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# SOME CONTAINMENT RELATIONS BETWEEN CLASSES OF IDEALS OF A COMMUTATIVE RING

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**The first section of this paper is devoted to proving the following theorem. Let $D$ be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of $D$, $\mathscr{V}$ the set of valuation ideals of $D$, and let $k$ be the quotient field of $D$. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold: (i) Each prime ideal $P$ of $D$ defines a $P$-adic valuation in the sense of van der Waerden, and (ii) every valuation of $k$ finite on $D$ is isomorphic to a $P$-adic valuation for some $P$.**

**The second section considers three additional sets of ideals; the set $\mathscr{Q}$ of primary ideals, the set $\mathscr{S}$ of semi-primary ideals, and the set $\mathscr{A}$ of ideals $A$ such that the complement of some prime ideal is prime to $A$.**

Commutative rings in which various containment relations exist between the sets $\mathscr{V}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{A}$, and $\mathscr{S}$ are also considered. Most of the results of this section represent applications of previous results of the author.

Let $D$ be an integral domain with identity having quotient field $K$. An ideal $A$ of $D$ is said to be a *valuation ideal* provided there exists a valuation ring $D_v$ with $D \subseteqq D_v \subseteqq K$ such that $AD_v \cap D = A$. More specifically, if $D_v$ is the valuation ring of the valuation $v$ of $K$, we may say $A$ is a *v-ideal*. We denote by $\mathscr{T}(D)$ the set of valuation ideals of the domain $D$ and by $\mathscr{Q}(D)$ the set of primary ideals of $D$. Where no ambiguity exists we may speak of $\mathscr{V}$ and $\mathscr{Q}$.

This paper is closely related to a paper of Gilmer and Ohm [5], and frequent reference is made to their results. In [5] the relations $\mathscr{V} \subseteqq \mathscr{Q}$, $\mathscr{V} = \mathscr{Q}$, and $\mathscr{Q} \subseteqq \mathscr{V}$ were investigated. That paper arose as a result of the following observation in [8, p. 341]:

*If $D$ is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$.* But if $D$ is Dedekind, the sets $\mathscr{P}(D)$ of prime powers of $D$ and $\mathscr{Q}(D)$ coincide. Hence if $D$ is Dedekind $\mathscr{V} = \mathscr{P}$. In § 2 necessary and sufficient conditions are given on a domain $D$ in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular it is shown that $\mathscr{V} \subseteqq \mathscr{P}$ implies $\mathscr{V} = \mathscr{P}$.

In § 3 we consider the set $\mathscr{A}(R)$ consisting of all ideals $A$ of the commutative ring $R$ such that $R - P$ is prime to $A$ for some prime ideal $P$ of $R$. It is always true that $\mathscr{Q}(R) \subseteqq \mathscr{A}(R)$ and if $R$ is an integral domain with identity, we also have $\mathscr{V}(R) \subseteqq \mathscr{A}(R)$. The

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relations $\mathscr{A}(R) \subseteqq \mathscr{Q}(R)$, $\mathscr{A}(R) \subseteqq \mathscr{P}(R)$ are investigated in §3. In particular, if $R$ is an integral domain with identity then $\mathscr{A} \subseteqq \mathscr{V}$ if and only if $R$ is a Prüfer domain[1] and $\mathscr{A} \subseteqq \mathscr{P}$ if and only if $R$ is almost Dedekind[1]. The latter is a natural conjecture which is false if $\mathscr{A}$ is replaced by $\mathscr{V}$.

**2. Valuation ideals and prime powers.** In [8; p. 341], it is observed that if $D$ is a Dedekind domain, then $\mathscr{V} = \mathscr{Q}$. The converse is clearly false. In fact, it is proved in [5; Th. 3.1, Th. 3.8] that the domain $D$ with identity has the property $\mathscr{V} = \mathscr{Q}$ if and only if $D$ is a one-dimensional Prüfer domain.

Because an ideal of a Dedekind domain is primary if and only if it is a prime power, we also have $\mathscr{V}(D) = \mathscr{P}(D)$, the set of prime powers of $D$, if $D$ is Dedekind. Theorem 1 gives necessary and sufficient conditions on a domain with identity in order that $\mathscr{V} \subseteqq \mathscr{P}$. In particular, an example in this section shows that such a domain need not be Dedekind.

