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CONVOLUTION IN FOURIER-WIENER TRANSFORM

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Let C be the Wiener space and K be the space of complex valued continuous functions on $0 \leqq t \leqq 1$ which vanish at $t = 0$. The Fourier-Wiener transform of a functional $F[x]$, $x \in K$, is by definition

$$G[y] = \int_C^w F[x + iy] d_w x \,, \quad y \in K \,.$$

Let E_0 be the class of functionals $F[x]$ of the type

$$F[x] = \Phi_F\left[\int_0^1 \alpha_1(t) dx(t), \cdots, \int_0^1 \alpha_n(t) dx(t)\right]$$

where $\Phi_F(\zeta_1, \cdots, \zeta_n)$ is an entire function of the n complex variables $\{\zeta_j\}$ of the exponential type and $\{\alpha_j\}$ are n linearly independent real functions of bounded variation on $0 \leqq t \leqq 1$. Let E_m be the class of functionals which are mean continuous, entire and of mean exponential type.

We define the convolution of two functionals F_1, F_2 to be

$$(F_1 * F_2)[x] = \int_C^w F_1\left[\frac{y + x}{2^{1/2}}\right] F_2\left[\frac{y - x}{2^{1/2}}\right] d_w y \,, \quad x \in K \,.$$

Then if $F_1, F_2 \in E_0$ or $F_1, F_2 \in E_m$, the convolution of F_1, F_2 exists for every $x \in K$ and furthermore

$$G_{F_1} * G_{F_2}[z] = G_{F_1}\left[\frac{z}{2^{1/2}}\right] G_{F_2}\left[-\frac{z}{2^{1/2}}\right], \quad z \in K \,.$$

Let K be the space of complex-valued continuous functions defined on $0 \leqq t \leqq 1$ which vanish at $t = 0$ and let C be the Wiener space, namely the subspace of K which consists of real-valued elements of K. Let $F[x] = F[x(\cdot)]$ be a functional which is defined throughout K. If it exists, the functional

$$G[y] = \int_C^w F[x + iy] d_w x \,, \quad y \in K \tag{1.1}$$

is called the Fourier-Wiener transform of $F[x]$.

The first class E_0 of functionals is defined as follows: A functional $F[x]$ belongs to E_0 if

$$F[x] = \Phi_F\left[\int_0^1 \alpha_1(t) dx(t), \cdots, \int_0^1 \alpha_n(t) dx(t)\right] \tag{1.2}$$

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where $\Phi_F(\zeta_1, \cdots, \zeta_n)$ is an entire function of the n complex variables $\{\zeta_j\}$ of exponential type

$$(1.3) \qquad |\Phi_F(\zeta_1, \cdots, \zeta_n)| < Me^{a(|\zeta_1|+\cdots+|\zeta_n|)}$$

and $\alpha_j(t)$ are n linearly independent real functions of bounded variation on $0 \leqq t \leqq 1$. The function Φ_F as well as the constants M and a depend on F.

The second class E_m consists of functionals $F[x]$ which are mean continuous, entire and of mean exponential type: that is, E_m is the class of functionals satisfying the following three conditions:

1° $\lim_{n\to\infty} F[x^{(n)}] = F[x]$ holds for all x and $x^{(n)}$ in K for which $\lim_{n\to\infty}\int_0^1 |x^{(n)}(t) - x(t)|^2 dt = 0$.

2° $F[x + \lambda y]$ is an entire function of the complex variable λ for all x and y in K; and

3° there exist positive constants A_F and B_F depending on F such that

$$(1.4) \qquad |F[x]| \leqq A_F \exp\left\{B_F\left(\int_0^1 |x(t)|^2 dt\right)^{1/2}\right\} \qquad \text{for all } x \in K .$$

According to Theorems 1 and A, [3], if $F[x]$ belongs to E_0 or E_m, its transform $G[y]$ exists for all $y \in K$ and belongs to the same class.

