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SOME GENERAL PROPERTIES OF MULTI-VALUED FUNCTIONS

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The object is to determine what theorems for single-valued functions can be extended to which class of multi-valued functions. It is shown that an arc cannot be mapped onto a circle by a continuous, monotone multi-valued function when the image of each point is an arc. On the other hand, the arc can be mapped onto a nonlocally connected space by a monotone, continuous function such that the image of each point is an arc. Characterizations of nonalternating functions analogous to the results in the single-valued theory are obtained, and it is shown that an nonalternating semi-single-valued continuous function on a dendrite is monotone. An analog of the monotone light factorization theorem is obtained for semi-single-valued continuous functions.

Some other results are: an open continuous function with finite images maps a regular curve onto a regular curve, and a continuous function with finite images maps a locally connected, compact space onto a locally connected compact space.

A number of definitions for continuity have been proposed for multi-valued or set-valued functions, and Wayman Strother studied the problem of continuity extensively [10, 11, 12]. Also Choquet [2] has studied upper and lower semi-continuous functions. Further, Berge, unlike most authors, allows functions to be multi-valued in [1]. However, much of the work that has been done on set-valued functions has been devoted to the discovery of fixed point theorems ([3], [7] through [9], [11], [13], and [15] through [17]). The purpose of this paper is to investigate properties of multi-valued functions which are similar to the properties of single-valued functions studied in G. T. Whyburn's book, Analytic Topology, [18].

We shall use the following topology on the set of closed subsets of a space Y. Let

$$S(Y) = \{E \subset Y: E \text{ is closed and nonempty}\}.$$

Let $S(Y)$ have the topology used by Michael [6]; i.e., if $V_1, \cdots, V_n$ are open subsets of Y, then the collection $\langle V_1, \cdots, V_n \rangle = \{E \in S(Y): E \cap V_i \neq \phi$ for all i, and $E \subset \bigcup_{i=1}^n V_i\}$ is a basis for the open sets of $S(Y)$. We shall call this topology the finite topology. This is equivalent to

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the topology used by Strother [3] and Frink [4]. Since we shall be dealing extensively with subspaces of $S(Y)$ we shall use $\langle V_1, \cdots, V_n\rangle$ to be either a basic open set in $S(Y)$ or a basic, relatively open set in the appropriate subspace of $S(Y)$. If $\mathscr{V} = \{V_1, \cdots, V_n\}$, then set $\langle \mathscr{V}\rangle = \langle V_1, \cdots, V_n\rangle$.

In this paper we shall assume that all spaces are Hausdorff.

Given a set-valued function $F: X \to Y$ with $F(x)$ closed and nonempty we define the induced function f on X into $S(Y)$ by setting $f(x) = F(x)$ for each $x \in X$. Note that f is single-valued, and f will always denote the function induced by F unless otherwise stated. Also we shall always use upper case letters to denote multi-valued functions.

If A is a subset of X, then the symbols $\bar{A}$ and $Cl(A)$ are used to denote the closure of A, and the symbol A° is used to denote the interior of A.

Henceforth, we assume that $S(Y)$ has the finite topology, and that $F: X \to Y$ is, unless otherwise stated, a function such that $F(x)$ is in $S(Y)$ for each x in X.

1. Preliminaries.

This section will be devoted mainly to gathering known results that are needed in the development of succeeding sections.

DEFINITION. A multi-valued function $F: X \to Y$ is called continuous in case the induced function $f: X \to S(Y)$ is continuous.

NOTATION. If $A \subset X$, then $F(A) = \cup\{F(x): x \in A\}$.

Now we have the following lemmas due to Strother [10].

LEMMA 1.1. *A function $F: X \to Y$ is continuous if and only if statements* (1) *and* (2) *hold.*

1. *If $x_0 \in X$, V is open in Y, and if $F(x_0) \cap V \neq \phi$, then there exists an open set U of X with $x_0 \in U$ such that $F(x) \cap V \neq \phi$ for all $x \in U$.*

2. *If $x_0 \in X$ and $F(x_0) \subset V$ where V is open in Y, then there exists an open set U containing x_0 such that $F(U) \subset V$.*

LEMMA 1.2. *Let Y be regular. If $F: X \to Y$ is continuous, if $\{x_d\}$ is a net in X converging to x_0, and if $y_d \in F(x_d)$ such that $\{y_d\}$ converges to y_0, then $y_0 \in F(x_0)$.*

LEMMA 1.3. *Let $F: X \to Y$ be continuous, and let X and Y be compact. Then F is closed; i.e., $F(A)$ is closed in Y whenever A is closed in X.*

We also need the following lemma from Michael [6].

LEMMA 1.4. *If $\mathscr{B}$ is a collection of subsets of Y which is disjoint (a subcollection of $S(Y)$) and connected in the factor (finite) topology and all (one) of whose elements are (is) connected, then* $\cup\{E: E \in \mathscr{B}\}$ *is connected.*

Set $\mathscr{F}(Y) = \{E \in S(Y); E \text{ finite}\}$, $\mathscr{C}(Y) = \{E \in S(Y): E \text{ is compact}\}$ and $\mathscr{F}_n(Y) = \{E: E \text{ has at most } n \text{ elements}\}$.

Now we can apply the above results to obtain some further lemmas. Lemma 1.5 is a variation of a theorem in Berge [1].

LEMMA 1.5. *Let F be continuous and onto, and let X be compact. Then Y is compact if and only if $F(x)$ is compact for each $x \in X$.*

Proof. Suppose Y is compact; then $F(x)$ closed implies $F(x)$ compact. Suppose that $F(x)$ is compact for each x, and let $\mathscr{V}$ be an open cover of Y. Then for each x we obtain a subcover $\mathscr{V}_x$ of $F(x)$, such that $F(x) \cap V \neq \phi$ for all $V \in \mathscr{V}_x$. Since $F(x)$ is compact, there is a finite subcover $\mathscr{V}'_x$ of $F(x)$ in $\mathscr{V}_x$, and $F(x) \in \langle \mathscr{V}'_x \rangle$. The collection $\{\langle \mathscr{V}'_x \rangle: x \in X\}$ is an open cover for $f(X)$ in $S(Y)$. Since $f(X)$ is compact, there is a finite subcover, say $\langle \mathscr{V}'_1 \rangle, \cdots, \langle \mathscr{V}'_n \rangle$ of $f(X)$; hence the collection $\mathscr{V}_0 = \bigcup_{i=1}^n \mathscr{V}'_i$ is a finite subcover of Y and $\mathscr{V}_0 \subset \mathscr{V}$.

LEMMA 1.6. *Let F be continuous and A a connected subset of X. Then, if $F(x)$ is connected for some $x \in A$, $F(A)$ is a connected subset of Y.*

Proof. Since F is continuous, $f(A)$ is connected in $S(Y)$, and for some x, $F(x) \in f(A)$ is connected. So by Lemma 1.4, $F(A) = \cup \{F(x): x \in A\}$ is connected.

COROLLARY 1.7. *If F is continuous, if X is connected, and if there is an $x \in X$ such that $F(x)$ is connected, then $F(X)$ is connected. Hence Y is connected if F is onto.*

COROLLARY 1.8. *Let F be continuous. Then $F(A)$ is connected for every connected subset A of X if and only if $F(x)$ is connected for each $x \in X$.*

Proof. Since $\{x\}$ is connected, $F(x)$ must be connected by hypothesis. On the other hand, if $A \neq \phi$, then for any $x \in A$, $F(x)$ is connected. So Lemma 1.6 applies.

