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PROPER ORDERED INVERSE SEMIGROUPS

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Let S be an ordered inverse semigroup, that is, an inverse semigroup with a simple order $<$ which satisfies the condition:

$$x < y \quad \text{implies} \quad xz \leqq yz \quad \text{and} \quad zx \leqq zy .$$

Let E be the subsemigroup of S constituted by all the idempotents of S. By a result of Munn, $\Gamma = S/\sigma$ is an ordered group, where σ is the congruence relation such that $x\sigma y$ if and only if $ex = ey$ for some $e \in E$. An ordered inverse semigroup S is called proper if the σ-class I which is the identity element of Γ contains only idempotents of S.

In a proper ordered inverse semigroup S, let $\Gamma(e)(e \in E)$ be the set of those members of Γ which intersect nontrivially with R_e. Each element of S can be represented in the form (α, e), where $e \in E$ and $\alpha \in \Gamma(e)$. We define $e^\alpha = a^{-1}a \in E$, where $a = (\alpha, e)$. Then $\Gamma(e)$ and e^α satisfy the following six conditions:

(i) $\bigcup_{e \in E} \Gamma(e) = \Gamma$;

(ii) $I \in \Gamma(e)$ **and** $e^I = e$;

(iii) if $f \leqq e$ in the semilattice with respect to the natural ordering of the commutative idempotent semigroup E and $\alpha \in \Gamma(e)$, then $\alpha \in \Gamma(f)$ and $f^\alpha \leqq e^\alpha$ in the semilattice E;

(iv) if $\alpha \in \Gamma(e)$ and $\beta \in \Gamma(e^\alpha)$, then $\alpha\beta \in \Gamma(e)$ and $e^{\alpha\beta} = (e^\alpha)^\beta$;

(v) if $\alpha \in \Gamma(e)$, then $\alpha^{-1} \in \Gamma(e^\alpha)$;

(vi) if $\alpha \in \Gamma(e) \cap \Gamma(f)$ and $e \leqq f$, then $e^\alpha \leqq f^\alpha$.

Also the product and the order in S determined by

$(\alpha, e)(\beta, f) = (\alpha\beta, (e^\alpha f)^{\alpha^{-1}})$;

$(\alpha, e) \leqq (\beta, f)$ **if and only if either** $\alpha < \beta$ **or** $\alpha = \beta, e \leqq f$.

Next we prove conversely a theorem asserting that, for an ordered commutative idempotent semigroup E and an ordered group Γ, if $\Gamma(e)$ and e^α satisfy the six conditions above, then the set $\{(\alpha, e); e \in E, \alpha \in \Gamma(e)\}$ is a proper ordered inverse semigroup with respect to the product and the order mentioned above. Besides this, we present other characterizations of special cases.

Ordered semigroups were studied systematically in [4], [5], [6]. In [4], we studied ordered idempotent semigroups. In an ordered semigroup, the set of all the idempotents always constitutes a subsemigroup and so the study of ordered idempotent semigroups will clarify the structure of this subsemigroup. In [5], we were essentially concerned with ordered regular semigroups. As the first step of the study of these semigroups, in that paper we determined all the types

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of subsemigroups generated by a regular pair. There are two important types of regular semigroups: completely regular semigroups and inverse semigroups. In [6], we characterized ordered completely regular semigroups. In the continuation of the study in this direction, in this note, as the first step of the study of ordered inverse semigroups, we shall be concerned with proper ordered inverse semigroups.

In the algebraic theory of semigroups, those inverse semigroups are of the most important kinds which are situated between groups and general semigroups.

In [3], Munn defined a relation σ in an algebraic inverse semigroup S and showed that S/σ is a group. In the case when S is an ordered inverse semigroup, S/σ is an ordered group. In §§2, 3 of this note, we shall characterize a proper ordered inverse semigroup S by the ordered subsemigroup E of all the idempotents of S and the ordered group S/σ where a collection of subsets $\Gamma(e)$ $(e \in E)$ of S/σ and an operation $e^{\alpha}(e \in E, \alpha \in \Gamma(e))$ are defined with some conditions. However, this characterization is not so satisfactory, since these conditions are too complicated to clarify sufficiently the structure of this semigroup. In §4, for a simple type of such semigroups, that is, for proper ordered $\mathscr{D}$-simple inverse semigroups in which the group S/σ is commutative, we shall give a more satisfactory characterization relating only to the group S/σ. As an appendix, in §5, we shall show that an important sort of ordered inverse semigroup belongs to this category.

1. Preliminaries. Those terminologies and notations which are found in the book of Clifford and Preston [2], shall be used in the sense defined there. (We note that some of them were used differently in the previous papers [4], [5], [6].) For convenience, we quote a lemma from [2], which will be repeatedly applied in the following discussion.

LEMMA 1 (Theorem 1.17 [2]). *The following three conditions on a semigroup S are equivalent*:

(i) *S is an inverse semigroup*;

(ii) *S is regular and any two idempotents of S commute with each other*;

(iii) *every $\mathscr{R}$-class and every $\mathscr{L}$-class contains one and only one idempotent.*

A semigroup S is called an *ordered semigroup*, if it has a simple order $<$ which satisfies the following condition:

$$\text{for } x, y, z \in S,\ x < y \text{ implies } xz \leqq yz \text{ and } zx \leqq zy\ .$$

Let S be an ordered inverse semigroup. An element x of S is called *positive* if $x^2 > x$, while x is called *negative* if $x^2 < x$. Since any two idempotents of S commute with each other, the set of all the idempotents of S forms a commutative idempotent subsemigroup, which we denote by E. Moreover, being a commutative idempotent semigroup, E turns out to be a semilattice with respect to the natural ordering of E (Theorem 1.12 [2]). We denote the partial order of the semilattice E by $\leqq$ in order to distinguish it from the original order $\leqq$ in S. When two elements e, f of E are comparable with respect to the order $\leqq$, we simply say e and f are comparable.

Here we give some lemmas from our previous papers.

LEMMA 2 (Lemma 2 [4]). *If $e, f \in E$ and $e \leqq f$, then $e \leqq ef \leqq f$ and $e \leqq fe \leqq f$.*

LEMMA 3 (Lemma 4 [4]). *If $e, f, g \in E$ and $e \leqq f \leqq g$, then $eg \leqq f$.*

LEMMA 4 (Theorem 3 [4]). *If $e, f, g \in E$ and $e, f \leqq g$, then e and f are comparable.*

LEMMA 5 (Theorem 1 and Lemma 6 [5]). *An element a of S is positive if and only if a^{-1}* is *negative.*

2. **The structure theory.** For two elements x, y of an ordered inverse semigroup S we define $x\sigma y$ if and only if there exists an element $e \in E$ such that $ex = ey$. This relation σ was introduced by Munn [3] in the investigation of algebraic inverse semigroups. Here we give a fundamental property of σ.

LEMMA 6 (Theorem 1 [3]). *σ is a congruence and S/σ is a group.*

Now we mention some further properties of σ.

LEMMA 7. *Each σ-class is a convex subset of S, that is, if $x\sigma y$ and $x \leqq z \leqq y$, then $x\sigma z$.*

Proof. By definition, $ex = ey$ for some $e \in E$. Moreover $ex \leqq ez \leqq ey$. Hence $ex = ez$, and so $x\sigma z$.

By Lemma 7, we can introduce an order into S/σ in natural way, that is, for $\bar{x}, \bar{y} \in S/\sigma$, we define $\bar{x} \leqq \bar{y}$ if and only if $x \leqq y$ for some $x \in \bar{x}, y \in \bar{y}$. With this order S/σ turns out to be an ordered group, or more explicitly a simply ordered group. In what follows we denote by Γ the ordered group S/σ with this order. The identity element of Γ is denoted by I. For $a \in S$, the σ-class which contains a is denoted by $\bar{a}$. For $e \in E$, we have $\bar{e}^2 = \bar{e}$ and so $\bar{e} = I$. Hence the σ-class I contains all the idempotents of S. For $a \in S$, we have $\bar{a}\overline{a^{-1}} = \overline{aa^{-1}} = I$. Hence $\overline{a^{-1}} = \bar{a}^{-1}$. If the σ-class $\bar{p}$ is positive in Γ, then $\overline{p^2} = \bar{p}^2 > \bar{p}$ and so $p^2 > p$. Hence every positive σ-class contains only positive elements of S. Similarly every negative σ-class contains only negative elements of S. However the σ-class I may contain non-idempotent elements. Indeed, if S contains zero, the σ-class I contains all the elements of S. An ordered inverse semigroup S is called *proper*, if the σ-class I contains only idempotents of S.

THEOREM 1. *In a proper ordered inverse semigroup S, the intersection of an $\mathscr{R}$-class and a σ-class consists of at most one element.*

Proof. Let $a\mathscr{R}b$ and $a\sigma b$. Then $aa^{-1}\mathscr{R}a\mathscr{R}b\mathscr{R}bb^{-1}$, and so, by Lemma 1, we have $aa^{-1} = bb^{-1}$. Since $a\sigma b$, we have $b^{-1}a\sigma b^{-1}b$ by Lemma 6. Since S is proper, $b^{-1}a$ is idempotent. On the other hand, since $a\mathscr{R}b$ we have $b^{-1}a\mathscr{R}b^{-1}b$. Hence, by Lemma 1, we have $b^{-1}a = b^{-1}b$. Hence

$$a = aa^{-1}a = bb^{-1}a = bb^{-1}b = b\ .$$

We denote, for $e \in E$,

$$\Gamma(e) = \{\alpha; \alpha \in \Gamma, \alpha \cap R_e \neq \square\}\ ,$$

where R_e is the $\mathscr{R}$-class in S containing e. For $a \in S$, the element aa^{-1} is called the *E-component* of a and is denoted by $e(a)$. Evidently $e(a) \in E$ and $e(a)\mathscr{R}a$. Moreover, by Lemma 1, $e(a)$ is the unique element $e \in E$ such that $e\mathscr{R}a$. For $a \in S$, the element $\bar{a}$ of Γ is called the *Γ-component* of a. Since $a \in \bar{a} \cap R_{e(a)}$, we have $\bar{a} \in \Gamma(e(a))$. Now we consider the mapping

$$\varphi : a \longrightarrow (\bar{a}, e(a))$$

of S into the set S' of all pairs (α, e) such that $e \in E, \alpha \in \Gamma(e)$. For every $(\alpha, e) \in S'$, we have $\alpha \in \Gamma(e)$ and so, by definition, there is an element $a \in S$ such that $a \in \alpha \cap R_e$. Hence $\bar{a} = \alpha$ and $e(a) = e$. Therefore the mapping φ is onto S'. If S is proper, then, by Theorem 1, φ is one-to-one. Thus in that case we can identify a with $(\bar{a}, e(a))$. We remark, under this identification, $e = (I, e)$.

