PSEUDO-FRATTINI SUBGROUPS

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1. Introduction. During the current interest in Frattini subgroups, $\Phi(G)$, of finite groups, G, two well-defined characteristic subgroups have been overlooked. They are the intersection of the normal maximal subgroups, $R(G)$, and the intersection of the self-normalizing maximal subgroups, $L(G)$; in each case one sets $R(G) = G$ or $L(G) = G$ if the respective maximal subgroups do not exist properly. This paper examines the properties of these subgroups and introduces an upper L-series which is defined as $L_0 = L(G)$, $L_1 = [L(G), G], \cdots$, $L_j = [L_{j-1}, G], \cdots$; its role being analogous to the upper central series. The terminal member of an L-chain, $L^*(G)$, is called an L-commutator. Whenever $L^*(G) = 1$, $L(G)$ coincides with the hypercenter of G. In conclusion it is shown that a group having $L^*(G) = 1$ and having all subgroups of $G/\Phi(G)$ the direct product of elementary Abelian p-group is equivalent to a group having each proper subgroup nilpotent.

It is assumed that the reader is familiar with the definitions and properties of ascending and descending central series, nilpotent groups, and Frattini subgroups. Moreover, the following properties (see Gaschütz [2]) will also be used.

P1. If N is a normal subgroup of a group G and $N \leqq \Phi(U)$, for a subgroup U of G, then $N \leqq \Phi(G)$.

P2. If N is a normal subgroup of a group G and T is the normalizer in G of a Sylow p-subgroup P of N, then $G = NT$.

P3. If $N \leqq \Phi(G)$ is a normal subgroup of G, then $\Phi(G/N) = \Phi(G)/N$.

All groups will be assumed finite.

2. Pseudo-Frattini subgroups. A maximal subgroup of a group is either normal or self-normalizing, i.e., a subgroup which coincides with its normalizer. Moreover, the two classes remain invariant under an automorphism of the group.

DEFINITION 2.1. For a group G denote by $R(G)$ the intersection of the normal maximal subgroups (defining $R(G) = G$ if no normal maximal subgroups exist), and denote by $L(G)$ the intersection of the self-normalizing maximal subgroups (defining $L(G) = G$ if no self-normalizing maximal subgroups exist).

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Clearly $R(G)$ and $L(G)$ are characteristic subgroups and $\Phi(G) = R(G) \cap L(G)$. One notes that $R(G)$ contains the derived group of G and that $L(G)$ contains the center of G; the containment need not be proper.

It is seen that $R(G) \leqq L(G)$ implies that G is nilpotent, i.e., $G = L(G)$. However, it is possible that $L(G) \leqq R(G)$ without implying that the group is perfect, i.e., $G = R(G)$. Moreover $R(G) = 1$ implies that G is an elementary Abelian p-group and if $L(G) = 1$, then G is centerless. In either case G is a Φ-free group (a group whose Frattini subgroup is the identity). For a discussion of the Φ-free groups, one is referred to the papers of Gaschütz [2] and Zacher [5].

THEOREM 2.1. *For each homomorphism θ of a group G,*
(i) $L(G)\theta \leqq L(G\theta)$,
(ii) $R(G)\theta \leqq R(G\theta)$, *and*
(iii) $\Phi(G)\theta \leqq \Phi(G\theta)$.

Proof. It is sufficient to note that if K is the kernel of the homomorphism G, then $L(G\theta)$ corresponds to the intersection of the self-normalizing maximal subgroups of G containing K. The other parts follow similarly.

Examples can be found to show that in general equality will not hold for any one of the parts in Theorem 2.1.

For the group G, denote the terminal member of the descending central series, the *hypercommutator*, by $D(G)$ and the terminal member of the ascending central series, the *hypercenter*, by $Z^*(G)$.

THEOREM 2.2. *In a group G*
(i) $D(G) \leqq R(G)$,
(ii) $Z^*(G) \leqq L(G)$,
(iii) $L(G) \cap [G, G] \leqq \Phi(G)$,
(iv) $[L(G), G] \leqq \Phi(G)$,
(v) $[G, G] \cap Z^*(G) \leqq \Phi(G)$,
(vi) $[G, G] \cap Z(G) \leqq \Phi(G)$,
(vii) $\Phi(G) = 1$ *implies* $L(G) = Z(G) = Z^*(G)$, *and*
(viii) $L(G)/\Phi(G) = Z(G/\Phi(G)) = Z^*(G/\Phi(G))$.

Proof. Since $[G, G] \leqq R(G)$, (i) is evident. On the other hand it is known that the hypercenter is contained in each self-normalizing subgroup and so (ii) results from this. For (iii) note that $L(G) \cap [G, G] \leqq R(G)$. The normality of $L(G)$ together with (iii) implies (iv), then (ii) and (iii) imply (v). (vi) is just a consequence of $Z(G) \leqq Z^*(G)$.