Another result from Michael's paper [6] we need is the following.

LEMMA 1.9. *If* $A \subset Y$ *is closed, the following hold.*
1. $\{E \in S(Y): E \subset A\}$ *is closed.*
2. $\{E \in S(Y): E \cap A \neq \phi\}$ *is closed.*

COROLLARY 1.10. *If* F *is continuous, the set* $\{x: y \in F(x)\}$ *is closed for each* y.

Proof. The set $\{x: y \in F(x)\} = f^{-1}\{F(x): F(x) \cap \{y\} \neq \phi\}$ and the latter is closed by part 2 of Lemma 1.9.

We call $\{x: y \in F(x)\}$ the inverse of y and write $F^{-1}(y)$. Similarly, for $A \subset Y$ we define

$$F^{-1}(A) = \{x: F(x) \cap A \neq \phi\} .$$

Note that if A is closed, so is $F^{-1}(A)$.

NOTATION. We write $E = A \cup B$, $A \mid B$ to denote a separation of E, and we say that A and B separate E.

Note. In general, for $A \subset Y$ we need not have $F(F^{-1}(A)) = A$.

We can generalize a lemma of Whyburn's.

LEMMA 1.11. *Let* X *be compact,* Y *regular,* $F: X \to Y$ *continuous, and let* $Y_0 \subset Y$. *If* $F^{-1}(Y_0) = A \cup B$, $A \mid B$ *with* $F(A)$ *and* $F(B)$ *intersecting the same quasi-component* Q *of* Y_0, *then there exists* $y_0 \in Y_0$ *such that* $F^{-1}(y_0)$ *intersects* A *and* B.

Proof. Let $A_1 = F(A) \cap Y_0$ and $B_1 = F(B) \cap Y_0$. Now, by hypothesis $A_1 \cap Q \neq \phi$ and $B_1 \cap Q \neq \phi$. Therefore, A_1 is not separated from B_1, so there is a net $\{y_\alpha\}$ in A_1, say, such that $y_\alpha \to y_0 \in B_1$.

Now let $x_\alpha \in F^{-1}(y_\alpha) \cap A$ for each α. This defines a net in $F^{-1}(A_1) \cap A$, and since X is compact $\{x_\alpha\}$ has a limit point x_0 and thus a convergent subnet $x_\gamma \to x_0$. By Lemma 1.2, $y_0 \in F(x_0)$ so $x_0 \in F^{-1}(y_0)$. Further, $A \mid B$ implies that $x_0 \mid B$ and so $x_0 \in A$ or $F^{-1}(y_0) \cap A \neq \phi$. Finally, $y_0 \in B_1$ implies that $F^{-1}(y_0) \cap B \neq \phi$.

Let X, Y, and Z be spaces and $F_1: X \to Y$, $F_2: Y \to Z$ be set-valued functions. The composition function $F = F_2 \circ F_1$ is defined by $F(x) = F_2(F_1(x))$ for each $x \in X$. Note that in this case $F(x)$ may not be a closed set. Also, if $z \in Z$, then $F^{-1}(z) = F_1^{-1}(F_2^{-1}(z))$. Consequently, we write $F^{-1} = F_1^{-1}F_2^{-1}$. When X, Y, and Z are compact we have the following result from [10].

LEMMA 1.12. *If* $F_1: X \to Y$ *and* $F_2: Y \to Z$ *are continuous and* *if X, Y, and Z are compact, then $F = F_2 \circ F_1$ is continuous.*

Let $F: X \to Y$ and let A be a subspace of X. Then the restriction of F to A, $F|A$, is defined by $F|A(x) = F(x)$ for all $x \in A$. An immediate consequence of Lemma 1.1 is:

LEMMA 1.13. *Let $F: X \to Y$ be continuous and let $A \subset X$. Then the restriction of F to A is continuous.*

2. Monotone functions. In this section we generalize the definition of monotone functions and investigate their elementary properties.

DEFINITION. A continuous function $F: X \to Y$ is called monotone if and only if $F^{-1}(y)$ is connected for each $y \in Y$.

Another generalization of a lemma in Whyburn [18] is:

LEMMA 2.1. *If X is compact, Y regular, and $F: X \to Y$ is continuous, then F is monotone if and only if $F^{-1}(A)$ is connected whenever A is a connected subset of Y.*

Proof. If $F^{-1}(A)$ is connected for each connected set in Y, then $F^{-1}(y)$ is connected for each $y \in Y$ and hence F is monotone.

On the other hand, suppose that F is monotone and that A is a connected subset of Y. Further, suppose that $F^{-1}(A) = C \cup D$ with $C | D$. Both $F(C)$ and $F(D)$ meet A, and A is a quasi-component of itself. Thus, by Lemma 1.11, there exists a $y \in A$ such that $F^{-1}(y) \cap C \neq \phi$ and $F^{-1}(y) \cap D \neq \phi$, a contradiction, as F is monotone. Hence $F^{-1}(A)$ is connected.

Whyburn shows the following properties are preserved by monotone, continuous, single-valued functions, the property of being, (1) a unicoherent continuum, (2) a hereditarily locally connected continuum, (3) a regular curve, and (4) a rational curve. However, the following examples show that these properties fail to be preserved by continuous, monotone, multi-valued functions, even when fairly stringent conditions are placed on the set $F(x)$, i.e., we may require $F(x)$ to be a locally connected continuum for each $x \in X$ and have X a locally connected continuum, but still not have Y locally connected. See Example 6.

EXAMPLE 1. Let X and Y be any two spaces. Define $F(x) = Y$ for each $x \in X$. Clearly F is continuous and Y need not possess any property that is not shared by all spaces. Thus we see the necessity of placing restrictions on the sets $F(x)$.

EXAMPLE 2. Let X be the closed interval $[(0, 0), (0, 1)]$ in the plane, let Y be the circle that is tangent to the x-axis at $x = 2$ and the line $y = 1$ at $(2, 1)$. Denote the points of X by their y-coordinate, and in Y let the closed arc $(x, y) - (2, 0) - (x', y)$ be denoted by yy', where we denote the point (x, y) by y and (x', y) by y'. Then define F by $F(0) = (2, 0)$, $F(y) = yy'$, $0 < y < 1$ and $F(1) = Y$. It is easily seen that F is continuous. In fact f is a homeomorphism and F is monotone. However, X is unicoherent and Y is not.

EXAMPLE 3. Let $X = [(0, 0), (0, 1)]$ as above and let Y be the unit square and its interior with corners $(1, 0)$, $(2, 0)$, $(1, 1)$, and $(2, 1)$. Let the closed horizontal lines $[(1, y), (2, y)]$ be denoted by $\hat{y}$ where y is the common y-coordinate. Again identify the points of X with their y-coordinate.

Then let $F(y) = \hat{y}$. Here again f is a homeomorphism. In fact F is monotone and the inverse of a single-valued continuous function of Y onto X. Further, X and $F(x)$ are locally connected continua for each x. Also, X and $F(x)$ are hereditarily locally connected and hereditarily unicoherent, but Y is neither, and Y is neither rational nor regular, but X is both.