From now on, we assume that S *is a proper ordered inverse semigroup* unless otherwise mentioned, and accept the above identification.

For $e \in E$ and $\alpha \in \Gamma(e)$, we write $e^{\alpha} = a^{-1}a$, where $a = (\alpha, e)$. Evidently $e^{\alpha} \in E$.

THEOREM 2. (i) $\bigcup_{e \in E} \Gamma(e) = \Gamma$.

(ii) *For every* $e \in E$, *we have* $I \in \Gamma(e)$ *and* $e^{I} = e$.

(iii) *If* $f \leqq e$ *in the semilattice* E *and* $\alpha \in \Gamma(e)$, *then* $\alpha \in \Gamma(f)$ *and* $f^{\alpha} \leqq e^{\alpha}$.

(iv) *If* $\alpha \in \Gamma(e)$ *and* $\beta \in \Gamma(e^{\alpha})$, *then* $\alpha\beta \in \Gamma(e)$ *and* $e^{\alpha\beta} = (e^{\alpha})^{\beta}$.

(v) *If* $\alpha \in \Gamma(e)$, *then* $\alpha^{-1} \in \Gamma(e^{\alpha})$.

(vi) *If* $\alpha \in \Gamma(e) \cap \Gamma(f)$ *and* $e \leqq f$, *then* $e^{\alpha} \leqq f^{\alpha}$.

Proof. (i) For $\alpha \in \Gamma$, we take an element a which belongs to the σ-class α. Then $\alpha \in \Gamma(e(a))$.

(ii) Since $e = (I, e)$, the assertion is evident.

(iii) Since $\alpha \in \Gamma(e)$, we can take $a = (\alpha, e)$. Then $a\mathscr{R}e$, and so, since $f \leqq e$, we have $fa\mathscr{R}fe = f$. Moreover $\overline{fa} = \bar{f}\bar{a} = I\alpha = \alpha$. Hence $fa \in \alpha \cap R_f$, and so $\alpha \in \Gamma(f)$ and $fa = (\alpha, f)$. Then $f^{\alpha} = (fa)^{-1}(fa) = a^{-1}fa$, and so $f^{\alpha}e^{\alpha} = a^{-1}faa^{-1}a = a^{-1}fa = f^{\alpha}$. Hence $f^{\alpha} \leqq e^{\alpha}$.

(iv) Since $\alpha \in \Gamma(e)$ and $\beta \in \Gamma(e^{\alpha})$, we can take $a = (\alpha, e)$ and $b = (\beta, e^{\alpha})$. Then $e^{\alpha} = a^{-1}a, a\mathscr{R}e, b\mathscr{R}e^{\alpha}$, and so $ab\mathscr{R}ae^{\alpha} = aa^{-1}a = a\mathscr{R}e$. Moreover we have $\overline{ab} = \bar{a}\bar{b} = \alpha\beta$. Hence $ab \in \alpha\beta \cap R_e$, and so $\alpha\beta \in \Gamma(e)$ and $ab = (\alpha\beta, e)$. Since $a^{-1}a = e^{\alpha} = e(b) = bb^{-1}$ and $(e^{\alpha})^{\beta} = b^{-1}b$, we have $e^{\alpha\beta} = (ab)^{-1}(ab) = b^{-1}a^{-1}ab = b^{-1}bb^{-1}b = b^{-1}b = (e^{\alpha})^{\beta}$.

(v) Since $\alpha \in \Gamma(e)$, we can take $a = (\alpha, e)$. Then $e^{\alpha} = a^{-1}a$ and $e^{\alpha}a^{-1} = a^{-1}aa^{-1} = a^{-1}$. Hence $a^{-1}\mathscr{R}e^{\alpha}$. Moreover we have $\overline{a^{-1}} = \bar{a}^{-1} = \alpha^{-1}$. Hence $a^{-1} \in \alpha^{-1} \cap R_e\alpha$ and so $\alpha^{-1} \in \Gamma(e^{\alpha})$.

(vi) First we consider the case when $f \leqq e$, and set $a = (\alpha, e)$. Then, in the proof of (iii), we have shown that $f^{\alpha} = a^{-1}fa$. Hence $e^{\alpha} = a^{-1}a = a^{-1}aa^{-1}a = a^{-1}ea \leqq a^{-1}fa = f^{\alpha}$. In the case when $e \leqq f$, we can prove $e^{\alpha} \leqq f^{\alpha}$ in a similar way. Finally, in the general case, we have $e \leqq ef \leqq f$ by Lemma 2. Moreover $ef \leqq e$, $ef \leqq f$, and so, by (iii), $\alpha \in \Gamma(ef)$. Hence $e^{\alpha} \leqq (ef)^{\alpha} \leqq f^{\alpha}$.

THEOREM 3. *For* $(\alpha, e), (\beta, f) \in S$, *we have* $\alpha^{-1} \in \Gamma(e^{\alpha}f)$, $\alpha\beta \in \Gamma((e^{\alpha}f)^{\alpha^{-1}})$ *and*

$$(\alpha, e)(\beta, f) = (\alpha\beta, (e^{\alpha}f)^{\alpha^{-1}}) .$$

Proof. We set $a = (\alpha, e)$ and $b = (\beta, f)$. In the proof of Theorem 2 (v), we have shown that $a^{-1}\mathscr{R}e^{\alpha}$, and so $fa^{-1}\mathscr{R}fe^{\alpha} = e^{\alpha}f$. Moreover we have $\overline{fa^{-1}} = \bar{f}\overline{a^{-1}} = I\alpha^{-1} = \alpha^{-1}$. Hence $\alpha^{-1} \in \Gamma(e^{\alpha}f)$ and $fa^{-1} = (\alpha^{-1}, e^{\alpha}f)$. Hence $(e^{\alpha}f)^{\alpha^{-1}}$ is definable and $(e^{\alpha}f)^{\alpha^{-1}} = (fa^{-1})^{-1}(fa^{-1}) = afa^{-1}$. Therefore $e(ab) = (ab)(ab)^{-1} = abb^{-1}a^{-1} = afa^{-1} = (e^{\alpha}f)^{\alpha^{-1}}$ and $\overline{ab} = \bar{a}\bar{b} = \alpha\beta$. Hence $\alpha\beta \in \Gamma((e^{\alpha}f)^{\alpha^{-1}})$ and $ab = (\alpha\beta, (e^{\alpha}f)^{\alpha^{-1}})$.

THEOREM 4. *For* $(\alpha, e), (\beta, f) \in S$,

$(\alpha, e) \leqq (\beta, f)$ *if and only if either* $\alpha < \beta$ *in* Γ *or* $\alpha = \beta, e \leqq f$.

Proof. First we suppose that $\alpha < \beta$. Then, since $(\alpha, e) \in \alpha$, $(\beta, f) \in \beta$, we have $(\alpha, e) < (\beta, f)$. Next we suppose that $\alpha = \beta$ and $e \leqq f$. We set $a = (\alpha, e)$ and $b = (\beta, f) = (\alpha, f)$. If $f \leqq e$ in the semilattice E, we have shown in the proof of Theorem 2 (iii) that $fa = (\alpha, f) = b$. Hence $a = aa^{-1}a = ea \leqq fa = b$. In the case when $e \leqq f$, we can prove $(\alpha, e) \leqq (\beta, f)$ in a similar way. Finally if e and f are noncomparable, then, by Lemma 2, we have $e \leqq ef \leqq f$ and $ef \leqq e, ef \leqq f$. Hence $(\alpha, e) \leqq (\alpha, ef) \leqq (\alpha, f) = (\beta, f)$. This proves the 'if' part of the theorem. Conversely suppose that $(\alpha, e) \leqq (\beta, f)$. If it were false that either $\alpha < \beta$ or $\alpha = \beta$, $e \leqq f$, then we have either $\alpha > \beta$ or $\alpha = \beta$, $e > f$. Hence, by the 'if' part just proved, we have $(\alpha, e) > (\beta, f)$, which is a contradiction. This proves the 'only if' part of the theorem.

3. The characterization theory. In this section, we argue conversely and prove that the theorems in § 2 really characterize proper ordered inverse semigroups. More precisely

THEOREM 5. *Let* E^* *be an ordered commutative idempotent semigroup and let* Γ^* *be an ordered group with the identity element* I. *Suppose that, for each* $e \in E^*, \Gamma^*(e)$ *is defined to be a subset of* Γ^*, *and, for each* $e \in E^*$ *and* $\alpha \in \Gamma^*(e)$, e^α *is defined to be an element of* E^*, *and suppose that these satisfy the following conditions*:

(i) $\bigcup_{e \in E^*} \Gamma^*(e) = \Gamma^*$;

(ii) *for every* $e \in E^*$, *we have* $I \in \Gamma^*(e)$ *and* $e^I = e$;

(iii) *if* $f \leqq e$ *in the semilattice* E^* *and* $\alpha \in \Gamma^*(e)$, *then* $\alpha \in \Gamma^*(f)$ *and* $f^\alpha \leqq e^\alpha$;

(iv) *if* $\alpha \in \Gamma^*(e)$ *and* $\beta \in \Gamma^*(e^\alpha)$, *then* $\alpha\beta \in \Gamma^*(e)$ *and* $e^{\alpha\beta} = (e^\alpha)^\beta$;

(v) *if* $\alpha \in \Gamma^*(e)$, *then* $\alpha^{-1} \in \Gamma^*(e^\alpha)$;

(vi) *if* $\alpha \in \Gamma^*(e) \cap \Gamma^*(f)$ *and* $e \leqq f$, *then* $e^\alpha \leqq f^\alpha$.