THEOREM 1. *Let $D$ be an integral domain with identity. Let $\mathscr{P}$ be the set of prime powers of $D$, $\mathscr{V}$ the set of valuation ideals of $D$, and let $k$ be the quotient field of $D$. $\mathscr{V} \subseteqq \mathscr{P}$ if and only if the following conditions hold:*

(i) *If $P$ is a nonzero proper prime ideal of $D$, $\bigcap_{n=0}^{\infty} P^n = (0)$ and the function $v_p: D - \{0\} \to Z$ defined by $v_p(x) = i$ if $x \in P^i - P^{i+1}$ can be extended to a valuation of $k$.*

(ii) *Every valuation of $k$ finite on $D$ is isomorphic to some $v_p$.*

*Proof.* We first show that $D$ is one-dimensional. Thus suppose $P_1, P_2$ are prime ideals of $D$ such that $(0) \subset P_1 \subset P_2 \subset D$. There exists a valuation ring $D'$ containing prime ideals $M_1, M_2$ such that $M_i \cap D = P_i$ [6; p. 37]. There is no loss of generality in assuming $M_1 = \sqrt{dD'} = \sqrt{P_1D'}$ for some element $d$ of $P_1$. This implies $M_1 = \sqrt{d^kD'}$ for any $k$. Now $d^2D' \cap D \subset dD' \cap D$ and $\sqrt{d^2D'} \cap D = P_1$. Because $\mathscr{V} \subseteqq \mathscr{P}$, $d^2D' \cap D = P_1^r \subset dD' \cap D = P_1^s$ for some $r, s$ with $s < r$. Hence, $P_1^rD' \neq P_1D'$ and in particular, $P_1 \supsetneqq P_1^2D'$. We choose $p \in P_1 - P_1^2D'$. Then $P_1^2 \subseteqq P_1^2D' \cap D \subset pD' \cap D \subseteqq P_1D' \cup D$. This implies $pD' \cap D = P_1$ and consequently $P_1D' = pD'$. Now if $r \in P_2 - P_1$ we have $rD' \supset pD'$. Hence $P_1D' = pD' \supset rpD' \supset p^2D' = P_1^2D'$. It follows that $P_1 \supset rpD' \cap D \supset p^2D' \cap D \supseteqq P_1^2$. This contradicts the assumption that $\mathscr{V} \subseteqq \mathscr{P}$. Hence $D$ is one-dimensional.

[1]An integral domain $J$ with identity is said to be a *Prüfer domain* if $J_P$ is a valuation ring for each prime ideal $P$ of $J$. $J$ is *almost Dedekind* if $J_P$ is a valuation ring for each prime $P$ of $J$.

Now let $P$ be a nonzero proper prime ideal of $D$ and let $v$ be a valuation of $k$ finite on $D$ and having center $P$ on $D$. If $D_v$ is the valuation ring of $v$ and if $P_v = \sqrt{PD_v}$, then by passage to $(D_v)_{P_v}$ we may assume $v$ is of rank one. If $p$ is a nonzero element of $P$, then $p^2D_v \cap D = P^s \subset P$ for some integer $s$. Thus $P^sD_v \subset PD_v$. This implies the powers of $PD_v$ properly descend, for if $P^tD_v = P^{t+1}D_v$, then $P^tD_v$ is an idempotent ideal of a valuation ring. Hence $P^tD_v$ is prime, [5; Lemma 2.10], $P^tD_v = PD_v$, and $PD_v = P^sD_v$ — a contradiction.

We next show that $\mathscr{P} \subseteqq \mathscr{V}$. In fact, we will show by induction that $P^n$ is a $v$-ideal for all $n$. Thus if $P^r$ is a $v$-ideal and if $t \in P^{r+1}D_v - P^{r+2}D_v$, then $P^r = P^rD_v \cap D \supset P^{r+1}D_v \cap D \supseteqq tD_v \cap D \supset P^{r+2}D_v \cap D \supseteqq P^{r+2}$. Hence, since $\mathscr{V} \subseteqq \mathscr{P}$, $tD_v \cap D$ must equal $P^{r+1}$ so that $P^{r+1}$ is a $v$-ideal. We have shown in the process of the proof that if $x \in P^t - P^{t+1}$, $y \in P^m - P^{m+1}$, then $xD_v = P^tD_v$, $yD_v = P^mD_v$ so that $xyD_v = P^{m+t}D_v \supset P^{m+t+1}$. Whence $xy \in P^{m+t} - P^{m+t+1}$. Hence (i) holds.

We proceed to show $D_{v_p} = D_v$. Since $D_v$ has rank one, it suffices to show $D_v \subseteqq D_{v_p}$. Thus let $x/y \in D_v$ where $y \in P^t - P^{t+1}$. Then $x = (x/y)y \in yD_v = P^tD_v$. Hence $v_p(x) \geqq t = v_p(y)$ so that $x/y \in D_{v_p}$. Therefore $D_{v_p} = D_v$.