We now define the convolution of two functionals $F_1[x]$ and $F_2[x]$ to be

$$(1.5) \qquad (F_1 * F_2)[x] = \int_C^w F_1\left[\frac{y + x}{2^{1/2}}\right]F_2\left[\frac{y - x}{2^{1/2}}\right]d_w y\ , \quad x \in K$$

if the integral in the right side exists.

The result of this paper is stated in the following two theorems:

THEOREM I. *If $F_1[x], F_2[x] \in E_0$, the convolution* (1.5) *exists for every $x \in K$. Morcover, the Fourier-Wiener transform $G_{F_1 * F_2}[z]$ of* (1.5) *exists and satisfies*

$$(1.6) \qquad G_{F_1 * F_2}[z] = G_{F_1}\left[\frac{z}{2^{1/2}}\right]G_{F_2}\left[-\frac{z}{2^{1/2}}\right] \quad \textit{for every } z \in K .$$

THEOREM II. *Exactly the same as in Theorem I holds for any two functionals belonging to E_m.*

Theorem I and II will be proved in §2 and §3 respectively. From these theorems follows the Parseval relation of [3].

2. NOTATION. We introduce the notation $\Phi([\zeta_j]_n)$ for the function $\Phi(\zeta_1, \cdots, \zeta_n)$ of n complex variables, $\Phi([\zeta_j]_n, [\zeta'_j]_m)$ for the function $\Phi(\zeta_1, \cdots, \zeta_n, \zeta'_1, \cdots, \zeta'_m)$ of $n+m$ complex variables. In particular, $\Phi([\zeta_j]_n, \zeta')$ stands for the function $\Phi(\zeta_1, \cdots, \zeta_n, \zeta')$ of $n+1$ complex variables.

We first make a few remarks on the entire functions of exponential type.

REMARK 1. If $\Phi_1([\zeta_j]_n)$, $\Phi_2([\zeta_j]_n)$ are two entire functions of exponential type, the two factors in the right hand and consequently the left hand of

$$\Phi([\zeta_j]_n, [\zeta'_j]_n) = \Phi_1([2^{-1/2}(\zeta_j + \zeta'_j)]_n)\Phi_2([2^{-1/2}(\zeta_j - \zeta'_j)]_n) \tag{2.1}$$

are entire functions of exponential type of the n complex variables $\zeta_1, \cdots, \zeta_n$ for fixed $\zeta'_1, \cdots, \zeta'_n$ and, similarly, of the n complex variables $\zeta'_1, \cdots, \zeta'_n$ for fixed $\zeta_1, \cdots, \zeta_n$.

REMARK 2. If $\varphi(u_1, \cdots, u_n, \zeta)$ is continuous in the $n+1$ variables for $-\infty < u_j < \infty$, $j = 1, 2, \cdots, n$ and $\zeta \in R$, a region in the complex plane, and is analytic in $\zeta \in R$ for fixed $u_1, \cdots, u_n$, the uniform convergence over R of the integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(u_1, \cdots, u_n, \zeta) du_1 \cdots du_n$$

implies that the integral is an analytic function of $\zeta \in R$.

REMARK 3. If $\Phi([\zeta_j]_n, [\zeta'_j]_n)$ is an entire function of exponential type of $2n$ complex variables, the integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j]_n) \exp\{-\zeta_1^2 - \cdots - \zeta_n^2\} d\zeta_1 \cdots d\zeta_n$$

is an entire function of exponential type of the n complex variables $\zeta'_1, \cdots, \zeta'_n$.

