Pr$ with $F_{\mu*}(x) = 0$, and W^* the set with $F_\mu^*(x) = \infty$. It is enough to prove that W and W^* are G_δ's with dense sections.

The complement of W in $[0, 1) \times Pr$ is $\bigcup_{n=1}^{\infty} C_n$, where C_n is the intersection over all rational s in $[0, 1)$ of $\{(x, \mu)\colon 0 \leqq x < 1,\ \mu \in Pr,$ and either $x \geqq s$ or $F_\mu(s) - F_\mu(x) \geqq n^{-1}(s - x)\}$. Since each C_n is closed, W is a G_δ. Being disjoint from the dense set on which F_μ has zero derivative, the section of C_n by μ in Pr has no interior. The section of C_n by x in $[0, 1)$ has no interior because, for μ in Pr, arbitrarily small translates modulo 1 of μ have distribution functions with 0 derivative at x. Then the sections of W are dense according to Baire's category argument.

The similar proof for W^* is omitted.

There is, of course, an analogous theorem for derivatives from the left.

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