As for (vii), note that $\Phi(G) = 1$ implies $[L(G), G] = 1$, $L(G) \leqq Z(G)$ and so by (iii), $L(G) = Z(G)$. Finally (viii) is a consequence of **P3**, in the introduction, together with (vii).

It is known that $L(G)$ is nilpotent (see Gaschütz [2] and Deskins [1]) and this can be proven directly. However, the next theorem permits this and several other known results as corollaries.

THEOREM 2.3. *Consider the normal subgroup N of a group G and denote its hypercommutator by $D(N)$ then either*
(i) $D(N) = 1$ *or*
(ii) $D(N) \not\leqq \Phi(G)$.

Proof. Suppose $D(N) \leqq \Phi(G)$ and denote a Sylow p-subgroup of N by P for a fixed prime p. Then $D(N)\cdot P/D(N)$ is a characteristic subgroup of $N/D(N)$ and hence normal in $G/D(N)$. This in turn implies that $D(N) \cdot P$ is normal in G. Consequently by **P2**, in Section 1, if T is the normalizer of P in G, then $G = D(N) \cdot P \cdot T = D(N)\cdot T = T$, i.e. P is normal in G. So P is normal in N. This holds for each prime p and it follows that N is nilpotent, i.e. $D(N) = 1$. From this, one concludes either $D(N) = 1$ or $D(N) \not\leqq \Phi(G)$.

COROLLARY 2.3.1. *The following properties are equivalent for the normal subgroup N of G:*
(i) *N is nilpotent,*
(ii) *$N/N \cap \Phi(G)$ is elementary Abelian,*
(iii) *$N/N \cap \Phi(G)$ is Abelian, and*
(iv) *$N/N \cap \Phi(G)$ is nilpotent.*

Proof. Since N is nilpotent, $[N, N] \leqq \Phi(N) \leqq \Phi(G)$, by **P1**, and thus (ii) is a consequence of (i). Clearly (ii) implies (iii) which in turn implies (iv). Then (iv) implies (i) since $D(N) \leqq N \cap \Phi(G)$.

COROLLARY 2.3.2. *If S and T are normal subgroups of a group G and $S \leqq T$ such that T/S is nilpotent, then $S \leqq \Phi(G)$ implies that T is nilpotent.*

COROLLARY 2.3.3. *$L(G)$ is nilpotent.*

Proof. It is sufficient to apply (viii) in Theorem 2.2.

THEOREM 2.4. *A necessary and sufficient condition that a subgroup H of G be contained in $L(G)$ is that $[H, G] \leqq \Phi(G)$.*

Proof. Suppose H is a subgroup of G having $[H, G] \leqq \Phi(G)$. If $H \leqq \Phi(G)$ then $\Phi(G)\cdot H/\Phi(G) \leqq Z(G/\Phi(G)) = L(G)/\Phi(G)$ i.e. $H \leqq L(G)$. Similarly for an arbitrary subgroup of G. Conversely if $H \leqq L(G)$, then $[H, G] < [L(G), G] \leqq \Phi(G)$. This completes the proof.

COROLLARY 2.4.1. *Let H be a subgroup of G. Then $H \leqq \Phi(G)$ if and only if*

(i) $H \leqq R(G)$ *and*

(ii) $[H, G] \leqq \Phi(G)$.

Proof. By the theorem, $[H, G] \leqq \Phi(G)$ implies $H \leqq L(G)$ and then (i) leads to $H \leqq \Phi(G)$. The converse is evident.

In the group $G = \{a, b \mid a^5 = b^4 = a^3bab^3 = 1\}$, $\Phi(G) = 1$ whereas $\Phi(\{b\}) = \{b^2\} \neq 1$. So in general $\Phi(H)$ is not necessarily contained in $\Phi(G)$ whenever H is an arbitrary subgroup of G. Of course, if H is normal in G, then $\Phi(H)$ is contained in $\Phi(G)$ upon application of **P1**.

THEOREM 2.5. *If H is a subgroup of G, then $\Phi(H) \leqq R(H) \leqq R(G)$.*

Proof. One must show that $R(H) \leqq R(G)$. This is trivial if $R(G) = G$. Hence assume $R(G) \neq G$ and let H denote a normal maximal subgroup of G. Then either $H \leqq N$ and $R(H) \leqq N$, or $H \not\leqq N$. For the latter note that $N \cap H$ is a normal maximal subgroup of H. Therefore $R(H) \leqq N \cap H \leqq N$. Combining the two cases one obtains $R(H) \leqq R(G)$ for an arbitrary subgroup H of G and the result follows.