We set $S^* = \{(\alpha, e); e \in E^*, \alpha \in \Gamma^*(e)\}$, *and define the product in* S^* *by*

$$(\alpha, e)(\beta, f) = (\alpha\beta, (e^\alpha f)^{\alpha^{-1}}) .$$

Also we define the order in S^* *by*

$(\alpha, e) \leqq (\beta, f)$ *if and only if either* $\alpha < \beta$ *in* Γ^* *or* $\alpha = \beta, e \leqq f$.

Then S^* *is a proper ordered inverse semigroup.*

Proof. We divide the proof into several steps.

1°. *If* $(\alpha, e), (\beta, f) \in S^*$, *then* $(\alpha\beta, (e^\alpha f)^{\alpha^{-1}}) \in S^*$. In fact, $\alpha \in \Gamma^*(e)$ and so, by (v), $\alpha^{-1} \in \Gamma^*(e^\alpha)$. Since $e^\alpha f \leqq e^\alpha$, we have $\alpha^{-1} \in \Gamma^*(e^\alpha f)$ by (iii). Hence $(e^\alpha f)^{\alpha^{-1}}$ is definable. Moreover, by (v), we have $\alpha \in \Gamma^*((e^\alpha f)^{\alpha^{-1}})$, and so, by (iv) and (ii), $((e^\alpha f)^{\alpha^{-1}})^\alpha = (e^\alpha f)^{\alpha^{-1}\alpha} = (e^\alpha f)^I = e^\alpha f \leqq f$. We have $\beta \in \Gamma^*(f)$, and so, by (iii), $\beta \in \Gamma^*(e^\alpha f)$. Hence, by (iv), $\alpha\beta \in \Gamma^*((e^\alpha f)^{\alpha^{-1}})$, and so $(\alpha\beta, (e^\alpha f)^{\alpha^{-1}}) \in S^*$.

2°. *If e and f are comparable and* $\alpha \in \Gamma^*(e) \cap \Gamma^*(f)$, *then* $(ef)^\alpha = e^\alpha f^\alpha$. In fact, if $e \leqq f$, then, by (iii), $e^\alpha \leqq f^\alpha$, and so $(ef)^\alpha = e^\alpha = e^\alpha f^\alpha$. In the case when $f \leqq e$, we can prove the assertion in a similar way.

3°. *S^* is a semigroup.* In fact,

$$\begin{aligned}((\alpha, e)(\beta, f))(\gamma, g) &= (\alpha\beta, (e^\alpha f)^{\alpha^{-1}})(\gamma, g)\\ &= (\alpha\beta\gamma, (((e^\alpha f)^{\alpha^{-1}})^{\alpha\beta} g)^{\beta^{-1}\alpha^{-1}})\\ &= (\alpha\beta\gamma, ((e^\alpha f)^\beta g)^{\beta^{-1}\alpha^{-1}}),\\ (\alpha, e)((\beta, f)(\gamma, g)) &= (\alpha, e)(\beta\gamma, (f^\beta g)^{\beta^{-1}})\\ &= (\alpha\beta\gamma, (e^\alpha (f^\beta g)^{\beta^{-1}})^{\alpha^{-1}}).\end{aligned}$$

Here we remark that $(e^\alpha f)^\beta g \leqq (e^\alpha f)^\beta$ and $\beta^{-1} \in \Gamma^*((e^\alpha f)^\beta)$. Hence, by (iii), $\beta^{-1} \in \Gamma^*((e^\alpha f)^\beta g)$ and $((e^\alpha f)^\beta g)^{\beta^{-1}} \leqq ((e^\alpha f)^\beta)^{\beta^{-1}} = e^\alpha f \leqq e^\alpha$. Hence $\alpha^{-1} \in \Gamma^*(((e^\alpha f)^\beta g)^{\beta^{-1}})$ and $(((e^\alpha f)^\beta g)^{\beta^{-1}})^{\alpha^{-1}} = ((e^\alpha f)^\beta g)^{\beta^{-1}\alpha^{-1}}$. Thus in order to prove 3°, it suffices to show that $((e^\alpha f)^\beta g)^{\beta^{-1}} = e^\alpha (f^\beta g)^{\beta^{-1}}$. Now

$$((e^\alpha f)^\beta g)^{\beta^{-1}} = ((e^\alpha f)^\beta f^\beta g)^{\beta^{-1}}, \text{ since } (e^\alpha f)^\beta \leqq f^\beta \text{ by (iii)}.$$

We have $(e^\alpha f)^\beta \leqq f^\beta, f^\beta g \leqq f^\beta$ and $\beta^{-1} \in \Gamma^*(f^\beta)$ by (v). Hence, by Lemma 4, $(e^\alpha f)^\beta$ and $f^\beta g$ are comparable and, by (iii), $\beta^{-1} \in \Gamma^*((e^\alpha f)^\beta) \cap \Gamma^*(f^\beta g)$. Therefore by 2°,

$$\begin{aligned}((e^\alpha f)^\beta f^\beta g)^{\beta^{-1}} &= ((e^\alpha f)^\beta)^{\beta^{-1}} (f^\beta g)^{\beta^{-1}}\\ &= e^\alpha f (f^\beta g)^{\beta^{-1}}\\ &= e^\alpha (f^\beta g)^{\beta^{-1}} \text{ since } (f^\beta g)^{\beta^{-1}} \leqq (f^\beta)^{\beta^{-1}} = f.\end{aligned}$$

4°. *S^* is simply ordered with respect to* $\leqq$. Evident.

5°. *S^* is an ordered semigroup.* In fact, by 3° and 4° it suffices to prove the monotone property: if $a \leqq b$, then $ac \leqq bc$ and $ca \leqq cb$. Let $a = (\alpha, e)$, $b = (\beta, f)$ and $c = (\gamma, g)$. Then

$$\begin{aligned}ac &= (\alpha\gamma, (e^\alpha g)^{\alpha^{-1}}), \qquad & bc &= (\beta\gamma, (f^\beta g)^{\beta^{-1}}),\\ ca &= (\gamma\alpha, (g^\gamma e)^{\gamma^{-1}}), \qquad & cb &= (\gamma\beta, (g^\gamma f)^{\gamma^{-1}}).\end{aligned}$$

If $\alpha < \beta$, then $\alpha\gamma < \beta\gamma$, and so $ac \leqq bc$. If $\alpha = \beta$, $e \leqq f$, then, by (vi), $(e^{\alpha}g)^{\alpha^{-1}} \leqq (f^{\alpha}g)^{\alpha^{-1}} = (f^{\beta}g)^{\beta^{-1}}$. Hence also in this case we have $ac \leqq bc$. We can prove $ca \leqq cb$ in a similar way.

6°. *S^* is a regular semigroup*, that is, for every $a \in S^*$, there exists an element $x \in S^*$ such that $axa = a$. In fact, for $a = (\alpha, e)$, we set $x = (\alpha^{-1}, e^{\alpha})$. By (iii), we have $x \in S^*$. Moreover $ax = (\alpha, e)(\alpha^{-1}, e^{\alpha}) = (I, (e^{\alpha}e^{\alpha})^{\alpha^{-1}}) = (I, e)$, and so $axa = (I, e)(\alpha, e) = (\alpha, (e^{I}e)^{I}) = (\alpha, e) = a$.

7°. *An element (α, e) of S^* is idempotent if and only if $\alpha = I$.* In fact, if (α, e) is idempotent, then $(\alpha, e) = (\alpha, e)^2 = (\alpha^2, (e^{\alpha}e)^{\alpha^{-1}})$. Hence $\alpha = \alpha^2$, and so $\alpha = I$. Conversely, $(I, e)^2 = (I^2, (e^{I}e)^{I}) = (I, e)$, and so (I, e) is idempotent.

8°. *S^* is an ordered inverse semigroup.* In fact, according to Lemma 1, by 5° and 6°, it suffices to show that two idempotents of S^* commute with each other. By 7°, let (I, e) and (I, f) be two idempotents of S^*. Then $(I, e)(I, f) = (I, ef) = (I, fe) = (I, f)(I, e)$.

9°. *In the ordered inverse semigroup S^*, $(\alpha, e)\sigma(\beta, f)$ if and only if $\alpha = \beta$.* In fact, if $(\alpha, e)\sigma(\beta, f)$, then there exists an idempotent (I, g) such that $(I, g)(\alpha, e) = (I, g)(\beta, f)$. Hence $(\alpha, ge) = (\beta, gf)$ and so $\alpha = \beta$. Conversely, for $(\alpha, e), (\alpha, f) \in S^*$, we can take $(I, ef) \in S^*$ be (ii). Then $(I, ef)(\alpha, e) = (\alpha, ef) = (I, ef)(\alpha, f)$, and so $(\alpha, e)\sigma(\alpha, f)$.

10°. *S^* is a proper ordered inverse semigroup.* In fact, by 8° it suffices to show that if $(\alpha, e)\sigma(I, f)$, then $\alpha = I$, which is a immediate consequence of 9°. This completes the proof of Theorem 5.