Proof of Theorem I. For $F_1[x], F_2[x] \in E_0$,

$$F_i[x] = \Phi_i\left(\left[\int_0^1 \alpha_j(t) dx(t)\right]_n\right), \qquad i = 1, 2 \tag{2.2}$$

where $\Phi_i([\zeta_j]_n)$, $i = 1, 2$, are two entire functions of exponential type of n complex variables. We first prove the theorem for the special case where $\{\alpha_j(t)\}$ are an orthonormal set on $0 \leqq t \leqq 1$. We quote a result by Paley and Wiener [7] which states that for any orthonormal set of real functions $\{\alpha_j(t)\}$ of bounded variation on $0 \leqq t \leqq 1$, the equality

$$\text{(2.3)}\quad \int_C^w \Psi\left(\left[\int_0^1 \alpha_j(t)dx(t)\right]_n\right)d_wx = \frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Psi([u_j]_n) \times \exp\{-u_1^2 - \cdots - u_n^2\}du_1\cdots du_n$$

holds for every function $\Psi([u_j]_n)$ for which the integral on the right side exists as an absolutely convergent Lebesgue integral. By (1.5), (2.2), (2.1), (2.3),

$$\text{(2.4)}\quad \begin{aligned}(F_1 * F_2)[x] &= \int_C^w \Phi\left(\left[\int_0^1 \alpha_j(t)dy(t)\right]_n, \left[\int_0^1 \alpha_j(t)dx(t)\right]_n\right)d_wy \\ &= \frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi\left([u_j]_n, \left[\int_0^1 \alpha_j(t)dx(t)\right]_n\right) \\ &\quad\times \exp\{-u_1^2 - \cdots - u_n^2\}du_1\cdots du_n\end{aligned}$$

for every $x \in K$, where the last integral exists becuase $\Phi([\zeta_j]_n, [\zeta_j']_n)$ is an entire function of exponential type in $\{\zeta_j\}$ for fixed $\{\zeta_j'\}$ according to Remark 1. This proves the existence of $(F_1 * F_2)[x]$ for every $x \in K$.

Now according to Remark 3,

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi([\zeta_j]_n, [\zeta_j']_n)\exp\{-\zeta_1^2 - \cdots - \zeta_n^2\}d\zeta_1\cdots d\zeta_n$$

is an entire function of exponential type of $\{\zeta_j'\}$, and hence, Theorem 1, [3] applies to the last member of (2.4). Thus the Fourier-Wiener transform of $(F_1 * F_2)[x]$ namely $G_{F_1 * F_2}[z]$, exists for every $z \in K$ and is given by (1.1) as

(2.5)

$$G_{F_1*F_2}[z] = \int_C^w \frac{1}{\pi^{n/2}}\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi\left([u_j]_n, \left[\int_0^1 \alpha_j(t)dx(t) + i\int_0^1 \alpha_j(t)dz(t)\right]_n\right) \times \exp\{-u_1^2 - \cdots - u_n^2\}du_1\cdots du_n\right\}d_wx\,.$$

Now since

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi([\zeta_j]_n, [\zeta_j' + \zeta_j'']_n)\exp\{-\zeta_1^2 - \cdots - \zeta_n^2\}d\zeta_1\cdots d\zeta_n$$

is an entire function of exponential type of $\{\zeta_j'\}$ for fixed $\{\zeta_j''\}$, (2.3) is applicable to the last integral of (2.5). Thus

$$\begin{aligned}G_{F_1*F_2}[z] &= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi\left([u_j]_n, \left[v_j + i\int_0^1 \alpha_j(t)dz(t)\right]_n\right) \\ &\quad\times \exp\{-u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2\}du_1\cdots du_n dv_1\cdots dv_n \\ &= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_1\left(\left[2^{-1/2}\left(u_j + v_j + i\int_0^1 \alpha_j(t)dz(t)\right)\right]_n\right) \\ &\quad\times \Phi_2\left(\left[2^{-1/2}\left(u_j - v_j - i\int_0^1 \alpha_j(t)dz(t)\right)\right]_n\right) \\ &\quad\times \exp\{-u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2\}du_1\cdots du_n dv_1\cdots dv_n\,.\end{aligned}$$