EXAMPLE 4. Let $X = [0, 1]$, and Y the area between and including two concentric circles C_0 and C_1. Let C_a, $0 \leqq a \leqq 1$, be the circle that has the same center as C_0 and C_1, and with radius $r_a = ar_1 + (1 - a)r_0$ where r_0, r_1 are the radii of C_0 and C_1, respectively. Define F by $F(x) = C_x$. Then F is monotone, continuous and $F(x)$ is a locally connected continuum for each x, and if $x_1 \neq x_2$, $F(x_1) \cap F(x_2) = \phi$. Yet X is unicoherent and Y is not.

In Whyburn [18] it is shown that the image of a simple arc under a continuous, monotone transformation is again a simple arc, and similarly for a simple closed curve. However, in the case of multi-valued functions neither of these results holds. Example 5 is a counterexample for the former, and the function that maps each point of the circle onto the entire unit interval serves nicely as a counterexample for the latter. We shall, however, subsequently show that the unit interval cannot be mapped onto the circle by a continuous, monotone, multi-valued function F, for which $F(x)$ is a simple arc for each x. (Here and in the following $F(x)$ may be degenerate, i.e., a point.)

EXAMPLE 5. Let I be the unit interval. Let I_1, I_2, and I_3 be copies of I. Form Y by erecting I_2 perpendicular to I_1 at $1/4$ and by erecting I_3 prependicular to I_1 at $3/4$ (the 0 of I_2 is identified with $1/4$ in I_1 and the 0 of I_3 is identified with $3/4$ in I_1). Define $F: I \rightarrow Y$ by

$F(0) = [0, 1/4] \cup I_2$, $F(1/4) = I_2$, $F(1/2) = I_2 \cup [1/4, 3/4] \cup I_3$, $F(3/4) = I_3$, and $F(1) = I_3 \cup [3/4, 1]$ (where intervals are subsets of I_1 unless otherwise stated). For other points in I, F is defined by ratios. The function F constructed in this manner is monotone and continuous. Also $F(x)$ is an arc for each $x \in I$. Note that the range of F is a space with two branch points and that F is also nonalternating (see § 3) but not open.

EXAMPLE 6. A construction similar to that of Example 5 can be used to define a continuous, monotone function with $F(x)$ an arc for each x on the unit interval onto the following nonlocally connected planar space. The space consists of the union of the following subsets of the plane: $\{(x, 0): 0 \leqq x \leqq 1\}$, $\{(0, y): 0 \leqq y \leqq 1\}$, and $\{(1/n, y): n \geqq 2, 0 \leqq y \leqq 1\}$.

DEFINITION. A continuum X is called a *multi-arc* in case there exists a continuous, monotone, set-valued function F on the unit interval onto X, such that $F(x)$ is a simple arc for each x in the interval. (Here $F(x)$ may be degenerate, i.e., a point.)

DEFINITION. A continuum X is called *circularly reducible* if and only if there exists a continuous, monotone function F from X onto the circle, such that $F(x)$ is a simple arc for each $x \in X$ ($F(x)$ may be a point).

REMARK. By extending the construction in Example 5, it can be shown that any dendrite with a finite number of branch points is a multi-arc. Note, however, that Example 6 shows that not all multi-arcs are locally connected, and that Example 3 shows that the disc is a multi-arc.

From Wallace [14] we have:

DEFINITION. A continuous function $F: X \to Y$ is *anarthric* if and only if for each $y \in Y$ no $x \in X - F^{-1}(y)$ separates $F^{-1}(y)$.

Then from the definition of monotone and anarthric we obtain

LEMMA 2.2. *Let X be a totally ordered, compact, connected space, and let $F: X \to Y$ be a continuous function on X into Y. Then F is anarthric if and only if F is monotone.*

Also from [14] we have

THEOREM (*Wallace*): *Let X be compact. A necessary and sufficient condition that a function F on X be anarthric is: If*

$X = M \cup N$, where M and N are continua meeting in a cutpoint x, and K is any continuum meeting M, then $F(M \cap K) = F(M) \cap F(K)$.

COROLLARY 2.3. *The circle is not a multi-arc.*

Proof. Suppose $F: [0, 1] \to C$ is a monotone continuous function on the unit interval onto a circle such that $F(x)$ is an arc for each $x \in [0, 1]$. By Lemma 2.2 F is anarthric. Thus if $x \in (0, 1)$ we have by the theorem $F(x) = F([0, x]) \cap F([x, 1])$. Also $F([0, x]) \cup F([x, 1]) = C$ and $F(x)$ is a subarc of C. Thus either $F([0, x])$ or $F([x, 1])$ is equal to C for otherwise their intersection would not be connected. Hence we may assume that there exists an x' such that $F([0, x']) = F(x')$ and $F([x, 1]) = C$. Let $x_0 = \sup \{x: F([0, x]) = F(x')$ and $F([x, 1]) = C\}$. If $y \in F(x_0) - F(x')$, and if U is an open set containing y which does not meet $F(x')$, then $F^{-1}(U)$ is an open set containing x_0 which does not meet $\{x: F([0, x]) = F(x')\}$. This contradicts the choice of x_0. Hence $F(x_0) = F(x')$ and $F([x_0, 1]) = C$. Note $F([x_0, 1]) = C$ implies that $x_0 \neq 1$. Now if $x > x_0$, $F([0, x]) = C$ since x_0 is the sup $\{x: F([x, 1]) = C$ and $F([0, x]) = F(x') = F(x_0)\}$. Thus for $y \in C - F(x_0)$ there is a decreasing sequence $\{x_n\}$ such that $x_n \to x_0$ and $y \in F(x_n)$ for all n. But this implies that $x_0 \in F^{-1}(y)$ since $F^{-1}(y)$ is closed, a contradiction.

We can derive more corollaries to Theorem 2.3.

COROLLARY 2.4. *A hereditarily unicoherent multi-arc is not circularly reducible.*

Proof. Suppose that x is circularly reducible, and that $F_2: X \to C$ is a continuous, monotone function on X onto the circle C such that $F_2(x)$ is a simple arc for each $x \in X$. Since X is a multi-arc there exists a continuous, monotone function F_1 on the unit interval I onto X such that $F_1(r)$ is a simple arc for each $r \in I$. Then by Lemma 2.1 the function $F = F_2 \circ F_1$ is continuous and monotone, and F maps I onto C. Now let M be an arc contained in X. Then $F_2 | M$ is continuous. Further, if $y \in C$, either $F_2^{-1}(y) \cap M = \phi$ or $F_2^{-1}(y) \cap M$ is connected since X is hereditarily unicoherent. Therefore $F_2 | M$ is monotone. Hence $F_2(M) \neq C$. Further, if $M \in I$, $F_1(r)$ is at most an arc, and hence, $F_2 \circ F_1(r) \neq C$. Note that $F_2 \circ F_1(r)$ is connected. Consequently, F is a continuous, monotone function on I onto C such that $F(r)$ is a simple arc for each $r \in I$; this is a contradiction. Hence the result holds.

COROLLARY 2.5. *A hereditarily unicoherent, arcwise connected continuum is not circularly reducible.*

Proof. We sketch the proof of this result. Let X be an hereditarily unicoherent, arcwise connected continuum. First observe that the set $\{F(x): x \in X\}$ has maximal elements, where $F: X \to C$ is a monotone function on X onto C such that $F(x)$ is an arc. If x', x'' are such that $F(x')$ and $F(x'')$ are maximal, then $F(x') \cap F(x'') \neq \phi$ and $F(x') \cup F(x'') \neq C$. From Corollary 2.4, if we have $x_1, x_2, \cdots, x_n$ such that $F(x_1), F(x_2), \cdots, F(x_n)$ are maximal, then $\bigcup_{i=1}^{n} F(x_i) \neq C$. Then the fact that X is compact is used to complete the proof.