THEOREM 6. *In Theorem 5, E^* is isomorphic as an ordered semigroup with E consisting of all the idempotents of the ordered inverse semigroup S^*, and Γ^* is isomorphic as an ordered group with $\Gamma = S^*/\sigma$. If we identify the corresponding elements in these isomorphisms, then $\Gamma^*(e)$ coincides with $\Gamma(e)$ defined in § 2, and e^{α} defined in the assumption of Theorem 5 coincides with e^{α} in the sense of § 2.*

Proof. By 7° of the proof of Theorem 5, $E = \{(I, e); e \in E^*\}$ is the set of all the idempotents of S^*. Moreover

$$(I, e)(I, f) = (I, ef)$$
$$(I, e) \leqq (I, f) \text{ if and only if } e \leqq f,$$

and so the mapping

$$E \ni (I, e) \rightarrow e \in E^*$$

is an isomorphism of E onto E^*. Next we consider the mapping

$$\Gamma = S^*/\sigma \ni \overline{(\alpha, e)} \rightarrow \alpha \in \Gamma^* .$$

By 9° of the proof of Theorem 5, this mapping is well defined irrespective of the choice of (α, e) in $\overline{(\alpha, e)}$ and moreover it is one-to-one. By the condition (i) in Theorem 5, this mapping is onto Γ^*. Furthermore, since

$$\overline{(\alpha, e)}\overline{(\beta, f)} = \overline{(\alpha\beta, (e^{\alpha}f)^{\alpha^{-1}})} ,$$
$$\overline{(\alpha, e)} < \overline{(\beta, f)} \text{ if and only if } \alpha < \beta ,$$

this mapping is an isomorphism of Γ onto Γ^*. Here we show that $(\alpha, e)\mathscr{R}(\beta, f)$ if and only if $e = f$. We have shown, in 6° of the proof of Theorem 5, that, for $(\alpha, e) \in S^*$, we have $(\alpha^{-1}, e^{\alpha}) \in S^*$ and $(\alpha, e)(\alpha^{-1}, e^{\alpha}) = (I, e), (I, e)(\alpha, e) = (\alpha, e)$. Hence $(\alpha, e)\mathscr{R}(I, e)$. Therefore, if $(\alpha, e)\mathscr{R}(\beta, f)$, then $(I, e)\mathscr{R}(\alpha, e) \mathscr{R}(\beta, f)\mathscr{R}(I, f)$, and so, by Lemma 1, we have $e = f$. Conversely, for $(\alpha, e), (\beta, e) \in S^*$, we have $(\alpha, e)\mathscr{R}(I, e)\mathscr{R}(\beta, e)$. Accordingly, for $(I, e) \in E$,

$$\Gamma((I, e)) = \{\alpha; \alpha \in \Gamma, \alpha \cap R_e \neq \square\} = \{\overline{(\alpha, e)}; \alpha \in \Gamma^*(e)\} ,$$

since $(\alpha, e) \in S^*$ if and only if $\alpha \in \Gamma^*(e)$. Hence, under the identification mentioned in the assumption of this theorem, for $e \in E$, we have $\Gamma(e) = \Gamma^*(e)$. In 6° of the proof of Theorem 5, we have shown that, for $(\alpha, e) \in S^*$, we have $(\alpha^{-1}, e^{\alpha}) \in S^*$ and $(\alpha, e)(\alpha^{-1}, e^{\alpha})(\alpha, e) = (\alpha, e)$. Replacing α and e by α^{-1} and e^{α} respectively, we also have $(\alpha^{-1}, e^{\alpha})(\alpha, e)(\alpha^{-1}, e^{\alpha}) = (\alpha^{-1}, e^{\alpha})$. Hence $(\alpha, e)^{-1} = (\alpha^{-1}, e^{\alpha})$. Now, under the identification, for $a = (\alpha, e) \in S^*$, the E-component of a is e and the Γ-component of a is α. Thus the identification of a with $(\bar{a}, e(a))$ mentioned in § 2, coincides with (α, e). Moreover, for $e \in E$ and $\alpha \in \Gamma(e)$, e^{α} in the sense of § 2 is

$$(\alpha, e)^{-1}(\alpha, e) = (\alpha^{-1}, e^{\alpha})(\alpha, e) = (I, e^{\alpha}) = e^{\alpha} .$$

4. **A special case.** We discuss in more detail the structure of S in the case when S is $\mathscr{D}$-simple and the ordered group Γ is commutative.

LEMMA 8. *Let (α, e) and (β, f) be elements of a proper ordered inverse semigroup S. Then*
(i) $(\alpha, e)\mathscr{R}(\beta, f)$ *if and only if* $e = f$;
(ii) $(\alpha, e)\mathscr{L}(\beta, f)$ *if and only if* $e^{\alpha} = f^{\beta}$;
(iii) $(\alpha, e)\mathscr{D}(\beta, f)$ *if and only if* $e^{\gamma} = f$ *for some* $\gamma \in \Gamma(e)$.

Proof. (i) has been shown in the proof of Theorem 6.

(ii) For $(\alpha, e) \in S$, we have shown in the proof of Theorem 6 that $(\alpha, e)^{-1} = (\alpha^{-1}, e^{\alpha})$. Hence $(\alpha, e)\mathscr{L}(\alpha, e)^{-1}(\alpha, e) = (\alpha^{-1}, e^{\alpha})(\alpha, e) = (I, e^{\alpha}) = e^{\alpha}$. Therefore, if $(\alpha, e)\mathscr{L}(\beta, f)$, then $e^{\alpha}\mathscr{L}(\alpha, e)\mathscr{L}(\beta, f)\mathscr{L}f^{\beta}$. Hence, by Lemma 1, we have $e^{\alpha} = f^{\beta}$. Conversely, if $e^{\alpha} = f^{\beta}$, then $(\alpha, e)\mathscr{L}e^{\alpha} = f^{\beta}\mathscr{L}(\beta, f)$.

(iii) If $(\alpha, e)\mathscr{D}(\beta, f)$, then there exists $(\delta, g) \in S$ such that $(\alpha, e)\mathscr{R}(\delta, g)\mathscr{L}(\beta, f)$. Hence $e = g$, $\delta \in \Gamma(g) = \Gamma(e)$ and $e^{\delta} = g^{\delta} = f^{\beta}$. Hence $\beta^{-1} \in \Gamma(f^{\beta}) = \Gamma(e^{\delta})$ and $\delta\beta^{-1} \in \Gamma(e)$, $e^{\delta\beta^{-1}} = f^{\beta\beta^{-1}} = f$. Conversely, if $e^{\gamma} = f$ for some $\gamma \in \Gamma(e)$, then $(\alpha, e)\mathscr{R}(\gamma, e)\mathscr{L}(I, f)\mathscr{R}(\beta, f)$, and so $(\alpha, e)\mathscr{D}(\beta, f)$.

In the rest of this section, we assume that S *is a proper ordered inverse* $\mathscr{D}$*-simple semigroup in which the ordered group* $\Gamma = S/\sigma$ *is commutative.* For $e \in E$, we denote

$$\Sigma(e) = \{\alpha; \alpha \in \Gamma(e), e \geqq e^{\alpha}\},$$
$$\Delta(e) = \{\alpha; \alpha \in \Gamma(e), e = e^{\alpha}\}.$$

Evidently $I \in \Delta(e) \subseteqq \Sigma(e)$.

LEMMA 9. *$\Sigma(e)$ is an algebraic subsemigroup of Γ, and, for $e, f \in E$, we have $\Sigma(e) = \Sigma(f)$.*

Proof. If $\alpha, \beta \in \Sigma(e)$, then $e \geqq e^{\alpha}$ and $\beta \in \Gamma(e^{\alpha})$. Hence $\alpha\beta \in \Gamma(e)$, $e \geqq e^{\beta} \geqq e^{\alpha\beta}$, and so $\alpha\beta \in \Sigma(e)$. Thus $\Sigma(e)$ is a subsemigroup of Γ. Since S is $\mathscr{D}$-simple, for $e, f \in E$, there exists, by Lemma 8, an element $\gamma \in \Gamma(e)$ such that $f = e^{\gamma}$. For $\alpha \in \Sigma(f)$, we have $\gamma\alpha \in \Gamma(e)$ and $e^{\gamma\alpha} = f^{\alpha} \leqq f$. Now $\gamma^{-1} \in \Gamma(f)$, and so $\gamma^{-1} \in \Gamma(e^{\gamma\alpha})$. Hence $\alpha = \gamma\alpha\gamma^{-1} \in \Gamma(e)$ and $e^{\alpha} = e^{\gamma\alpha\gamma^{-1}} \leqq f^{\gamma^{-1}} = e$. Therefore $\Sigma(f) \subseteqq \Sigma(e)$. The converse inclusion can be proved in a similar way.

By Lemma 9, $\Sigma(e)$ is determined irrespective of the choice of $e \in E$. This common subsemigroup of Γ is denoted simply by Σ.

THEOREM 7. *Σ is a subsemigroup of Γ containing I. Moreover, for each $\alpha \in \Gamma$, we have either $\alpha \in \Sigma$ or $\alpha^{-1} \in \Sigma$.*

Proof. It suffices to prove the second assertion. For $\alpha \in \Gamma$, by Theorem 2 (i), we can take $e \in E$ such that $\alpha \in \Gamma(e)$. We set $f = ee^{\alpha}$. Then $f \leqq e$, $f \leqq e^{\alpha}$ and so $\alpha, \alpha^{-1} \in \Gamma(f)$. Now $ff^{\alpha} \leqq f$, $ff^{\alpha^{-1}} \leqq f$, and so, by Lemma 4, ff^{α} and $ff^{\alpha^{-1}}$ are comparable. If $ff^{\alpha} \leqq ff^{\alpha^{-1}}$, then $\alpha \in \Gamma(ff^{\alpha^{-1}})$, $(ff^{\alpha^{-1}})^{\alpha} \leqq f^{\alpha}$, $(ff^{\alpha^{-1}})^{\alpha} \leqq f^{\alpha^{-1}\alpha} = f$, and so $(ff^{\alpha^{-1}})^{\alpha} \leqq ff^{\alpha} \leqq ff^{\alpha^{-1}}$. Hence $\alpha \in \Sigma(ff^{\alpha^{-1}}) = \Sigma$. If $ff^{\alpha^{-1}} \leqq ff^{\alpha}$, then we can similarly prove that $\alpha^{-1} \in \Sigma(ff^{\alpha}) = \Sigma$.