COROLLARY 2.5.1. *A necessary and sufficient condition that $\Phi(H) \leqq \Phi(G)$, for an arbitrary subgroup H of G, is that $[\Phi(H), G] \leqq \Phi(G)$.*

Proof. Combine Theorem 2.5 with Corollary 2.4.1.

A counterexample to an analogue for $L(G)$ and $L(N)$ for a subgroup N of a group G is provided by the symmetric group on three symbols, S_3, $L(S_3) = 1$ and $L(A_3) = A_3$, A_3 the alternating subgroup. Moreover for each subgroup H of order two, $L(H) = H \neq 1$. Consequently there is no proper subgroup T of S_3 for which $L(T) \leqq L(S_3)$.

3. L-series. This section will be given to defining an L-series and to the development of its properties.

DEFINITION 3.1. For a group G

(i) an *L-series* is a series $L(G) = B_0 \geqq B_1 \geqq \cdots B_j \geqq \cdots$ such that B_j is normal in G and $B_i/B_{i+1} \leqq Z(G/B_{i+1})$ i.e. $[B_i, G] \leqq B_{i+1}$ for $i = 0, 1, 2, \cdots$,

(ii) an *upper L-series* is a series $L(G) = L_0 \geqq L_1 \geqq \cdots \geqq L_j \geqq \cdots$ in which $L_j = [L_{j-1}, G]$, for $j = 1, 2, \cdots$, and

(iii) a *lower L-series* is a series $1 = Z_0 \leqq Z_1 \leqq \cdots Z_j \leqq \cdots$ in which $Z_j/Z_{j-1} = Z(G/Z_{j-1})$, for $j = 1, 2, \cdots$, (i.e. a lower L-series is precisely the ascending central series.)

The terminal member of the lower L-series is the hypercenter $Z^*(G)$ whereas we will call the terminal member of the upper L-series the *L-commutator* and designate it by $L^*(G)$, i.e. there exists an integer j such that $L_0 > L_1 > \cdots > L_j > L^*(G) = L_{j+1} = L_{j+2} = \cdots$.

The immediate properties of the L-series are the following:

(i) The L-series are a normal series.

(ii) The upper and lower L-series are a characteristic series.

(iii) An upper L-series coincides with the descending central series if and only if G is nilpotent.

(iv) For an integer j, $j \geqq 1$, $L_j \leqq \Phi(G)$. Since $L_j \leqq [\Phi(G), G] \leqq \Phi(G)$, $j \geqq 1$, the result follows.

As for the L-commutator, one has these properties:

(i) $L^*(G) \leqq \Phi(G)$.

This follows since $L_j \leqq \Phi(G)$ for all integers $j \geqq 1$ and in the case $L^*(G) = L(G)$, $L(G) = \Phi(G)$.

(ii) $L^*(G)$ is nilpotent.

(iii) $L^*(G)$ contains no subgroup H, H normal in G, such that $L^*(G)/H \leqq Z(G/H)$.

(iv) For a homorphism G of a group G, $(L^*(G)\theta \leqq L^*(G\theta)$.

It is sufficient to note that $(L(G)\theta \leqq L(G\theta)$.

Then by the same method of proof as used in showing the relationship between the elements of the central series of a nilpotent group with the elements of either the descending or ascending central series (see M. Hall, [3], p. 151) one has

THEOREM 3.1. *In a group G possessing an L-series $L(G) = B_0 \geqq B_1 \cdots \geqq B_k = 1$, $L_i \leqq B_i$, $i = 0, 1, \cdots, k$* and *$B_{k-j} \leqq Z_j$, $j = 0, 1, \cdots, k-1$.*

COROLLARY 3.1.1.

(i) *If there exist integers j and k such that $L_j \leqq Z_k$, then $L(G) = Z^*(G)$ and $k = n - j$ where $L_n = 1$.*

(ii) *If $L(G) = Z^*(G)$, the upper and lower L-series have the same length.*

(iii) *$L(G) = Z^*(G)$ if and only if $L^*(G) = 1$.*

COROLLARY 3.1.2. *The following statements are equivalent for the group G.*

(i) *An L-series of G terminates with the identity of G.*

(ii) *The lower L-series of G terminates with $L(G)$.*

(iii) $L^*(G) = 1$.

Note that (i) of Corollary 3.1.1 cannot be extended to include $Z_k < L_j$ for the same pair of integers j and k in order to obtain a similar result. Even if $Z^*(G) < L^*(G)$, all that can be said is that $G/Z^*(G)$ is centerless. For an example consider the group $G = \{a, b \mid a^9 = b^2 = baba = 1\}$. In this group $\Phi(G) = L(G) \neq 1$ whereas $Z(G) = Z^*(G) = 1$.