3. Nonalternating functions. The purpose of this section is to generalize the definition of nonalternating functions to set-valued functions and to derive some characterizations of such functions.

DEFINITION. A function $F: X \to Y$ is called nonalternating if and only if for any pair $y_1, y_2 \in F(X)$ there does not exist a separation $X - F^{-1}(y_1) = A \cup B$ such that $y_2 \in F(A) \cap F(B)$.

EXAMPLE 7. Let $X = [0, 1]$ and define $F: X \to X$ by $F(\frac{1}{2}) = \{0\}$, $F(x) = [0, 2(x - \frac{1}{2})]$ for $x > \frac{1}{2}$ and $F(x) = [0, 2(\frac{1}{2} - x)]$ for $x < \frac{1}{2}$. Then F is continuous and nonalternating, but not monotone. Further, this serves as a counterexample to theorems which are true for single-valued functions [18, pp. 138–140].

DEFINITION. A multi-valued function $F: X \to Y$ is called *semi-single-valued* (s.s.v.) if and only if $F(x_1) \cap F(x_2) \neq \phi$ implies that $F(x_1) = F(x_2)$.

A very small change will allow us to get the counterpart to Theorem 2.1 [18, p. 138].

THEOREM 3.1. *Let $F: X \to Y$ be continuous. Then F is nonalternating if and only if for each $y \in Y$, and each quasi-component Q of $X - F^{-1}(y)$, $F^{-1}(F(Q)) \cap (X - F^{-1}(y)) = Q$.*

Proof. Suppose that F is nonalternating and that Q is a quasi-component of $X - F^{-1}(y)$ for $y \in Y$. Then, if

$$x \in F^{-1}(F(Q)) \cap (X - F^{-1}(y)) - Q ,$$

there exists a separation $X - F^{-1}(y) = A \cup B$ such that $x \in A$ and $Q \subset B$, as Q is a quasi-component. However, this implies that $F(A) \cap F(B) \neq \phi$, as $x \in F^{-1}(F(Q))$ implies $F(x) \cap F(Q) \neq \phi$ which implies there exists an $x' \in Q$ such that $F(x) \cap F(x') \neq \phi$. This is contrary to the assumption that F is nonalternating. If F is not nonalternating, there exist points $y_1, y_2 \in Y$, and a separation $X - F^{-1}(y_1) = A \cup B$ such that $y_2 \in F(A) \cap F(B)$. Let $x \in A$ with $y_2 \in F(x)$ and let Q be the quasi-component of $X - F^{-1}(y_1)$ containing x. Since $y_2 \in F(B)$, there exists $x' \in B$ such that $y_2 \in F(x')$. Hence,

$$x' \in F^{-1}(F(Q)) \cap (X - F^{-1}(y_1)) - Q ,$$

and the condition fails.

We also obtain

THEOREM 3.2. *Let $F: X \to Y$ be continuous, and let $y \in Y$. Let Q be any quasi-component of $Y - \{y\}$. Then if $F^{-1}(Q) \cap (X - F^{-1}(y))$ is contained in a quasi-component of $X - F^{-1}(y)$, F is nonalternating.*

Proof. Let $y_1, y_2 \in Y$, and let Q be the quasi-component of $Y - y_1$ which contains y_2. Then, since

$$F^{-1}(y_2) \cap (X - F^{-1}(y_1)) \subset F^{-1}(Q) \cap (X - F^{-1}(y_1)) ,$$

the hypothesis implies that for any separation

$$X - F^{-1}(y_1)\ A \cup B,\ A \mid B ,$$

$F^{-1}(y_2) \cap (X - F^{-1}(y_1))$ is contained in A or in B. Thus, F is nonalternating.

DEFINITIONS. Denote the set of all points that separate a and b by $E(a, b)$. Then call a, b conjugate in case $E(a, b) = \phi$. Then, if x is neither a cutpoint nor an end point, the set containing x and all points which are conjugate to x is called a *simple link*. Finally, a *cyclic element* of X is either a cutpoint, an end point, or a simple link.

DEFINITION. A connected space is called *semi-locally-connected* (s.l.c.) at a point x in case x has arbitrarily small neighborhoods whose complements have only a finite number of components. If X is s.l.c. at each of its points, it is called s.l.c.

Using a result of Wallace [14] we can generalize a result on single-valued functions in [18] to multi-valued functions.

THEOREM (Wallace). *A function $F: X \to Y$ on a continuum X into a continuum Y is anarthric if and only if for any subcontinuum H of Y and any subcontinuum K of X such that $K \cap F^{-1}(H) = P \cup Q$, $P \mid Q$, there exist points $p \in P$, $q \in Q$, such that p and q are conjugate.*

THEOREM 3.3. *Let $F: X \to Y$ be continuous and semi-single valued, and let X be a semi-locally-connected, metric continuum and Y a* *metric continuum. Then F is nonalternating if and only if the following hold.*

(i) *F is anarthric,*

(ii) *F is nonalternating on each cyclic element of X.*

Proof. Suppose that F is nonalternating. Let $y \in Y$, and suppose there exists a point $p \in E(a, b) - F^{-1}(y)$, where $a, b \in F^{-1}(y)$. Now $y \notin F(p)$, thus $(F(a) \cup F(b)) \cap F(p) = \phi$ since F is semi-single-valued. Moreover, there exists a separation $X - p = A \cup B$, $A \mid B$, with $a \in A$ and $b \in B$. Let $y' \in F(p)$. Then, there exists a separation A', B' of $X - F^{-1}(y')$ such that $a \in A'$ and $b \in B'$, which implies that $F(A') \cap F(B') \neq \phi$. This contradicts the hypothesis that F is nonalternating. Thus, (i) holds.

In order to show (ii) holds, let E be a true cyclic element of X (i.e., a simple link). Let $F(E) = E' \subset Y$, and let $y_1, y_2 \in E'$. If $E - F^{-1}(y_1) \cap E = A \cup B$, $A \mid B$ such that $y_2 \in F(A) \cap F(B)$, then by [18, IV, 3.22 and 6.81], there exists a separation of $X - F^{-1}(y_1) = A' \cup B'$, $A' \mid B'$, with $y_2 \in F(A') \cap F(B')$, a contradiction.

Suppose (i) and (ii) hold. Let $y \in Y$. If $X - F^{-1}(y) = A \cup B$, $A \mid B$, and if $x_1 \in A$, $x_2 \in B$ such that $F(x_1) \cap F(x_2) \neq \phi$, then by the result of Wallace there exist x_1' and x_2' which are separated by $F^{-1}(y)$ and which are contained in the same cyclic element, but this contradicts (ii). Thus F is nonalternating.

COROLLARY 3.4. *Any nonalternating semi-single-valued fnnction on a dendrite is monotone.*

Proof. If $a, b \in F^{-1}(y)$ then by (i), $E(a, b) \subset F^{-1}(y)$, and $E(a, b)$ is a simple arc from a to b.

4. Composite functions and factorization.

In this section some of the properties of composite functions are investigated and a factorization theorem is obtained.

DEFINITION. A function $F: X \rightarrow Y$ is called *open* in case whenever U is open in X, $F(U)$ is open in Y.

Let X, Y and Z be compact spaces, and let F, F_1, and F_2 be continuous functions such that $F_1: X \rightarrow Z$, $F_2: Z \rightarrow Y$ and $F = F_2 \circ F_1$, $F: X \rightarrow Y$.