LEMMA 10. *For $e \in E$, $\Delta(e)$ it the group of units of Σ.*

Proof. Let α be a unit of Σ. Then $\alpha, \alpha^{-1} \in \Sigma$, and so $e^\alpha \leqq e$ and $e^{\alpha^{-1}} \leqq e$. Hence we also have $e = e^{\alpha^{-1}\alpha} \leqq e^\alpha$. Therefore $e = e^\alpha$, and so $\alpha \in \Delta(e)$. Conversely, if $\alpha \in \Delta(e)$, then trivially $\alpha \in \Sigma(e) = \Sigma$. Moreover $\alpha^{-1} \in \Gamma(e^\alpha) = \Gamma(e)$, $e = e^{\alpha\alpha^{-1}} = e^{\alpha^{-1}}$, and so $\alpha^{-1} \in \Delta(e) \subseteqq \Sigma$. Hence α is a unit of Σ.

By Lemma 10, $\Delta(e)$ is determined irrespective of the choice of $e \in E$. This common subgroup of Γ is denoted simply by Δ.

We rewrite Lemma 10 as the following

THEOREM 8. *Δ is the group of units of Σ.*

THEOREM 9. *$\Gamma(e)$ is a subset of Γ containing Σ. If $\alpha \in \Sigma$, $\beta \in \Gamma(e)$, then $\beta \in \Gamma(e^\alpha)$, $\alpha\beta \in \Gamma(e)$ and $(e^\alpha)^\beta = e^{\alpha\beta}$.*

Proof. The first assertion is trivial. Now we suppose that $\alpha \in \Sigma$ and $\beta \in \Gamma(e)$. Then we have $e^\alpha \leqq e$. Hence $\beta \in \Gamma(e^\alpha)$, and so $\alpha\beta \in \Gamma(e)$ and $(e^\alpha)^\beta = e^{\alpha\beta}$.

THEOREM 10. *If $e, f \in E$, $\alpha \in \Gamma(e)$ and $f = e^\alpha$, then $\Gamma(f) = \alpha^{-1}\Gamma(e)$.*

Proof. Since $f = e^\alpha$, we have $\alpha^{-1} \in \Gamma(f)$ and $f^{\alpha^{-1}} = e$. Hence, for $\beta \in \Gamma(e)$, we have $\beta \in \Gamma(f^{\alpha^{-1}})$, and so $\alpha^{-1}\beta \in \Gamma(f)$. Thus $\alpha^{-1}\Gamma(e) \subseteqq \Gamma(f)$. From $e = f^{\alpha^{-1}}$, we can prove $\alpha\Gamma(f) \subseteqq \Gamma(e)$ in a similar way. Hence $\Gamma(f) \subseteqq \alpha^{-1}\Gamma(e)$.

LEMMA 11. *If $\alpha, \beta \in \Gamma(e)$ and $e^\alpha = e^\beta$, then $\alpha\beta^{-1} \in \Delta$. Conversely, if $\alpha\beta^{-1} \in \Delta$ and $\beta \in \Gamma(e)$, then $\alpha \in \Gamma(e)$ and $e^\alpha = e^\beta$.*

Proof. If $\alpha, \beta \in \Gamma(e)$ and $e^\alpha = e^\beta$, then $\beta^{-1} \in \Gamma(e^\beta) = \Gamma(e^\alpha)$, $\alpha\beta^{-1} \in \Gamma(e)$ and $e^{\alpha\beta^{-1}} = e^{\beta\beta^{-1}} = e$. Hence $\alpha\beta^{-1} \in \Delta$. Conversely suppose that $\alpha\beta^{-1} \in \Delta$ and $\beta \in \Gamma(e)$. Then $\alpha\beta^{-1} \in \Sigma$ and so, by Theorem 9, $\alpha = \alpha\beta^{-1}\beta \in \Gamma(e)$ and $e^\alpha = (e^{\alpha\beta^{-1}})^\beta = e^\beta$.

The algebraic factor group Γ/Δ is denoted by $\bar{\Gamma}$. For $\alpha \in \Gamma$, the element of $\bar{\Gamma}$ which contains α is denoted by $\bar{\alpha}$. We write

$$\bar{\Sigma} = \{\bar{\alpha}; \alpha \in \Sigma\}\ , \qquad \bar{\Gamma}(e) = \{\bar{\alpha}; \alpha \in \Gamma(e)\}\ .$$

By Lemma 11, if $\Gamma(e)$ contains at least one element of Γ in a coset $\bar{\alpha}$, then it contains all the elements of Γ in $\bar{\alpha}$. Also, since Σ is a subsemigroup containing Δ, Σ contains with at least one element of Γ in a coset $\bar{\alpha}$ all the elements in $\bar{\alpha}$. Moreover $\bar{\Sigma}$ is a subsemigroup of $\bar{\Gamma}$.

Now we take $e \in E$ and make it fixed. Since S is $\mathscr{D}$-simple, every element of E has a form e^{α} for some $\alpha \in \Gamma(e)$. We consider the mapping

$$\psi_e : E \ni e^{\alpha} \longrightarrow \bar{\alpha} \in \bar{\Gamma}(e) .$$

By Lemma 11, ψ_e is well defined irrespective of the choice of α for an idempotent e^{α} and moreover it is one-to-one. It is evident that ψ_e is a mapping onto $\bar{\Gamma}(e)$.

THEOREM 11. *For* $\alpha, \beta \in \Gamma(e)$, *the following conditions are equivalent*:

(i) $e^{\alpha} \leqq e^{\beta}$;
(ii) $\alpha\beta^{-1} \in \Sigma$;
(iii) $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}$.

Proof. If $e^{\alpha} \leqq e^{\beta}$, then $\beta^{-1} \in \Gamma(e^{\alpha})$, and so $\alpha\beta^{-1} \in \Gamma(e)$ and $e^{\alpha\beta^{-1}} \leqq e^{\beta\beta^{-1}} = e$. Hence $\alpha\beta^{-1} \in \Sigma$. It is evident that (ii) implies (iii). Finally, if $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}$, then $\alpha\beta^{-1} \in \Sigma$ and, since $\beta \in \Gamma(e)$, we have $e^{\alpha} = (e^{\alpha\beta^{-1}})^{\beta} \leqq e^{\beta}$, by Theorem 9.

We define the order $\leqq$ in $\bar{\Gamma}(e)$ by

$$\bar{\alpha} \leqq \bar{\beta} \text{ if and only if } \bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma} .$$

Then, by Theorem 11, ψ_e is an order-isomorphism of E onto $\bar{\Gamma}(e)$ with respect to $\leqq$.

LEMMA 12. *Both E and* $\bar{\Gamma}(e)$ *are simply ordered with respect to* $\leqq$.

Proof. It suffices to prove the assertion for $\bar{\Gamma}(e)$. For $\alpha, \beta \in \Gamma(e)$, by Theorem 7, we have either $\alpha\beta^{-1} \in \Sigma$ or $\beta\alpha^{-1} = (\alpha\beta^{-1})^{-1} \in \Sigma$. If $\alpha\beta^{-1} \in \Sigma$, then $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}$ and so $\bar{\alpha} \leqq \bar{\beta}$. If $\beta\alpha^{-1} \in \Sigma$, then we have $\bar{\beta} \leqq \bar{\alpha}$ in a similar way.

In the simply ordered set $\bar{\Gamma}(e)$, we define $\bar{\alpha} \circ \bar{\beta}$ as the lesser of $\bar{\alpha}$ and $\bar{\beta}$ with respect to the order $\leqq$. Then $\bar{\Gamma}(e)$ turns out to be a commutative idempotent semigroup with respect to the operation $\circ$. Since $\bar{\Gamma}(e)$ is order-isomorphic to E, the semigroup $\bar{\Gamma}(e)$ is isomorphic to the abstract semigroup E.

THEOREM 12. *If* $\alpha \in \Gamma(f)$, *then* $(f^{\alpha})\psi_e = (f\psi_e)\bar{\alpha}$.

Proof. We set $f\psi_e = \bar{\beta}$. Then $f = e^{\beta}$ and so $f^{\alpha} = e^{\beta\alpha}$. Hence $(f^{\alpha})\psi_e = \overline{\beta\alpha} = \bar{\beta}\bar{\alpha} = (f\psi_e)\bar{\alpha}$.

We write

$$P\Sigma = \{\alpha; \alpha \in \Sigma, e \leqq e^{\alpha}\}, \qquad N\Sigma = \{\alpha; \alpha \in \Sigma, e \geqq e^{\alpha}\}.$$

It is easily seen that both $P\Sigma$ and $N\Sigma$ contains with at least one element of Γ in a coset $\bar{\alpha}$ all the elements in $\bar{\alpha}$.

LEMMA 13. *Both $P\Sigma$ and $N\Sigma$ are subsemigroups of Σ and $\Sigma = P\Sigma \cup N\Sigma$, $P\Sigma \cap N\Sigma = \Delta$.*

Proof. Suppose that $\alpha, \beta \in P\Sigma$. Then $\alpha, \beta \in \Sigma$ and so, by Theorem 7, we have $\alpha\beta \in \Sigma$. Moreover, by Theorem 9, we have $\beta \in \Gamma(e^{\alpha})$ and $(e^{\alpha})^{\beta} = e^{\alpha\beta}$. Hence, by Theorem 2 (vi), we have $e \leqq e^{\beta} \leqq (e^{\alpha})^{\beta} = e^{\alpha\beta}$ and so $\alpha\beta \in P\Sigma$. Hence $P\Sigma$ is a subsemigroup of Σ. It can be proved similarly that $N\Sigma$ is a subsemigroup and it is trivial that $\Sigma = P\Sigma \cup N\Sigma$, $P\Sigma \cap N\Sigma = \Delta$.