As is known, the hypercenter can be defined as the intersection of the self-normalizing subgroups and this can even be improved to the intersection of the normalizers of the Sylow p-subgroups of the group G for each prime $p \mid ord\ (G)$. Hence on the basis of Corollary 3.1.1 it appears that the latter may be the best possible "intersection property" of $Z^*(G)$ since $Z^* = L(G)$ if and only if $L^*(G) = 1$.

THEOREM 3.2. *If N is a normal subgroup of a group G having $L^*(G) = 1$, then $L^*(N) = 1$.*

Proof. If N is nilpotent, the result is trivially true. Since G possesses an L-series, then $L(G) = Z^*(G)$. Thus if $\Phi(N) = 1$, then $L(N) = Z(N) = Z^*(N)$. On the other hand since $\Phi(N) \leqq \Phi(G) \leqq Z^*(G)$, the element x of order p in $\Phi(N)$ commutes with all elements of G of order prime to p and so with those in N. This is because x is an element of the hypercenter $Z^*(G)$ and hence $x \in Z^*(N)$. Therefore $\Phi(N) \leqq Z^*(N)$ which in turn implies $Z^*(N) = L(G)$.

COROLLARY 3.2.1. *If N is a normal subgroup of G and $L(G) = Z^*(G)$, then $L(N) = Z^*(N)$.*

As the symmetric group on three symbols clearly indicates, the existence of an L-series does not imply $L(N) \leqq L(G)$.

Moreover the example preceding Theorem 3.2 points out that the converse is not true.

THEOREM 3.3. *A necessary and sufficient condition that $L^*(G) = 1$ for a group G is that for each prime p, $p \mid ord\ (G)$, the p-component P of $\Phi(G)$ commutes with the elements of the Sylow q-subgroups Q of G, $p \neq q$.*

Proof. Clearly if $L(G) = Z^*(G)$, the relationship exists from out of the properties of the hypercenter. On the other hand, suppose that the p-components of $\Phi(G)$ commute with the elements of the Sylow q-subgroups of Q of G for $p \neq q$. Then again $\Phi(G) \leqq Z^*(G)$ which implies $Z^*(G) = L(G)$.

4. **Property** (NN). Since simple groups have $L^*(G) = 1$ trivially, it is evident that in general this property may not alone enlighten us on further structural properties of the group unless additional conditions are included. Analogous to a nilpotent group, let us define a group property as follows:

DEFINITION 4.1. A group G has property (NN), $G \in (NN)$, provided that

(i) $L^*(G) = 1$ and

(ii) each subgroup of $G/\Phi(G)$ is the direct product of elementary Abelian p-groups.

It should be noted that $L^*(G) = 1$ does not alone imply nilpotency of the proper subgroups, e.g. the alternating group on five symbols. Nor does the condition that each proper subgroup of $G/\Phi(G)$ is the direct product of elementary Abelian p-groups imply the same result, e.g. the group $G = \{a, b \mid a^9 = b^2 = ba\ ba = 1\}$.

THEOREM 4.1. *A group G has property (NN) if and only if each proper subgroup of G is nilpotent.*

Proof. The theorem is obviously true when G is nilpotent and so for the remainder of the proof we will assume that G is not nilpotent.

First suppose $G \in (NN)$ and let H be a proper subgroup of G. Then consider the quotient $\Phi(G) \cdot H/\Phi(G) = H/H \cap \Phi(G)$. If $H \cdot \Phi(G) = G$ one has $G = H$ and hence a contradiction to H being a proper subgroup of G. Thus $H/H \cap \Phi(G)$ Abelian implies that $[H, H]$ is contained in $\Phi(G)$. Thus if H_j represents a term in the descending central series of H, $H_2 = [[H, H], H] \leqq [L(G), G] = L_1$. Assume $H_{j+1} = [H_j, H] < L_j$ for $j \geqq 1$. Then $H_{j+2} = [H_{j+1}, H] < [L_j, G] = L_{j+1}$. So by induction, the descending central series of H exists which implies the nilpotency of H.

On the other hand suppose each proper subgroup of a group G is nilpotent. The structure of these groups is known (see P. Hall and Higman [4]). Such a group is of the form $G = QP$, Q normal in G, P cyclic of prime-power order, $\Phi(P) \leqq Z(G)$, and Q either an elementary Abelian q-group, $q \neq p$, or $\Phi(Q) = Z(Q) = [Q, Q]$, Q a q-group,

$p \neq q$ with both having P acting irreducibly on $Q/\Phi(Q)$, i.e. $[P, Q] = Q$. Then since $\Phi(P) \leqq \Phi(G)$, it follows that $G/\Phi(G)$ has each proper subgroup the direct product of elementary Abelian p-groups. Moreover since $\Phi(G) = \Phi(Q) \otimes \Phi(P) = Z(G)$, then $L(G) = Z^*(G)$ which implies $L^*(G) = 1$.

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