Let

$$u'_j = 2^{-1/2}(u_j + v_j)\ ,$$
$$v'_j = 2^{-1/2}(u_j - v_j)\ , \qquad j = 1, 2, \cdots, n$$

and apply (2.3) to the result of this transformation. By (2.2), (1.1) we have

$$\begin{aligned} G_{F_1 * F_2}[z] &= \left\{\int_C^w \Phi_1\left(\left[\int_0^1 \alpha_j(t)dx(t) + \frac{i}{2^{1/2}}\int_0^1 \alpha_j(t)dz(t)\right]_n\right)d_w x\right\} \\ &\quad \times \left\{\int_C^w \Phi_2\left(\left[\int_0^1 \alpha_j(t)dx(t) - \frac{i}{2^{1/2}}\int_0^1 \alpha_j(t)dz(t)\right]_n\right)d_w x\right\} \\ &= G_{F_1}\left[\frac{z}{2^{1/2}}\right]G_{F_2}\left[-\frac{z}{2^{1/2}}\right]. \end{aligned}$$

This proves Theorem I for the special case.

In the general ease where $\alpha_j(t)$ are n linearly independent real valued functions of bounded variation on $0 \leqq t \leqq 1$, according to the argument on p. 493, [3], we can write $F_i[x]$, $i = 1, 2$ defined by (2.2) as

$$F_i[x] = \Phi_i^{\star}\left(\left[\int_0^1 \alpha'_j(t)dx(t)\right]_n\right)\ , \qquad i = 1, 2$$

where $\Phi_i^{\star}([\zeta_j]_n)$ are entire functions of exponential type of $\{\zeta_j\}$ and $\alpha'_j(t)$ are n orthonormal functions of bounded variation on $0 \leqq t \leqq 1$. Now the result for the special case applies and the theorem is proved.

3. LEMMA. *Let* $\{F_{1,n}[x]\}$, $F_1[x]$, $\{F_{2,n}[x]\}$, $F_2[x]$ *be such that*

1° (3.1) $\lim_{n\to\infty} F_{i,n}[x] = F_i[x]$ *for every* $x \in K$, $i = 1, 2$.

2° *the Fourier-Wiener transform exists for every* $F_{i,n}[x]$ $n = 1, 2, \cdots, i = 1, 2$; *the convolution* $(F_{1,n} * F_{2,n})[x]$ *exists, its Fourier-Wiener transform also exists and satisfies*

$$G_{F_{1,n}*F_{2,n}}[z] = G_{F_{1,n}}\left[\frac{z}{2^{1/2}}\right]G_{F_{2,n}}\left[-\frac{z}{2^{1/2}}\right], \tag{3.2}$$

for every $z \in K$, *for* $n = 1, 2, \cdots$; *and*

3° (3.3) $|F_{i,n}[x]| \leqq A\exp\{B\,|||x|||^{2-\varepsilon}\}$, $\qquad n = 1, 2 \cdots, i = 1, 2$ *where* $A, B, > 0, 2 > \varepsilon > 0$ *and* $|||x||| = \max_{0\leqq t\leqq 1}|x(t)|$. *Then the Fourier-Wiener transforms of* $F_1[x]$, $F_2[x]$, *the convolution of* $F_1[x]$, $F_2[x]$ *and the Fourier-Wiener transform of the convolution exist and* (1.6) *holds.*

Proof of the lemma. By (1.5), (1.1), the equality (3.2) can be written as

$$\tag{3.4} \begin{aligned} &\int_C^w \left\{ \int_C^w F_{1,n}\left[\frac{y+x+iz}{2^{1/2}}\right] F_{2,n}\left[\frac{y-x-iz}{2^{1/2}}\right] d_w y \right\} d_w x \\ &= \left\{ \int_C^w F_{1,n}\left[x + \frac{iz}{2^{1/2}}\right] d_w x \right\} \left\{ \int_C^w F_{2,n}\left[x - \frac{iz}{2^{1/2}}\right] d_w x \right\}, \quad n = 1, 2, \cdots . \end{aligned}$$