LEMMA 14. *For $\alpha, \beta \in \Gamma(e)$, $e^{\alpha} \leqq e^{\beta}$ if and only if either $\beta\alpha^{-1} \in P\Sigma$ or $\alpha\beta^{-1} \in N\Sigma$.*

Proof. If $\beta\alpha^{-1} \in P\Sigma$, then $\beta\alpha^{-1} \in \Sigma$ and $e \leqq e^{\beta\alpha^{-1}}$. Hence by Theorem 9, $e^{\beta} = (e^{\beta\alpha^{-1}})^{\alpha} \geqq e^{\alpha}$. In the case when $\alpha\beta^{-1} \in N\Sigma$, we can prove $e^{\alpha} \leqq e^{\beta}$ in a similar way. Conversely suppose that $e^{\alpha} \leqq e^{\beta}$. Then, by Theorem 7, we have either $\beta\alpha^{-1} \in \Sigma$ or $\alpha\beta^{-1} \in \Sigma$. If $\beta\alpha^{-1} \in \Sigma$, then, by Theorem 9, we have $e^{\beta} = (e^{\beta\alpha^{-1}})^{\alpha} \geqq e^{\alpha}$ and so $\alpha^{-1} \in \Gamma(e^{\beta}) \cap \Gamma(e^{\alpha})$. Hence $e^{\beta\alpha^{-1}} \geqq e^{\alpha\alpha^{-1}} = e$ and so $\beta\alpha^{-1} \in P\Sigma$. If $\alpha\beta^{-1} \in \Sigma$, then we can prove $\alpha\beta^{-1} \in N\Sigma$ in a similar way.

An order $\leqq$ is said to be monotone to an order $\preceqq$, if $a \leqq b$ implies $a \preceqq b$ and conversely. An order $\leqq$ is said to be antitone to $\preceqq$, if $a \leqq b$ implies $b \preceqq a$ and conversely.

THEOREM 13. *We have either $P\Sigma = \Sigma$ or $N\Sigma = \Sigma$. If $P\Sigma = \Sigma$, then, in E, the order $\leqq$ is antitone to the order $\preceqq$. If $N\Sigma = \Sigma$, then, in E, $\leqq$ is isotone to $\preceqq$.*

Proof. Since $P\Sigma \cup N\Sigma = \Sigma$ and $P\Sigma \cap N\Sigma = \Delta$, in order to prove the first assertion it suffices to show that $\alpha \in P\Sigma$ and $\beta \in N\Sigma$ imply either $\alpha \in \Delta$ or $\beta \in \Delta$. Since $\alpha \in P\Sigma \subseteqq \Sigma$ and $\beta \in N\Sigma \subseteqq \Sigma$, we have $\alpha\beta \in \Sigma \subseteqq \Gamma(e)$. Since $(\alpha\beta)\alpha^{-1} = \beta \in N\Sigma \subseteqq \Sigma$ and $(\alpha\beta)\beta^{-1} = \alpha \in P\Sigma \subseteqq \Sigma$, we have, by Theorem 11 and Lemma 14, $e^{\alpha\beta} \preceqq e^{\alpha}$, $e^{\alpha\beta} \preceqq e^{\beta}$, $e^{\beta} \leqq e^{\alpha\beta} \leqq e^{\alpha}$. Hence $e^{\alpha\beta} \preceqq e^{\alpha}e^{\beta}$ and, by Lemma 3, $e^{\alpha}e^{\beta} \preceqq e^{\alpha\beta}$. Therefore $e^{\alpha\beta} = e^{\alpha}e^{\beta}$. By Lemma 12, we have either $e^{\alpha} \preceqq e^{\beta}$ or $e^{\beta} \preceqq e^{\alpha}$. If $e^{\alpha} \preceqq e^{\beta}$, then $e^{\alpha\beta} = e^{\alpha}e^{\beta} = e^{\alpha}$ and so, by Lemma 11, we have $\beta = (\alpha\beta)\alpha^{-1} \in \Delta$. If $e^{\beta} \preceqq e^{\alpha}$, we can prove $\alpha \in \Delta$ in a similar way. Now we suppose that $P\Sigma = \Sigma$. Since S is $\mathscr{D}$-simple, every element of E has a form e^{α} for some $\alpha \in \Gamma(e)$. For $e^{\alpha}, e^{\beta} \in E$, let $e^{\alpha} \lesssim e^{\beta}$. Then, by Theorem 11, we have $\alpha\beta^{-1} \in \Sigma = P\Sigma$, and so, by Lemma 14, we have $e^{\beta} \leqq e^{\alpha}$. Conversely, if $e^{\beta} \leqq e^{\alpha}$, then we have either $\alpha\beta^{-1} \in P\Sigma$ or $\beta\alpha^{-1} \in N\Sigma$. But, since $P\Sigma = \Sigma$, we have $N\Sigma = \Delta$. Therefore, if $\beta\alpha^{-1} \in N\Sigma = \Delta$, then $\alpha\beta^{-1} \in \Delta \subseteqq \Sigma$, and, if $\alpha\beta^{-1} \in P\Sigma$, we evidently have $\alpha\beta^{-1} \in \Sigma$. Hence, by Theorem 11, we have always $e^{\alpha} \lesssim e^{\beta}$. Thus the order $\leqq$ is antitone to the order $\lesssim$. In the case when $N\Sigma = \Sigma$, we can prove that the order $\leqq$ is isotone to the order $\lesssim$ in a similar way.

We define the order $\leqq$ in $\bar{\Gamma}(e)$ by

(i) in the case when $P\Sigma = \Sigma$, $\bar{\alpha} \leqq \bar{\beta}$ if and only if $\bar{\beta} \lesssim \bar{\alpha}$, that is, if and only if $\bar{\beta}\bar{\alpha}^{-1} \in \bar{\Sigma}$;

(ii) in the case when $N\Sigma = \Sigma$, $\bar{\alpha} \leqq \bar{\beta}$ if and only if $\bar{\alpha} \lesssim \bar{\beta}$, that is, if and only if $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}$.

By Theorem 13, the order $\leqq$ in $\bar{\Gamma}(e)$ is really defined in all cases and $\bar{\Gamma}(e)$ is an ordered semigroup with respect to the multiplication $\circ$ and the order $\leqq$, which is isomorphic to the ordered semigroup E.

Now let us consider conversely and prove the following

THEOREM 14. *Let Γ^* be an ordered commutative group with the identity element I, let Σ^* be a subsemigroup of Γ^* such that, for each $\alpha \in \Gamma^*$, we have either $\alpha \in \Sigma^*$ or $\alpha^{-1} \in \Sigma^*$ and let Γ_1^* be a subset of Γ^* containing Σ^* such that $\alpha \in \Sigma^*$ and $\beta \in \Gamma_1^*$ imply $\alpha\beta \in \Gamma_1^*$. The group of units of Σ^* is denoted by Δ^* and the factor group Γ^*/Δ^* is denoted by $\bar{\Gamma}^*$. For $\alpha \in \Gamma^*$ we denote by $\bar{\alpha}$ the element of $\bar{\Gamma}^*$ which contains α. The image set of Σ^* and Γ_1^* by the natural mapping of Γ^* onto $\bar{\Gamma}^*$ is denoted by $\bar{\Sigma}^*$ and $\bar{\Gamma}_1^*$ respectively. We set*

$$S^* = \{(\alpha, \bar{\beta}); \bar{\beta} \in \bar{\Gamma}_1^*, \alpha \in \beta^{-1}\Gamma_1^*\}$$

and define the product in S^* by

$$(\alpha, \bar{\beta})(\gamma, \bar{\delta}) = \begin{cases} (\alpha\gamma, \bar{\beta}) & \text{if } \bar{\alpha}\bar{\beta}\bar{\delta}^{-1} \in \bar{\Sigma}^* ; \\ (\alpha\gamma, \bar{\delta}\bar{\alpha}^{-1}) & \text{if } \bar{\delta}\bar{\alpha}^{-1}\bar{\beta}^{-1} \in \bar{\Sigma}^* . \end{cases}$$

Moreoverwe we define the order in S^* by either of the two ways:

$$(\alpha, \bar{\beta}) \leqq (\gamma, \bar{\delta}) \text{ if and only if either } \alpha < \gamma \text{ in } \Gamma^* \text{ or } \alpha = \gamma,\ \bar{\delta}\bar{\beta}^{-1} \in \bar{\Sigma}^* ;$$

$$(\alpha, \bar{\beta}) \leqq (\gamma, \bar{\delta}) \text{ if and only if either } \alpha < \gamma \text{ in } \Gamma^* \text{ or } \alpha = \gamma,\ \bar{\beta}\bar{\delta}^{-1} \in \bar{\Sigma}^* .$$

Then S^* is a proper ordered $\mathscr{D}$-simple inverse semigroup in which the group S^*/σ is commutative.

Proof. We prove this theorem in several steps.

1°. *If* $\alpha \in \Sigma^*, \beta \in \Gamma^*$ *and* $\bar{\alpha} = \bar{\beta}$, *then* $\beta \in \Sigma^*$. In fact, since $\bar{\alpha} = \bar{\beta}$, we have $\beta\alpha^{-1} \in \Delta^* \subseteqq \Sigma^*$. Since Σ^* is a subsemigroup of Γ^*, we have $\beta = (\beta\alpha^{-1})\alpha \in \Sigma^*$.