Lemmas 4.1 and 4.2 are extensions of results which hold for single-valued functions. The proofs are straightforward and are omitted.

LEMMA 4.1. *If F_1 is single valued:*

(i) *F open implies that F_2 is open;*

(ii) *F monotone implies that F_2 is monotone;*

(iii) *F nonalternating implies that F_2 is nonalternating.*

In addition to this we can obtain:

LEMMA 4.2. *The following statements hold.*

(i) *F_1, F_2 open implies F is open;*

(ii) *F_1, F_2 monotone implies F is monotone;*

(iii) *F_1 monotone and s.s.v., and F_2 nonalternating imply F is nonalternating.*

We now turn to the problem of factoring functions. First we have the known Theorem A, Whyburn [18, pp. 141–142], which is stated below. (Note that Theorem A holds for any compact Hausdorff space, as well as for metric spaces.)

DEFINITION. A function $F: X \to Y$ is called *light* in case $F^{-1}(y)$ is totally disconnected for each $y \in Y$.

THEOREM A. *Let g be a single-valued, continuous function from X onto Y. Then there exist a space Z and continuous functions g_1, g_2; $g_1: X \to Z$, $g_2: Z \to Y$, such that g_1 is monotone, g_2 is light, and $g = g_2 \circ g_1$.*

We can extend this theorem to semi-single-valued functions, but first we need the following lemma.

LEMMA 4.3. *Let $\mathscr{S} \subset S(X)$, and let $\mathscr{S}$ have the finite topology. Define a function $F: \mathscr{S} \to X$ by $F(S) = S$ for all $S \in \mathscr{S}$. Then F is continuous.*

Proof. Let U be an open set contained in X. If $S \in \mathscr{S}$ and $S \cap U \neq \phi$, the set $\langle U, X \rangle = \{S \in \mathscr{S}: S \cap U \neq \phi\}$ is an open set in $\mathscr{S}$ such that $F(S) \cap U \neq \phi$ for all $S \in \langle U, X \rangle$. If $S \subset U$, then $\langle U \rangle = \{S \in \mathscr{S}: S \subset U\}$ is an open set in $\mathscr{S}$ such that $F(\langle U \rangle) \subset U$. Thus, by Lemma 1.1, F is continuous.

Note. If $\mathscr{S}$ is a decomposition, then $F^{-1}(x) = \{S\}$ where $x \in S$, and if $F: X \to Y$ is semi-single-valued, then $\mathscr{S} = \{F(x): x \in X\}$ is a decomposition.

THEOREM 4.5. *Let $F: X \to Y$ be continuous and semi-single-valued. Then there exist a space Z and continuous functions F_1, F_2 with $F_1: X \to Z$ single-valued, $F_2: Z \to Y$, $F = F_2 \circ F_1$, and such that F_1 is monotone, and F_2 is light.*

Proof. Let f be the induced single-valued function on X into $S(Y)$. Then $f(X) = \{F(x): x \in X\}$ is a decomposition of Y. Then by Theorem A there exist a space Z and continuous functions f_1, f_2 such that f_1 is monotone, f_2 is light, and $f = f_2 \circ f_1$. Let F^* be the function of Lemma 4.3. Then set $F_1 = f_1$ and $F_2 = F^* \circ f_2$. Thus, F_1 is single valued and monotone and, from the remark following Lemma 4.3, F_2 is continuous and light. Finally, $F = F_2 \circ F_1$.

Finally, with Lemma 4.1 (i), we get:

COROLLARY 4.6. *If $F: X \to Y$ is semi-single-valued, continuous and open, then there exist continuous functions F_1, F_2 such that F_1 is single-valued and monotone, and F_2 is light and open, and such that $F = F_2 \circ F_1$.*

5. Semi-single-valued functions. Let $F: X \to Y$ be a semi-single-valued continuous function from X onto Y, and define the collections $Q = \{F(x): x \in X\}$, and $P = \{F^{-1}(y): y \in Y\}$. That P and Q are decompositions into disjoint closed sets follows from the definition of a continuous, semi-single-valued function.

Let $q: Y \to Q$ and $p: X \to P$ be the projections of Y onto Q and X onto P, respectively. Define $F^\#$ on P onto Y by $F^\#(D) = F(x)$ for $D \in P$ and $x \in D$, and define f' on X onto Q by $f'(x) = F(x)$. Note that f' and f are essentially the same but Q as a decomposition has the quotient topology rather than the finite topology. When F is the inverse of a single-valued function, we have by Theorem 5.10 [6], that the quotient and the finite topologies are equivalent. We shall generalize this result in Corollary 5.3. Finally, define $f^*: P \to Q$ by $f^*(D) = F(x)$ for $D \in P$ and $x \in D$.

THEOREM 5.1. *If X and Y are compact, the decompositions P and Q are upper semi-continuous. Further, P and Q are Hausdorff in the quotient topology.*

Proof. Let V_1, V_2 be disjoint open subsets of Y such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Then, for $i = 1, 2$, $Y - FF^{-1}(Y - V_i)$ is an open set containing $F(x_i)$ which is contained in V_i and which is the union of members of Q. Similarly, if $F^{-1}(y_1) \neq F^{-1}(y_2)$ are in P and if U_1 and U_2 are open and disjoint with $F^{-1}(y_i) \subset U_i$, then $X - F^{-1}F(X - U_i)$, $i = 1, 2$, are the required open sets. This shows that P and Q are upper semi-continuous, and Hausdorff in the quotient topology.

THEOREM 5.2. *The functions $F^\#$ and f' are continuous when P and Q have the quotient topology.*

Proof. Since $F^{\#}(D) = F(p^{-1}(D))$ for $D \in P$, Theorem 5.1 implies that $F^{\#}$ is continuous. Also $f' = q \circ F$ and hence is continuous by Lemma 1.12.

Now we obtain a generalization of Theorem 5.10 [6].

COROLLARY 5.3. *If X and Y are compact and $F: X \to Y$ is a semi-single-valued continuous function, then the finite and quotient topologies agree on $Q = \{F(x): x \in X\}$, and f and f' are equivalent functions.*

Proof. The function $F^{\#}$ is the inverse of a single-valued function. Hence, Theorem 5.10 [6] applies.

THEOREM 5.4. *The function $f^*: P \to Q$ is a homeomorphism onto, when X and Y are compact.*

Proof. That f^* is a single-valued function which is 1 to 1 and onto follows immediately from the fact that F is semi-single-valued. That f^* is continuous follows from $f^* = q \circ F^{\#}$, Theorem 5.2 and Corollary 5.3.

We associate with each multi-valued function $F: X \to Y$ the induced function f on X into $S(Y)$ and we can define a function F^* on $f(X)$ into Y by $F^*(f(x)) = F(x)$. Then $F = F^* \circ f$. We consider briefly the relationships between F, f and F^* and the properties of being monotone, open, and nonalternating. A typical question is: "Does F monotone imply that f is monotone, and conversely?" Simple examples show that f monotone does not imply that F is monotone, and Example 8 shows that the converse fails.