We prove the lemma by justifying the passing to the limit under the integral signs on both sides of (3.4). To do this, we observe that for any p complex numbers $\zeta_1, \cdots, \zeta_p$,

$$\tag{3.5} \left| \sum_{k=1}^p \zeta_k \right|^{2-\varepsilon} \leqq \left(p \max_k \{ |\zeta_1|, \cdots, |\zeta_p| \} \right)^{2-\varepsilon} \leqq p^2 \sum_{k=1}^p |\zeta_k|^{2-\varepsilon} .$$

An estimate of the first integrand on the right hand side of (3.4) is given by (3.3) and (3.5) with $p = 2$:

$$\tag{3.6} \left| F_{1,n}\left[x + \frac{iz}{2^{1/2}}\right] \right| \leqq A \exp \{ 4B(|||x|||^{2-\varepsilon} + |||z|||^{2-\varepsilon}) \} .$$

Since $\int_C^w \exp\{4B\, |||x|||^{2-\varepsilon}\} d_w x$ is finite according to [4], the right side of (3.6) is integrable with respect to x over the entire Wiener space for fixed z. By (3.1) with dominated convergence and by (1.1)

$$\tag{3.7} \lim_{n\to\infty} \int_C^w F_{1,n}\left[x + \frac{iz}{2^{1/2}}\right] d_w x = G_{F_1}\left[\frac{z}{2^{1/2}}\right]$$

for every $z \in K$ and similarly

$$\tag{3.8} \lim_{n\to\infty} \int_C^w F_{2,n}\left[x - \frac{iz}{2^{1/2}}\right] d_w x = G_{F_2}\left[-\frac{z}{2^{1/2}}\right],$$

for every $z \in K$. From (3.3) and (3.5) with $p = 3$, the integrand of the left side of (3.4) is seen to be bounded by $A^2 \exp\{18B(|||x|||^{2-\varepsilon} + |||y|||^{2-\varepsilon} + |||z|||^{2-\varepsilon})\}$. The repeated integral of the above expression with respect to y and then with respect to x over the entire Wiener space is finite for every $z \in K$. Thus by (3.1) with dominated convergence and by (1.5), (1.1),

$$\tag{3.9} \lim_{n\to\infty} \int_C^w \left\{ \int_C^w F_{1,n}\left[\frac{y+x+iz}{2^{1/2}}\right] F_{2,n}\left[\frac{y-z-iz}{2^{1/2}}\right] d_w y \right\} d_w x = G_{F_1 * F_2}[z]$$

for every $z \in K$. By letting $n \to \infty$ on both sides of (3.4) and by (3.7), (3.8) and (3.9), the lemma is established.

Proof of Theorem II. Let $F_i[x] \in E_m, i = 1, 2$, and let $\varphi_1(t), \varphi_2(t), \cdots$ be a complete orthonormal set of real valued continuous functions on the interval $0 \leqq t \leqq 1$ which vanish when $t = 0$. Let

$$F_{i,n}[z] = F_i\Big[\sum_{j=1}^{n} \varphi_j(\cdot) \int_0^1 x(t)\varphi_j(t)dt\Big] \qquad n = 1, 2, \cdots, i = 1, 2 , \tag{3.10}$$

and let

$$x^{(n)} = \sum_{j=1}^{n} \varphi_j(\cdot) \int_0^1 x(t)\varphi_j(t)dt , \qquad n = 1, 2, \cdots .$$

By 1° in the definition of E_m,

$$\lim_{n\to\infty} F_{i,n}[x] = F_i[x] , \tag{3.11}$$

for every $x \in K$, $i = 1, 2$, and $F_{i,n}[x]$, $i = 1, 2$, satisfy 1° of the lemma.