2°. *If* $\alpha \in \Gamma_1^*, \beta \in \Gamma^*$ *and* $\bar{\alpha} = \bar{\beta}$, *then* $\beta \in \Gamma_1^*$. In fact, $\beta\alpha^{-1} \in \Delta^* \subseteqq \Sigma^*$, $\alpha \in \Gamma_1^*$ and so $\beta = (\beta\alpha^{-1})\alpha \in \Gamma_1^*$.

For $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*$, we write $\bar{\alpha} \leqq \bar{\beta}$ if and only if $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}^*$.

3°. $\bar{\Gamma}_1^*$ *is simply ordered by the relation* $\leqq$. In fact, we have clearly $I \in \Sigma^*$. Hence, for $\bar{\alpha} \in \bar{\Gamma}_1^*$, we have $\bar{\alpha}\bar{\alpha}^{-1} = \bar{I} \in \bar{\Sigma}^*$ and so $\bar{\alpha} \leqq \bar{\alpha}$. Next suppose that, for $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*$, we have $\bar{\alpha} \leqq \bar{\beta}$ and $\bar{\beta} \leqq \bar{\alpha}$. By definition, $\bar{\alpha}\bar{\beta}^{-1}, \bar{\beta}\bar{\alpha}^{-1} \in \bar{\Sigma}^*$ and so, by 1°, $\alpha\beta^{-1}, \beta\alpha^{-1} \in \Sigma^*$. Hence $\alpha\beta^{-1}$ is a unit of semigroup Σ^*, and so $\alpha\beta^{-1} \in \Delta^*$. Therefore $\bar{\alpha} = \bar{\beta}$. Thirdly, suppose that $\bar{\alpha} \leqq \bar{\beta}$ and $\bar{\beta} \leqq \bar{\gamma}$. Then $\bar{\alpha}\bar{\beta}^{-1}, \bar{\beta}\bar{\gamma}^{-1} \in \bar{\Sigma}^*$ and so $\bar{\alpha}\bar{\gamma}^{-1} = (\bar{\alpha}\bar{\beta}^{-1})(\bar{\beta}\bar{\gamma}^{-1}) \in \bar{\Sigma}^*$, since Σ^* is a subsemigroup. Hence $\bar{\alpha} \leqq \bar{\gamma}$. Fourthly, for $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*$, we have either $\alpha\beta^{-1} \in \Sigma^*$ or $\beta\alpha^{-1} = (\alpha\beta^{-1})^{-1} \in \Sigma^*$. If $\alpha\beta^{-1} \in \Sigma^*$, then $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}^*$ and so $\bar{\alpha} \leqq \bar{\beta}$. If $\beta\alpha^{-1} \in \Sigma^*$, we have $\bar{\beta} \leqq \bar{\alpha}$ in a similar way.

By 3° $\bar{\Gamma}_1^*$ is a simply ordered set with respect to $\leqq$. For $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*$, we denote the lesser of $\bar{\alpha}$ and $\bar{\beta}$ with respect to $\leqq$ by $\bar{\alpha} \circ \bar{\beta}$. Then $\bar{\Gamma}_1^*$ turns out to be a commutative idempotent semigroup with respect to the operation $\circ$.

4°. $\bar{\Gamma}_1^*$ *is an ordered commutative idempotent semigroup with respect to the operation* $\circ$ *and the order* $\leqq$. In fact, it suffices to show that $\bar{\alpha} \leqq \bar{\beta}$ implies $\bar{\alpha} \circ \bar{\gamma} \leqq \bar{\beta} \circ \bar{\gamma}$, which can easily be proved.

For $\bar{\alpha} \in \bar{\Gamma}_1^*$, we write $\Gamma^*(\bar{\alpha}) = \alpha^{-1}\Gamma_1^*$. It is readily seen that, for $\bar{\alpha} \in \bar{\Gamma}_1^*$, we have $\beta \in \Gamma^*(\bar{\alpha})$ if and only if $\alpha\beta \in \Gamma_1^*$.

5°. $\Gamma^*(\bar{\alpha})$ *is well defined irrespective of the choice of* α *in* $\bar{\alpha}$. In fact, if $\bar{\alpha} = \bar{\beta}$, then $\alpha\beta^{-1} \in \Delta^* \subseteqq \Sigma^*$ and so, for $\gamma \in \Gamma_1^*$, we have $\alpha\beta^{-1}\gamma \in \Gamma_1^*$. Hence $\beta^{-1}\gamma = \alpha^{-1}\alpha\beta^{-1}\gamma \in \alpha^{-1}\Gamma_1^*$ and so $\beta^{-1}\Gamma_1^* \subseteqq \alpha^{-1}\Gamma_1^*$. The converse inclusion can be proved in a similar way and so $\alpha^{-1}\Gamma_1^* = \beta^{-1}\Gamma_1^*$.

6°. $\bar{I} \in \bar{\Gamma}_1^*$ *and* $\Gamma^*(\bar{I}) = \Gamma_1^*$. In fact, since $I \in \Sigma^* \subseteqq \Gamma_1^*$, we have $\bar{I} \in \bar{\Gamma}_1^*$. Moreover, $\Gamma^*(\bar{I}) = I^{-1}\Gamma_1^* = \Gamma_1^*$.

7°. $\bigcup_{\bar{\alpha} \in \bar{\Gamma}_1^*} \Gamma^*(\bar{\alpha}) = \Gamma^*$. In fact, for $\alpha \in \Gamma^*$, we have either $\alpha \in \Sigma^*$ or $\alpha^{-1} \in \Sigma^*$. If $\alpha \in \Sigma^*$, then $\alpha \in \Sigma^* \subseteqq \Gamma_1^* = \Gamma^*(\bar{I})$. If $\alpha^{-1} \in \Sigma^*$, then $\overline{\alpha^{-1}} \in \bar{\Gamma}_1^*$ and $\alpha = \alpha I \in \alpha\Gamma_1^* = \Gamma^*(\overline{\alpha^{-1}})$.

8°. *If* $\bar{\alpha} \in \bar{\Gamma}_1^*$ *and* $\beta \in \Gamma^*(\bar{\alpha})$, then $\bar{\alpha}\bar{\beta} \in \bar{\Gamma}_1^*$. In fact, since $\bar{\alpha} \in \bar{\Sigma}_1^*$ and $\beta \in \Gamma^*(\bar{\alpha})$, we have $\alpha\beta \in \Gamma_1^*$ and so $\bar{\alpha}\bar{\beta} \in \bar{\Gamma}_1^*$.

9°. *For every* $\bar{\beta} \in \bar{\Gamma}_1^*$, *we have* $I \in \Gamma^*(\bar{\beta})$ *and* $\bar{\beta}\bar{I} = \bar{\beta}$. In fact, by 2°, we have $\beta \in \Gamma_1^*$ and so $I = \beta^{-1}\beta \in \beta^{-1}\Gamma_1^* = \Gamma^*(\bar{\beta})$. It is trivial that $\bar{\beta}\bar{I} = \bar{\beta}$.

10°. *If* $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*, \bar{\beta} \leqq \bar{\alpha}$ *and* $\gamma \in \Gamma^*(\bar{\alpha})$, *then* $\gamma \in \Gamma^*(\bar{\beta})$ *and* $\bar{\beta}\bar{\gamma} \leqq \bar{\alpha}\bar{\gamma}$. In fact, since $\bar{\beta} \leqq \bar{\alpha}$, we have $\bar{\beta}\bar{\alpha}^{-1} \in \bar{\Sigma}^*$ and so $\beta\alpha^{-1} \in \Sigma^*$. Moreover, since $\gamma \in \Gamma^*(\bar{\alpha})$, we have $\alpha\gamma \in \Gamma_1^*$. Therefore $\beta\gamma = (\beta\alpha^{-1})(\alpha\gamma) \in \Gamma_1^*$ and so $\gamma \in \Gamma^*(\bar{\beta})$. Moreover $(\bar{\beta}\bar{\gamma})(\bar{\alpha}\bar{\gamma})^{-1} = \bar{\beta}\bar{\alpha}^{-1} \in \bar{\Sigma}^*$ and so $\bar{\beta}\bar{\gamma} \leqq \bar{\alpha}\bar{\gamma}$.

11°. *If* $\bar{\alpha} \in \bar{\Gamma}_1^*, \beta \in \Gamma^*(\bar{\alpha})$ *and* $\gamma \in \Gamma^*(\bar{\alpha}\bar{\beta})$, *then* $\beta\gamma \in \Gamma^*(\bar{\alpha})$ *and* $\overline{\alpha\beta\gamma} = (\bar{\alpha}\bar{\beta})\bar{\gamma}$. In fact, we have $\bar{\alpha}\bar{\beta} \in \bar{\Gamma}_1^*$, by 8°. Since $\gamma \in \Gamma^*(\bar{\alpha}\bar{\beta})$, we have $\alpha\beta\gamma \in \Gamma_1^*$ and so $\beta\gamma \in \Gamma^*(\bar{\alpha})$. It is trivial that $\bar{\alpha}\beta\gamma = (\overline{\bar{\alpha}\bar{\beta}})\bar{\gamma}$.

12°. *If* $\bar{\alpha} \in \bar{\Gamma}_1^*, \beta \in \Gamma^*(\bar{\alpha})$, *then* $\beta^{-1} \in \Gamma^*(\bar{\alpha}\bar{\beta})$. In fact, since $\bar{\alpha} \in \bar{\Gamma}_1^*$, we have $\alpha \in \Gamma_1^*$. Hence $(\alpha\beta)\beta^{-1} = \alpha \in \Gamma_1^*$. Moreover, by 8°, $\bar{\alpha}\bar{\beta} \in \bar{\Gamma}_1^*$ and so $\beta^{-1} \in \Gamma^*(\bar{\alpha}\bar{\beta})$.

We introduce in $\bar{\Gamma}_1^*$ the order $\leqq$ by either of the two ways:

(i) $\bar{\alpha} \leqq \bar{\beta}$ if and only if $\bar{\alpha}\bar{\beta}^{-1} \in \bar{\Sigma}^*$;

(ii) $\bar{\alpha} \leqq \bar{\beta}$ if and only if $\bar{\beta}\bar{\alpha}^{-1} \in \bar{\Sigma}^*$.