EXAMPLE 8. Let X be the rectangle with corners $(0, -1)$, $(1, -1)$, $(1, 1)$ and $(0, 1)$ together with its interior. Let Y be the unit interval. Let $(x, y) \in X$ and define $r_1 = \frac{1}{2}(1 - x)$, $r_2 = \frac{1}{2}(1 + x)$. Then define $z_1 = r_1(1 - |y|)$, $z_2 = r_1 + |y|(\frac{1}{2} - r_1)$, $z_3 = r_2 - |y|(r_2 - \frac{1}{2})$ and $z_4 = r_2 + |y|(1 - r_2)$ with $r_i, z_j \in Y$. Define $F: X \to Y$ by $F((x, y)) = [z_1, z_2] \cup [z_3, z_4]$. Then F is monotone and continuous but f is not monotone.

However, if F is semi-single-valued, we have

THEOREM 5.5. *If $F: X \to Y$ is a semi-single-valued, continuous function from X onto Y, then F is monotone if and only if f is monotone.*

Proof. If $y \in Y$, then there exists a unique S in $F(X)$ such that

$y \in S$. Thus $F^{-1}(y) = \{x: F(x) = S\} = f^{-1}(S)$. So $F^{-1}(y)$ is connected if and only if $f^{-1}(S)$ is connected.

THEOREM 5.6. *The following statements hold.*
(i) *F monotone implies F^* is monotone.*
(ii) *F open implies F^* is open.*
(iii) *If F is semi-single-valued, F open implies f is open.*
(iv) *F nonalternating implies F^* is nonalternating.*
Further, we may state a partial converse to (i), (ii) and (iv).

THEOREM 5.7. *If f is monotone, then*
(i) *F^* monotone implies that F is monotone; and*
(ii) *F^* nonalternating implies that F is nonalternating.*

THEOREM 5.8. *If f is open, F^* open implies F is open.*

6. Open functions. The purpose of this section is to show that certain results in Whyburn [18] on open mappings can be generalized to semi-single-valued functions and in some cases to arbitrary multi-valued functions. In this section all spaces will be separable, metric spaces.

REMARK 1. The definition of terms used in this section are those of Whyburn [18].

REMARK 2. If X is compact, then a collection of subsets G of X is continuous if and only if it is continuous in the limit sense.

THEOREM 6.1. *Let $F: X \to Y$ be a continuous, semi-single-valued function of X onto Y. If F is open, then the collection $\{F^{-1}(y): y \in Y\}$ is continuous in the limit sense. Conversely, if X is compact, and if the collection $\{F^{-1}(y): y \in Y\}$ is continuous, then F is open.*

Proof. By Theorem 5.6, F open implies f open and since F is s.s.v., $f^{-1}(F(x)) = F^{-1}(y)$, $y \in F(x)$. Thus, the first statement follows from the theorem for single-valued functions [18, Theorem 4.31, p. 130], and minor modifications of the proof in [18] will yield a proof of the converse.

COROLLARY 6.2. *Let X be compact and let F be as in Theorem 6.1. Then F is open if and only if the collection $\{F^{-1}(y): y \in Y\}$ is continuous.*

We can also generalize a theorem due to Eilenberg, [18, p. 138].

THEOREM 6.3. *Let $F: X \to Y$ be continuous, semi-single-valued, and onto. Then F is open if and only if for each sequence $\{y_n: n = 1, \cdots\}$ in Y such that $\lim_n y_n = y_0$, $\lim F^{-1}(y_n) = F^{-1}(y_0)$.*

Proof. Suppose that F is open, and that $\{y_n\}$ is a sequence in Y such that $\lim_n y_n = y_0$. In view of Theorem 6.1 we need only show that $F^{-1}(y_0) \cap \lim\inf F^{-1}(y_n) \neq \phi$. If $x \in F^{-1}(y_0)$, if U is an open set containing x, and if $U \cap F^{-1}(y_n) = \phi$ for infinitely many n, then $F(U)$ is an open set containing y_0 such that $y_n \notin F(U)$ for infinitely many n, a contradiction to $\lim_n y_n = y_0$.

Now suppose that $\lim_n y_n = y_0$ implies $\lim F^{-1}(y_n) = F^{-1}(y_0)$, and let U be open in X. If $F(U)$ is not open in Y, there exists $y_0 \in U$ and a sequence $\{y_n\} \subset Y - F(U)$ such that $\lim y_n = y_0$. Now $y_0 \in F(U)$ implies that there exists an $x \in F^{-1}(y_0) \cap U$, and from the hypothesis $U \cap F^{-1}(y_n) \neq \phi$ for all but finitely many n. Thus $y_n \in F(U)$ for all but finitely many n, a contradiction. Hence F is open.

The proof of the following lemma is straightforward and is omitted. Note that in many of the following results the restriction to separable metric spaces is unnecessary.

LEMMA 6.4. *Let $F: X \to Y$ be continuous and onto. Then $Q \subset X$ is an inverse set if and only if $F(A \cap Q) = F(A) \cap F(Q)$ for each $A \subset X$.*

LEMMA 6.5. *Let $F: X \to Y$ be continuous and open. If $Q \subset X$ is an inverse set, then F restricted to Q is open.*

Proof. Let V be open in Q. Then there exists an open set U in X such that $V = Q \cap U$. Then, by Lemma 6.4, $F(V) = F(U \cap Q) = F(U) \cap F(Q)$, which is open in $F(Q)$ since $F(U)$ is open.

In order to establish the next result we need a theorem of Michael's [6, Theorem 2.5.1].

THEOREM B. *If X is regular, and $B \subset S(X)$ is compact, then $\cup\{E: E \in B\}$ is closed.*

THEOREM 6.6. *Let $F: X \to Y$ be onto and continuous. Then, if $A \subset X$ is conditionally compact:*

(i) $\overline{F(A)} = F(\bar{A})$;

(ii) $\overline{F(A)} - F(A) \subset F(\bar{A} - A)$.

Further, if F is an open function, and A is an open set,

(iii) $b(F(A)) \subset F(b(A))$

where $b(A)$ denotes the boundary of A.

Proof.

(i) Let $A \subset X$ be conditionally compact. Then, by Theorem B, $F(\bar{A})$ is closed. Hence $\overline{F(A)} \subset F(\bar{A})$. Also $F(\bar{A}) \subset \overline{F(A)}$ since F is continuous.

(ii) From (i), $\overline{F(A)} - F(A) = F(\bar{A}) - F(A) \subset F(\bar{A} - A)$.

(iii) With A open and F open this is immediate from (ii).

LEMMA 6.7. *Let U, U_1, U_2 be open sets such that $U = U_1 \cup U_2$. If $U_1 \cap U_2 = \phi$, then $b(U) = b(U_1) \cup b(U_2)$.*

Proof. If $x \in b(U)$, then $x \in \bar{U}_1$ or $x \in \bar{U}_2$ and $x \notin U_1 \cup U_2$. Therefore $x \in b(U_1)$ or $x \in b(U_2)$. On the other hand $x \in b(U_i)$ implies $x \in \bar{U}_i$ and $x \notin U_1 \cup U_2$. Thus $x \in b(U)$.

THEOREM 6.8. *Let X and Y be continua, and let $F: X \to Y$ be continuous, open and onto. Then:*

(i) *If X is a curve of order less than or equal to n, and if $F(x)$ contains at most m points for each $x \in X$, then Y is a curve of order less than or equal to nm;*

(ii) *If X is a regular curve and if $F(x)$ is finite for each x, then Y is a regular curve; and*

(iii) *If X is a rational curve and $F(x)$ countable for each x, then Y is a rational curve.*

Proof.