To show that 2° of the lemma is satisfied, let us define $\Phi_{i,n}([\zeta_j]_n)$ by

$$\Phi_{i,n}([\zeta_j]_n) = F_i\Big[\sum_{j=1}^{n} \zeta_j\varphi_j(\cdot)\Big] , \qquad n = 1, 2, \cdots, i = 1, 2 . \tag{3.12}$$

To show that each $\Phi_{i,n}$ is an entire function of exponential type of n complex variables, we set

$$x(t) = \zeta_1\varphi_1(t) + \cdots + \zeta_{j-1}\varphi_{j-1}(t) + \zeta_{j+1}\varphi_{j+1}(t) + \cdots + \zeta_n\varphi_n(t) ,$$
$$y(t) = \varphi_j(t) .$$

From (3.12) it follows that $\Phi_{i,n}([\zeta_j]_n) = F_i[x(t) + \zeta_i y(t)]$ and by 2° in the definition of E_m, $\Phi_{i,n}$ is an entire function of ζ_j. From the arbitrariness of the choice of ζ_j from $\{\zeta_j\}$ and by Hartogs' regularity theorem, $\Phi_{i,n}$ is an entire function of the n complex variables $\{\zeta_j\}$ for $n = 1, 2, \cdots, i = 1, 2$. That $\Phi_{i,n}$ is of exponential type follows from (3.12) and 3° of the definition of E_m:

$$\begin{aligned} |\Phi_{i,n}([\zeta_j]_n)| &\leqq A_{F_i} \exp\Big\{B_{F_i}\Big(\int_0^1 \Big|\sum_{j=1}^{n} \zeta_j\varphi_j(t)\Big|^2 dt\Big)^{1/2}\Big\} \\ &\leqq A_{F_i} \exp\Big\{B_{F_i}\Big(\sum_{j=1}^{n} |\zeta_j|^2\Big)^{1/2}\Big\} \\ &\leqq A_{F_i} \exp\Big\{B_{F_i} \sum_{j=1} |\zeta_j|\Big\} . \end{aligned}$$

This proves the asserted property of $\Phi_{i,n}$. On the other hand from (3.10), (3.12)

$$F_{i,n}[x] = \Phi_{i,n}\Big(\Big[\int_0^1 x(t)\varphi_j(t)dt\Big]_n\Big) , \qquad n = 1, 2, \cdots, i = 1, 2 . \tag{3.13}$$

Now if we let $\alpha_j(t) = \int_t^1 \varphi_j(t)dt$, $n = 1, 2, \cdots$, then by integration by parts $\int_0^1 x(t)\varphi_j(t)dt = \int_0^1 \alpha_j(t)dx(t)$, and (3.13) becomes

$$F_{i,n}[x] = \Phi_{i,n}\Big(\Big[\int_0^1 \alpha_j(t)dx(t)\Big]_n\Big) , \qquad n = 1, 2, \cdots, 1 = 1, 2$$

where by definition $\alpha_j(t)$ are of bounded variation on $0 \leqq t \leqq 1$. Therefore each $F_{i,n}[x]$ satisfies the conditions of Theorem I, [3] and hence its Fourier-Wiener transform exists. Moreover by Theorem I the convolution $(F_{i,n} * F_{2,n})[x]$ exists and satisfies (3.2) for every $z \in K$ for $n = 1, 2, \cdots$. Thus 2° of the lemma is satisfied.

Finally, let A be the greater of A_{F_1}, A_{F_2} and B be the greater of B_{F_1}, B_{F_2} in 3° of the definition of E_m. By (3.10), (3.14)

$$\begin{aligned} |F_{i,n}[x]| &\leqq A \exp\left\{B\left(\int_0^1 \left|\sum_{j=1}^n \varphi_j(s) \int_0^1 x(t)\varphi_j(t)dt\right|^2 ds\right)^{1/2}\right\} \\ &\leqq A \exp\left\{B\left(\int_0^1 |x(t)|^2 dt\right)^{1/2}\right\} \\ &\leqq A \exp\{B \, |||x|||^{2-\varepsilon}\} \end{aligned}$$

for $1 > \varepsilon > 0$ and 3° of the lemma is satisfied.

By the conclusion of the lemma, Theorem II is proved.

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