13°. $\bar{\Gamma}_1^*$ *is an ordered commutative idempotent semigroup with respect to the operation* $\circ$ *and the order* $\leqq$. In fact, when we define the order $\leqq$ by (i), then, by definition, the order $\leqq$ is isotone to the order $\leqq$, and so, by 4°, we obtain the assertion. When we define the order $\leqq$ be (ii), then the order $\leqq$ is antitone to the order $\leqq$, and so $\bar{\Gamma}_1^*$ with the operation $\circ$ and the order $\leqq$ is isomorphic to the order-dual of $\bar{\Gamma}_1^*$ with the operation $\circ$ and the order $\leqq$. Hence also in this case we obtain the assertion.

14°. *If* $\bar{\alpha}, \bar{\beta} \in \bar{\Gamma}_1^*, \gamma \in \Gamma^*(\bar{\alpha}) \cap \Gamma^*(\bar{\beta})$ *and* $\bar{\alpha} \leqq \bar{\beta}$, *then* $\bar{\alpha}\bar{\gamma} \leqq \bar{\beta}\bar{\gamma}$. In fact, this assertion is an immediate consequence of 3° and 10°.

By what we have discussed, we see, replacing $\bar{\Gamma}_1^*$ by E^*, $\Gamma^*(\bar{\alpha})$ $(\bar{\alpha} \in \bar{\Gamma}_1^*)$ by $\Gamma^*(e)(e \in E)$ and $\bar{\alpha}\bar{\beta}(\bar{\alpha} \in \bar{\Gamma}_1^*, \beta \in \Gamma^*(\bar{\alpha}))$ by $e^{\beta}(e \in E^*, \beta \in \Gamma^*(e))$, that all the assumptions of Theorem 5 are satisfied. Also we see that the set S^*, the product in S^* and the order in S^* in this theorem are defined to be only the rewriting by the above replacement of these in Theorem 5. Thus S^* *is a proper ordered inverse semigroup.*

15°. S^* *is* $\mathscr{D}$*-simple.* In fact, in the proof of Theorem 6 we have shown that the expression $(\alpha, \bar{\beta})$ of an element of S^* can be identified with the expression $(\bar{\alpha}, e(\alpha))$ which is defined in § 2. Let

$(\alpha, \bar{\beta})$ and $(\gamma, \bar{\delta})$ be elements of S^*. Then, since $\delta \in \Gamma_1^*$, we have $\beta^{-1}\delta \in \beta^{-1}\Gamma_1^* = \Gamma^*(\bar{\beta})$ and $\overline{\beta\beta^{-1}\delta} = \bar{\delta}$. Hence, by Lemma 8, $(\alpha, \bar{\beta})\mathscr{D}(\gamma, \bar{\delta})$.

16°. *In S^*, the group S^*/σ is commutative.* In fact, by Theorem 6, the group S^*/σ is isomorphic to the group Γ^* and so it is commutative. This completes the proof of Theorem 14.

THEOREM 15. *In Theorem 14, Γ^* is isomorphic as an ordered group with $\Gamma = S^*/\sigma$ and $\bar{\Gamma}_1^*$ is isomorphic as an ordered semigroup with E consisting of all the idempotents of S^*. If we identify the corresponding elements in these isomorphisms, then Σ^*, Δ^*, Γ_1^* coincide with Σ, Δ, $\Gamma(\bar{I})$, respectively, defined for the ordered semigroup S^*.*

Proof. The first assertion follows from Theorem 6. Under the identification mentioned in the assumption, we have, by Theorem 6, $\Gamma(\bar{I}) = \Gamma^*(\bar{I}) = \Gamma_1^*$. Let $\bar{\alpha} \in \bar{\Gamma}_1^*$ and $\gamma \in \Sigma^*$. Then $\bar{\alpha}\bar{\gamma} \in \bar{\Gamma}_1^*$ and $\bar{I}\bar{\alpha}\bar{\gamma}\bar{\alpha}^{-1} = \bar{\gamma} \in \bar{\Sigma}^*$. Hence, by the definition of the product in S^*, we have $(\bar{\alpha}\bar{\gamma})\bar{\alpha} = (I, \bar{\alpha}\bar{\gamma})(I, \bar{\alpha}) = (I, \bar{\alpha}\bar{\gamma}) = \bar{\alpha}\bar{\gamma}$ and so $\bar{\alpha}\bar{\gamma} \leqq \bar{\alpha}$ with respect to the natural ordering $\leqq$ defined in the commutative idempotent subsemigroup $\bar{\Gamma}_1^*$ of S^*. Therefore $\gamma \in \Sigma(\bar{\alpha}) = \Sigma$ and so $\Sigma^* \subseteqq \Sigma$. Conversely let $\bar{\alpha} \in \bar{\Gamma}_1^*$ and $\gamma \in \Sigma$. Then $(I, \bar{\alpha}\bar{\gamma})(I, \bar{\alpha}) = (I, \bar{\alpha}\bar{\gamma})$. Now we have either $\gamma \in \Sigma^*$ or $\gamma^{-1} \in \Sigma^*$. If $\gamma^{-1} \in \Sigma^*$, then $\bar{\alpha}\bar{I}^{-1}(\bar{\alpha}\bar{\gamma})^{-1} = \bar{\gamma}^{-1} \in \bar{\Sigma}^*$ and so $(I, \bar{\alpha}\bar{\gamma})(I, \bar{\alpha}) = (I, \bar{\alpha}\bar{I}^{-1}) = (I, \bar{\alpha})$. Hence $\bar{\alpha}\bar{\gamma} = \bar{\alpha}, \bar{\gamma} = \bar{I}$ and so $\gamma \in \Delta^* \subseteqq \Sigma^*$. Thus always we have $\gamma \in \Sigma^*$ and so $\Sigma \subseteqq \Sigma^*$. Finally, since both Δ and Δ^* are the group of units of $\Sigma^* = \Sigma$, we have $\Delta^* = \Delta$.

5. **Appendix.** In § 4 we discussed proper ordered $\mathscr{D}$-simple inverse semigroups S in which the group S/σ is commutative. In this section, we shall show that an important sort of ordered inverse semigroup belongs to this category.

Let T be an ordered inverse semigroup. T is called *o-archimedean*, if, for each pair of positive elements p, q of T, there exists a natural number n such that $q \leqq p^n$.

LEMMA 15. *If T is o-archimedean, then T/σ is isomorphic as an ordered group with a subgroup of the ordered additive group of all real numbers, and so is commutative.*

Proof. Let $\bar{p}, \bar{q}$ be positive elements of T/σ. Then p and q are positive elements of T and so $q \leqq p^n$ for some natural number n. Hence $\bar{q} \leqq \bar{p}^n$ and so T/σ is an o-archimedean ordered group. Therefore, by Theorem 15 of Chapter 14 [1], T/σ is isomorphic to a subgroup of the ordered additive group of all real numbers.

LEMMA 16. *Suppose that T is o-archimedean and that, for every*

$e \in E$ which is not the identity element of T, there exists an element $f \in E$ such that $e \neq f$ and $e\mathscr{D}f$. Then T is proper.

Proof. We suppose that T were not proper. Then there exists a nonidempotent element a such that $a \in I$. Since $aa^{-1} \in E$, we have $aa^{-1}\sigma a$ and so there exists an element $e \in E$ such that $eaa^{-1} = ea$. We set $f = eaa^{-1}$. Then $f \in E, f \leqq aa^{-1}, fa = eaa^{-1}a = ea = eaa^{-1} = f$ and $fa^{-1} = faa^{-1} = f$. Since $fa = f$, the element f is not the identity element of T and so, by assumption, there exists $g \in E$ such that $f \neq g$ and $f\mathscr{D}g$. Hence there exists $b \in T$ such that $f\mathscr{L}b\mathscr{R}g$. By Lemma 1, b is not idempotent and $f = b^{-1}b$. Hence $ba = bb^{-1}ba = bfa = bf = b$, $ba^{-1} = bb^{-1}ba^{-1} = bfa^{-1} = bf = b$, $a^{-1}b^{-1} = (ba)^{-1} = b^{-1}$, $ab^{-1} = (ba^{-1})^{-1} = b^{-1}$. By Lemma 5, either a or a^{-1} is positive and also either b or b^{-1} is positive. First we consider the case when both a and b are positive. Since T is o-archimedean, there exists a natural number n such that $b \leqq a^n$. Hence $b = ba = ba^2 = \cdots = ba^n \geqq b^2$, which contradicts that b is positive. In the remaining cases, we obtain a contradiction in a similar way.

As an immediate corollary of these lemmas we have

THEOREM 16. *If an ordered inverse semigroup T is o-archimedean and $\mathscr{D}$-simple, then T is proper and the group T/σ is commutative.*

REFERENCES

1. G. Birkhoff, *Lattice theory* (*revised edition*), American Mathematical Society Colloquium Publications, vol. 25, 1948.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups* I, American Mathematical Society Mathematical Surveys, No. 7, 1961.
3. W. D. Munn, *A class of irreducible matrix representations of an arbitrary semigroup*, Proc. Glasgow Math. Association, **5** (1961), 41-48.
4. T. Saitô, *Ordered idempotent semigroups*, J. Math. Soc. Japan, **14** (1962), 150-169.
5. T. Saitô, *Regular elements in an ordered semigroup*, Pacific J. Math. **13** (1963), 263-295.
6. T. Saitô, *Ordered completely regular semigroups*, Pacific J. Math. **14** (1964), 295-308.
7. V. V. Vagner. *The theory of generalized heapes and generalized groups*, Mat. Sb. **32** (1953), 545-632 (Russian).
8. J. M. Howie, *The Maximum idempotent-separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. **14** (1964), 71-79.

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