Finally, we show $\{v_p\}$ is the set of nontrivial valuations of $k$ finite on $D$. Thus suppose $D_w$ is the valuation ring of a valuation $w$ of $k$ having center $P \subset D$ on $D$. As shown previously, if $P_w = \sqrt{PD_w}$, $P_w$ is minimal in $D_w$ and $(D_w)_{P_w} = D_{v_p}$. Consequently, $P_w = M_{v_p}$, the maximal ideal of $D_{v_p}$. We show that the assumption $D_w \subset D_{v_p}$ leads to a contradiction. Thus if $M_w$ is the maximal ideal of $D_w$, then $M_w \supset M_{v_p}$. Hence there exists $\xi = a/b \in D_w$ such that $\xi$ is a unit of $D_{v_p}$, but not of $D_w$. This implies there exists $r > 0$ such that $a, b \in P^r - P^{r+1}$ and $a^2D_w$ 56 $baD_w \subset b^2D_w \subseteqq P^{2r}D_w$. To complete the proof we notice $a^2D_w \supseteqq P^{2r+1}D_w$. This follows from a more general result: For any $k$, $P^kD_w \cap D = P^k$ since $P^kD_w \cap D \subseteqq P^kD_{v_p} \cap D = P^k$. Hence $P^{2r+1} = P^{2r+1}D_w \cap D \subseteqq a^2D_w \cap D$ 56 $baD_w \cap D \subset b^2D_w \cap D \subseteqq P^{2r}$. This contradiction to the assumption $\mathscr{V} \subseteqq \mathscr{P}$ shows $D_w = D_{v_p}$ so that $w$ and $v_p$ are isomorphic.

This shows (i) and (ii) are necessary in order that $\mathscr{V} \subseteqq \mathscr{P}$. Obviously (i) and (ii) are sufficient.

COROLLARY 1. *Using the notation of Theorem* 1, *if* $\mathscr{V} \subseteqq \mathscr{P}$, *then* $\mathscr{V} = \mathscr{P}$ *and $D$ is one-dimensional.*

The following example shows that $\mathscr{V} \subseteqq \mathscr{P}$ does not imply $D$ is Dedekind. In fact, $D$ need not be *almost Dedekind* in the sense of [3].

Let $R$ be a rank one discrete valuation ring with maximal ideal

$M$. Suppose also the $R = K + M$ where $K$ is a proper algebraic extension field over the subfield $k$ (we may take $R\,4\,(K[X])_{(X)}$, for example). If $D = k + M$, then $D$ is a one-dimensional quasi-local domain with maximal ideal $M$, but $D$ is not a valuation ring [5; Prop. 5.1]. Clearly (i) holds in $D$. Because $K$ is algebraic over $k$, $R$ is the integral closure of $D$. Since $R$ has rank one, $R$ is the only nontrivial valuation ring containing $D$ and contained in the quotient field of $D$. Hence (ii) holds. But $R = D_{v_M} \cap D$.

By a slight modification of the example just given we see that (ii) is independent of (i). For if we take $K = F(Y)$ where $F$ is a field and $Y$ is an indeterminate over $F$, then $F + M$ satisfies (i) but not (ii).

**3. A certain set of ideals containing $\mathscr{V}$.** The first example of §2 shows that a domain in which $\mathscr{V} \subseteq \mathscr{P}$ need not be almost Dedekind. Also, numerous examples shows that $\mathscr{Q} \subseteq \mathscr{V}$ does not imply $D$ is Prüfer. But by considering a certain set, to be denoted by $\mathscr{A}$, which contains both $\mathscr{V}$ and $\mathscr{Q}$, we obtain both these results by replacing $\mathscr{V}$ by $\mathscr{A}$ and $\mathscr{Q}$ by $\mathscr{A}$, respectively. The set $\mathscr{A}$ to which we refer consists of all ideals $A$ such that the complement of $P$ is prime to $A$ for some prime ideal $P$[2]. We shall consistently use the fact that if $A$ and $P$ are ideals of the commutative ring $R$ such that $A \subseteq P$ and $P$ is prime, then the smallest ideal $B$ of $R$ such that $B$ contains $A$ and such that $R - P$ is prime to $B$ is $B = A_P = \{x \mid x \in R,\ xm \in A$ for some $m \notin P\}$. More to the point as far as we are concerned, $R - P$ is prime to the ideal $A$ if and only if $AD_P \cap D = A$ ($D$ a domain).

The following theorem gives the relationship between the sets $\mathscr{A}$ and $\mathscr{V}$.

THEOREM 2. *Let $D$ be an integral domain with identity. Then $\mathscr{V} \subseteq \mathscr{A}$. $\mathscr{V} = \mathscr{A}$ if and only if $D$ is a Prüfer domain.*

*Proof.* It is easy to see that if $A$ is a $v$-ideal, the complement of the center of $v$ on $D$ is prime to $A$. Hence $\mathscr{V} \subseteq \mathscr{A}$.