(i) Let $y \in Y$, and let V be an open set containing y. Since F is onto, there exists an $x \in X$ such that $y \in F(x)$. Let $F(x) = \{y_1, \cdots, y_k\}$, $k \leqq m$. Suppose $y = y_1$. Then there exist open sets V, V' of Y such that $y_1 \in V$, $\{y_2, \cdots, y_k\} \subset V'$ and $V \cap V' = \phi$. Further there exists an open set U containing x such that $F(U) \subset V \cup V'$ and such that $b(U)$ contains at most n points (as X is a curve of order less than or equal to n). Let $V_1 = F(U) \cap V$ and $V_2 = F(U) \cap V'$, V_1 and V_2 are open and disjoint. Thus, by Lemma 6.7, $b(F(U)) \subset b(V_1) \cup b(V_2)$. Therefore by Theorem 6.6, $b(V_1) \subset b(F(U)) \subset F(b(U))$ and this latter set contains at most nm points. Thus V_1 is the required open set containing y.

(ii) A proof similar to the proof of (i) will establish (ii).

(iii) Let $x \in Y$, and let V be an open set containing y. Pick an $x \in X$ such that $y \in F(x)$. Since $F(x)$ is countable, we may assume that $F(x) \subset V \cup (Y - \bar{V})$. Since X is rational, there exists an open set U containing X such that $F(U) \subset V \cup (Y - \bar{V})$, with $b(U)$ countable. By part (iii) of Theorem 6.6, $b(F(U)) \subset F(b(U))$ and $F(b(U))$ is countable. Then $F(U) \cap V$ is an open set containing y

with countable boundary. This last since

$$b(F(U)) = b(V \cap F(U)) \cup [b(Y - \bar{V}) \cap F(U)],$$

by Lemma 6.7.

Following Whyburn's proof [18, p. 147, 7.4], we can prove

THEOREM 6.9. *Let X be compact and let $F: X \rightarrow Y$ be continuous, open and onto. If A is a connected open set in Y, and if Q is any quasi-component of $F^{-1}(A)$, then $A \subset F(Q)$.*

COROLLARY 6.10. *Let X and Y be locally connected, compact spaces, $F: X \rightarrow Y$ open and onto, and let A be any closed set in Y. If C is any component of $Y - A$, then $F^{-1}(C)$ has only a finite number of components and each of these maps onto all of C under F.*

Proof. It follows from the hypothesis that any quasi-component of $F^{-1}(C)$ is also a component of $F^{-1}(C)$. Then if $F^{-1}(C)$ has an infinite number of quasi-components, a sequence constructed by choosing one element from each quasi-component must have a limit point. However, each quasi-component is open; hence no subsequence can converge to the limit point, a contradiction. Finally, C is open so Theorem 6.5 implies that $C \subset F(Q)$ for any quasi-component $Q \subset F^{-1}(C)$.

PROPOSITION 6.11. Let $F: X \rightarrow Y$ be open and onto, and let Y be connected. If X_0 is an inverse set in X which is open and closed, then $F(X_0) = Y$.

Proof. Since X_0 is an inverse set and F is open, $F(X_0)$ and $F(X - X_0)$ are disjoint open sets whose union is Y. Therefore, $F(X_0) = Y$.

REMARK. Let $F: X \rightarrow Y$ be continuous. If C is a subset of Y, then $F^{-1}(C)$ need not be an inverse set. However, if F is an s.s.v. function, we have:

LEMMA 6.12. *Let $F: X \rightarrow Y$ be an* s.s.v. *function. If $C \subset Y$, then $F^{-1}(C)$ is an inverse set.*

Proof. If $x \in F^{-1}FF^{-1}(C)$, then $F(x) \cap F(F^{-1}(C)) \neq \phi$. Thus there exists an $x' \in F^{-1}(C)$ such that $F(x) \cap F(x') \neq \phi$. Then since F is s.s.v., $F(x) = F(x')$. Therefore, $F(x) \cap C \neq \phi$, and hence $x \in F^{-1}(C)$. Consequently, $F^{-1}(C)$ is an inverse set.

THEOREM 6.13. *Let X be compact, and let $F: X \rightarrow Y$ be a con-*

tinuous, open semi-single-valued function on X onto Y. Let C be any compact, connected set in Y. Then for any component K of $Q = F^{-1}(C)$, it follows that $C \subset F(K)$.

Proof. Since F is s.s.v., $F^{-1}(Q)$ is an inverse set in X by Lemma 6.12. Hence, by Lemma 6.5, F restricted to Q is open and the result follows by applying Theorem 6.9 to F restricted to Q.

Note. Single-valued open, continuous functions map nodal sets onto nodal sets (A is nodal in case $A \cap \overline{X - A}$ is at most a single point), but Example 3 is a counterexample to this result for s.s.v. mappings. In fact, in Example 3, F is the inverse of a continuous single-valued function.

7. Quasi-monotone functions. In this section X and Y are compact and connected, and $F: X \to Y$ will always denote a continuous function of X onto Y.

DEFINITION. A function F is called *quasi-monotone* in case for each continuum $Y_0 \subset Y$ with nonvoid interior, $F^{-1}(Y_0)$ has only a finite number of components C_n and $Y_0 \subset F(C_n)$ for each component C_n of $F^{-1}(Y_0)$. Note that any monotone function on a continuum is quasi-monotone.

REMARK. If g is a continuous single-valued function on a compact, connected, locally connected space X, then $g(X)$ is also compact, connected and locally connected. However, when F is multi-valued this may not be the case, so it is sometimes necessary to assume that Y as well as X is compact, connected, and/or locally connected.

The proof of Theorem 7.1 is very much like the proof of the corresponding theorem for single-valued functions [18, p. 152, Th. 8.1] and is omitted.

THEOREM 7.1. *Let X and Y be locally connected continua, and let $F: X \to Y$. Then F is quasi-monotone if and only if for each component C of the inverse of any connected open set V of Y, $V \subset F(C)$.*

COROLLARY 7.2. *Every open function on a locally connected continuum onto a locally connected continuum is quasi-monotone.*

Proof. Corollary 6.10 implies that the hypotheses of Theorem 7.1 are satisfied.

THEOREM 7.3. *If X and Y are locally connected continua, and if F is light, then F is quasi-monotone if and only if F is open.*

Proof. If F is open, then F is quasi-monotone by Corollary 7.2. Suppose that F is quasi-monotone, let U be open in X, and let $y \in F(U)$. If $x \in U \cap F^{-1}(y)$, then since F is light there exists a connected open set $U' \subset U$ such that $x \in U'$ and $b(U') \cap F^{-1}(y) = \phi$. Let Q be the component of $Y - F(b(U'))$ containing y, and let C be the component of $F^{-1}(Q)$ containing x. Then $C \subset U'$ since $C \cap b(U') = \phi$, and by Theorem 7.1, $Q \subset F(C)$. Then $Q \subset F(C) \subset F(U') \subset F(U)$ and Q is an open set containing y. Thus $F(U)$ is open.

THEOREM 7.4. *Let X, Y and Z be locally connected, and let $F = F_2 \circ F_1$, $F_1: X \to Z$, $F_2: Z \to Y$ with F_1 and F_2 continuous and onto. Then:*

(i) *If F is quasi-monotone and F_1 is single-valued, F_2 is quasi-monotone; and*

(ii) *If F_1 and F_2 are quasi-monotone, F is quasi-monotone.*

Proof.