Obviously $\mathscr{V} = \mathscr{A}$ if $D$ is Prüfer. Conversely, if $\mathscr{A} \subseteq \mathscr{V}$ and if $P$ is a proper prime ideal of $D$, we shall show $D_P$ is a valuation ring and hence that $D$ is Prüfer. Thus if $x$, $y$ are nonzero elements of $D$, we let $A = (xy)_P$. $A \in \mathscr{A}$, so $A \in \mathscr{V}$ and therefore $x^2 \in A$ or $y^2 \in A$. If, say, $x^2 \in A$, then $x^2 m = dxy$ for some $m \in D - P$, $d \in D$. Hence $x/y = d/m \in D_P$. This proves the theorem.

[2]If $A$ is an ideal of the commutative ring $R$ and $x \in R$, we say $x$ *is prime to* $A$ if $ax \in A$, $a \in R$, implies $a \in A$ [7; p. 223]. A subset $N$ of $R$ is prime to $A$ if each element of $N$ is prime to $A$.

Before proceeding to consider the relation $\mathscr{A} \subseteqq \mathscr{P}$ we note that this condition is meaningful in a ring with zero divisors. Also, the relation $\mathscr{A} \subseteqq \mathscr{Q}$ is meaningful for arbitrary commutative rings. We consider this case. First we need some definitions.

Suppose $R$ is a commutative ring. $R$ is a *primary ring*[3] if $R$ contains at most two prime ideals [1]. A *primary domain* is a primary ring without proper divisors of zero. $R$ is called a *u-ring* if the only ideal $A$ of $R$ such that $\sqrt{A} = R$ is $R$ itself. *$R$ satisfies Condition* (*) if $\mathscr{S}(R)$, the set of ideals of $R$ with prime radical, is a subset of $\mathscr{Q}(R)$.

Theorem 1 of [2] states: A ring $R$ satisfies (*) if and only if $R$ is one of the following:

(a) a primary domain.

(b) a ring, every element of which is nilpotent.

(c) a zero-dimensional $u$-ring.

or (d) a one-dimensional $u$-ring having the property that if $P$ and $M$ are prime ideals of $R$ such that $P \subset M \subset R$, then $(0)_M = P$.

From this result, it is clear that if $R$ satisfies (*), then every ideal of $R_P$ is primary for each prime ideal $P$ of $R$. But because of the one-to-one correspondence between primary ideals of $R$ contained in $P$ and primary ideals of $R_P$, we see that $\mathscr{A} \subseteqq \mathscr{Q}$ if and only if every ideal of $R_P$ is primary for each prime $P$ of $R$. Hence, if $R$ satisfies (*), then $\mathscr{A} \subseteqq \mathscr{Q}$. The converse is false, as can be seen by considering the ring of even integers. The converse is true, however, in a ring with identity or, more generally, in a $u$-ring as the following theorem shows:

THEOREM 3. *Let $R$ be a u-ring. If $\mathscr{A} \subseteqq \mathscr{Q}$, then $R$ satisfies* (*).

*Proof.* Suppose $P$ and $M$ are prime ideals of $R$ such that $P \subset M \subset R$. We let $p \in P$ and $m \in M - P$. The ideal $A = (mp)_M$ is $a$ in $\mathscr{A}$ and is therefore primary. Since $m \notin P \supseteqq \sqrt{A}$, $p \in A$. Therefore $py = rmp + kmp$ for some $y \notin M$, $r \in R$, $k \in Z$ and $p(y - rm - km) = 0$. Further $y - rm - km \equiv y \not\equiv 0 \pmod{M}$ and because $P$ and $M$ are arbitrary, $R$ has dimension $\leqq 1$. That $R$ satisfies (*) now follows.

Similarly, if $\mathscr{P}$ denotes the set of prime powers of the ring $R$, then because any ideal of $R_P$ is the extension of its contraction in $R$ [7; p. 223], every ideal of $R_P$ is a prime power for each prime ideal $P$ of $R$ if $\mathscr{A} \subseteqq \mathscr{P}$.

In view of Theorem 12 and 14 of [4], we may then state

[3]For the case of a ring with identity, this definition agrees with terminology of Zariski-Samuel [7; p. 204]. But unlike the case of a ring with identity, an ideal of a primary ring need not be a primary ideal.

THEOREM 4. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) *every ideal of R with prime radical is a prime power*

*and* (c) *R satisfies* (*) *and primary ideals of S are prime powers.*

COROLLARY 2. *Let D be an integral domain with identity.* $\mathscr{A} \subseteqq \mathscr{P}$ *if and only if D is almost Dedekind.*

In terms of $\mathscr{S}$, the set of ideals of $R$ having prime radical, Theorem 4 can be stated thusly:

THEOREM 5. *Suppose R is a u-ring. The following are equivalent conditions*:

(a) $\mathscr{A} \subseteqq \mathscr{P}$,

(b) $\mathscr{S} \subseteqq \mathscr{P}$,

(c) $\mathscr{A} \subseteqq \mathscr{Q} \subseteqq \mathscr{P}$.

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