(i) Let V be an open connected set in Y, and let C be a component of $F_2^{-1}(V)$. Let C' be a component of $F^{-1}(V)$ contained in $F_1^{-1}(C)$. Then, since F_1 is single-valued, $F_1(C') \subset C$, and since F is quasi-monotone, $V \subset F_2 \circ F_1(C') = F(C)$. Therefore, $V \subset F_2(C)$, as $F_2 \circ F_1(C') \subset F_2(C)$, and F_2 is quasi-monotone by Theorem 7.1.

(ii) Let V be an open connected set in Y. Let C be a component of $F^{-1}(V)$, and let Q be a component of $F_2^{-1}(V)$ such that C contains a component of $F_1^{-1}(Q)$. Then, since F_1 is quasi-monotone, $Q \subset F_1(C)$. Further, F_2 quasi-monotone implies that $V \subset F_2(Q)$. Thus $V \subset F_2 \circ F_1(C) = F(C)$.

THEOREM 7.5. *Let X and Y be locally connected and let $F: X \to Y$ be* s.s.v. *Then F is quasi-monotone if and only if there exists a locally connected continuum Z, a continuous monotone function F_1 of X onto Z and a continuous, light, open function F_2 of Z onto Y such that $F = F_2 \circ F_1$.*

Proof. If such a Z, F_1, and F_2 exist, then F is quasi-monotone by Corollary 7.2 and by Theorem 7.4, Part (ii). If F is quasi-monotone, then by Theorem 4.5 there exists a continuum Z, and a monotone, single-valued function F_1 of X onto Z, and there exists a continuous, light function F_2 of Z onto Y, such that $F = F_2 \circ F_1$. By Theorem 7.4, F_2 is quasi-monotone and therefore, by Theorem 7.3, F_2 is open.

Finally, combining the results of Theorem 7.3, the fact that monotone functions are quasi-monotone, and Theorem 7.5, we have the following result for semi-single-valued functions.

THEOREM 7.6. *A topological property of locally connected continua is invariant under quasi-monotone, semi-single-valued functions if and only if it is invariant under both monotone and light, open, semi-single-valued functions.*

8. Local properties and functions with finite images. In previous sections we have exhibited examples of functions that did not preserve local properties. We saw that even if $F(x)$ was an arc for each x, the image of the unit interval need not be locally connected. The purpose of this section is to show that if $F(x)$ is finite for each $x \in X$, then local properties may be preserved. The main theorem is this: If F is defined on a locally connected metric continuum X onto a metric continuum Y, and if $F(x)$ is finite for each x, then Y is locally connected.

NOTATION. Designate the number of points in $F(x)$ by $N(F(x))$, and if $N(F(x)) \leqq n$ for all x, then write $N(F) \leqq n$. $N(F) = n$ means that $N(F) \leqq n$ and there is at least one x such that $N(F(x)) = n$. If $F(x)$ is finite for each x, write $N(F) < \infty$. Finally, $N(F) \equiv n$ means $N(F(x)) = n$ for all $x \in X$.

LEMMA 8.1. *Let $F: X \rightarrow Y$ be continuous with $N(F) < \infty$. If K is a connected subset of X, then $F(K)$ has at most n components, where $n = \min N(F(x))$. If C is a component of $F(K)$ and if $x \in K$, then $F(x) \cap C \neq \phi$.*

Proof. Let C be a component of $F(K)$ and let $x \in K$. Suppose $F(x) \cap C = \phi$. Define $K_1 = \{x \in K: F(x) \cap C = \phi\}$ and $k_2 = \{x \in K: F(x) \cap C \neq \phi\}$. Clearly K_1, $K_2 \neq \phi$ and $K = K_1 \cup K_2$. Also $K_2 \subset F^{-1}(\bar{C})$ and so $\bar{K}_2 \cap K_1 = \phi$. If $x \in \bar{K}_1 \cap K_2$, then $F(x) \cap C \neq \phi$ and $x \in \bar{K}_1$ implies there is an $x' \in K_1$ such that $F(x') \cap C \neq \phi$, a contradiction. Hence $F(x) \cap C \neq \phi$. Finally since $n = \min N(F(x))$, $x \in K$, there can be at most n components of $F(K)$.

PROPOSITION 8.2. Let $F: X \rightarrow Y$ be open, continuous, and onto with $N(F) < \infty$. Then the following statements hold:

(i) X locally compact implies Y locally compact;

(ii) X locally connected implies Y locally connected.

Proof. Both proofs are done at once. Let $y \in Y$ and $x \in X$ such that $y \in F(x)$. Let $F(x) = \{y, y_1, \cdots, y_k\}$, and let $V_0, V_1, \cdots, V_k$ be disjoint open sets containing $y, y_1, \cdots, y_k$, respectively. Then there exists an open set U with $\bar{U}$ compact (or U connected) such that $F(\bar{U}) \subset \bigcup_{i=0}^{k} V_i$, and $F(\bar{U}) \cap V_j \neq \phi$ for all j. Then since F is open, $F(U)$ and hence $F(U) \cap V_0$ is open. Further, $F(\bar{U}) \subset \bigcup_{i=0}^{k} V_i$. Thus, when $\bar{U}$ is compact, $F(\bar{U}) \cap V_0$ is compact and (i) is proved. Moreover, when U is connected, $F(U)$ has exactly $k+1$ components C_i, each of which is open. If C_0 is the component of $F(U)$ containing y, then $C_0 \subset V_0$ and C_0 is connected. Hence Y is locally connected.

We now state one of the main results of this section.

THEOREM 8.3. *Let X and Y be compact metric spaces and let $F: X \to Y$ be continuous and onto with $N(F) < \infty$. If X is locally connected, then Y is locally connected.*

Proof. We shall show that Y has property S. Let $\varepsilon > 0$. Let $x \in X$ and $F(x) = \{y_1, \cdots, y_k\}$. There exist open sets $V_1, \cdots, V_k$ in Y with $d(V_i) < \varepsilon$ for each i, and $V_i \cap V_j = \phi$, $i \neq j$, such that $y_i \in V_i$ for each i. Since X is compact and locally connected, there exists an open connected set U_x containing x such that $F(U_x) \subset \bigcup_{i=1}^{k} V_i$. Thus, by Lemma 8.1, $F(U_x)$ has k components each of which has diameter less than ε. We obtain such a U_x for each x and extract a finite subcover $U_{x_1}, \cdots, U_{x_q}$. Then, if $F(U_{x_j})$ has components $C_{j1}, \cdots, C_{jn_j}$,, the collection $\{C_{ij}: i = 1, \cdots, q, j = 1, \cdots, n_i\}$ is a finite cover of Y by connected sets of diameter less than ε. Hence, Y has property S and is locally connected.

COROLLARY 8.4. *Let X be a locally connected, metric continuum, Y a metric space, and let $F: X \to Y$ be continuous. If $N(F) < \infty$, and* min $N(F(x)) = n$, *then $F(X)$ is the union of at most n locally connected, metric continua.*

PROPOSITION 8.5. If $F: X \to Y$ is a continuous function with $N(F) \leqq n$ and if C is a component of $F(X)$, then $F^*: X \to C$ defined by $F^*(x) = F(x) \cap C$ is continuous with $N(F^*) \leqq n - r$, where r is the number of other components of $F(X)$.

Proof. Since C is a component of $F(X)$ there is an open subset of Y which contains C and does not meet any other component of $F(X)$. By Lemma 8.1, $F(x) \cap C \neq \phi$ for all $x \in X$. Thus the result follows by Lemma 1.1.

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