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ON THE DETERMINATION OF CONFORMAL IMBEDDING

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Two imbedding fundamental forms determine (up to motions) the smooth imbedding of an oriented surface in E^3. The situation is, however, substantially different for the sufficiently smooth conformal imbedding of a Riemann surface R in E^3. Conventionally such an imbedding is achieved by a conformal correspondence between R and the Riemann surface R_1 determined on a smoothly imbedded oriented surface S in E^3 by its first fundamental from I. We show that except where $H \cdot K = 0$ on S, such an R_1 conformal imbedding of R in E^3 is determined (up to motions) by the second fundamental form II on S, expressed as a form on R. In particular, I is determined by II on R_1, where $H \cdot K \neq 0$ on S.

Similar remarks are valid for two less standard methods of conformal imbedding. If an oriented surface S is smoothly imbedded in E^3 so that $H > 0$ and $K > 0$, then II defines a Riemann surface R_2 on S. And, if S is imbedded so that $K < 0$, then II' given by

$$H'II' = KI - HII$$

with

$$H' = -\sqrt{H^2 - K}$$

defines a Riemann surface R_2' on S. Thus a conformal correspondence between R and R_2 (or R_2') is called an R_2 (or R_2') conformal imbedding of R in E^3. We show that I on S, expressed as a form on R, determines the R_2 or (wherever $H \neq 0$ and sign H is know) the R_2' imbedding of R in E^3 (up to motions). In particular, I determines II on R_2, and (where $H \neq 0$, and sign H is known) on R_2' as well. Finally, we give restatements of the fundamental theorem of surface theory in forms appropriate to R_1, R_2 and R_2' conformal imbeddings in E^3.

The two fundamental forms which determine (up to motions) the smooth imbedding of an oriented surface in E^3 are, of course, related by various equations. But neither form determines the other, except in very special cases. Thus, for instance, isometric imbeddings of a surface in E^3 may differ essentially unless (to cite a famous example) the surface is compact, and the common metric imposed by imbedding has positive Gaussian curvature.

2. Consider an oriented surface S which is C^3 imbedded in E^3.

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We may introduce C^3 isothermal coordinates x, y locally on S, so that

$$I = \lambda(dx^2 + dy^2) ,$$

with $x + iy$ a conformal parameter on R_1, and $\lambda > 0$ a C^2 function of x and y. The Codazzi-Mainardi equations involving λ and the C^1 coefficients L, M and N of II become

$$(1) \qquad \begin{aligned} L_y - M_x &= \frac{\lambda_y}{2\lambda}(L + N) , \\ N_x - M_y &= \frac{\lambda_x}{2\lambda}(L + N) . \end{aligned}$$

The theorem egregium formula for

$$K = \frac{LN - M^2}{\lambda^2}$$

may be written in the form

$$(2) \qquad LN - M^2 = \frac{-\lambda}{2}\left\{\left(\frac{\lambda_x}{\lambda}\right)_x + \left(\frac{\lambda_y}{\lambda}\right)_y\right\} .$$

Moreover, since $\lambda > 0$ while

$$H = \frac{L + N}{2\lambda} ,$$

the equations (1) may be solved for λ_x/λ and λ_y/λ, provided that $H \neq 0$. Substitution in (2) of the expressions so obtained yields

$$(LN - M^2) = \lambda\left\{\left(\frac{M_y - N_x}{L + N}\right)_x + \left(\frac{M_x - L_y}{L + N}\right)_y\right\} .$$

If we add the assumption that $K \neq 0$, making $LN - M^2 \neq 0$, then

$$(3) \qquad \lambda = \frac{LN - M^2}{\left\{\left(\frac{M_y - N_x}{L + N}\right)_x + \left(\frac{M_x - L_y}{L + N}\right)_y\right\}} .$$

Thus, we have established our original claim that II on R_1 determines I wherever $H \cdot K \neq 0$. It will be convenient to refer to the expression on the right side of (3) as $\lambda(L, M, N)$. Of course, when L_{yy}, N_{xx} and M_{xy} exist,

$$\lambda = \lambda(L, M, N) = \\ = \frac{(LN - M^2)(L + N)^2}{(L+N)(2M_{xy} - L_{yy} - N_{xx}) + (L_y + N_y)(L_y - M_x) + (L_x + N_x)(N_x - M_y)} .$$

In any case, substitution of $\lambda = \lambda(L, M, N)$ in (1) yields conditions

$$(4)\qquad \begin{aligned} L_y - M_x &= \frac{\{\lambda(L, M, N)\}_y}{2\lambda(L, M, N)}(L + N) \\ N_x - M_y &= \frac{\{\lambda(L, M, N)\}_x}{2\lambda(L, M, N)}(L + N) \end{aligned}$$

on L, M and N wherever $H \cdot K \neq 0$, or, equivalently, wherever

$$(L + N)(LN - M^2) \neq 0 .$$

Suppose now that a C^1 quadratic form

$$\Omega = Ldx^2 + 2Mdxdy + Ndy^2$$

is given on a Riemann surface R. (Here $x + iy$ is a conformal parameter on R.) Suppose also that $(L + N)(LN - M^2) \neq 0$. Then the previous discussion establishes

$$\lambda(L, M, N)(dx^2 + dy^2)$$

as the only possible I for a C^3 R_1 conformal imbedding of R in E^3 with $\Omega = II$. Thus, if such an imbedding exists, $\lambda(L, M, N)$ must be positive and C^2, while (4) must be valid. On the other hand, if $\lambda(L, M, N)$ is a positive C^2 function, and if (4) does hold, then both (1) and (2) are valid with $\lambda = \lambda(L, M, N)$. Thus the fundamental theorem of surface theory (see p. 124 of [3]) immediately implies the following result.

Theorem 1. *If $\Omega = Ldx^2 + 2Mdxdy + Ndy^2$ is a C^1 quadratic form on R with $(L + N)(LN - M^2) \neq 0$, then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions) of a C^3R_1 conformal imbedding of R in E^3 with $\Omega = II$ are that $\lambda(L, M, N)$ be positive and C^2, and that (4) be valid.*

3. Consider an oriented surface S which is C^4 imbedded in E^3 so that $H > 0$ and $K > 0$. We may introduce C^3 bisothermal coordinates x, y locally on S, so that

$$II = \mu(dx^2 + dy^2)$$

with $x + iy$ a conformal parameter on R_2, and $\mu > 0$ a C^1 function of x and y. Here the Codazzi-Mainardi equations involving μ, and the Christoffel symbols for the coefficients E, F and G of I become

$$(5)\qquad \begin{aligned} \mu_x &= \mu(\Gamma^2_{12} - \Gamma^1_{22}) , \\ \mu_y &= \mu(\Gamma^1_{12} - \Gamma^2_{11}) . \end{aligned}$$

And the theorem egregium yields a complicated expression for

$$K = \frac{\mu^2}{EG - F^2} > 0$$

as a function of E, F, G and their first and second partial derivatives, which we refer to for convenience as $K(E, F, G)$. Thus

$$\mu = \sqrt{K(E, F, G)(EG - F^2)}\,, \tag{6}$$

and we have established our original claim that I on R_2 determines II. We will refer to the expression on the right side of (6) as $\mu(E, F, G)$. Here, substitution of $\mu = \mu(E, F, G)$ in (5) yields conditions

$$\begin{aligned} \{\mu(E, F, G)\}_x &= \mu(E, F, G)(\Gamma^2_{12} - \Gamma^1_{22})\,, \\ \{\mu(E, F, G)\}_y &= \mu(E, F, G)(\Gamma^1_{12} - \Gamma^2_{11})\,, \end{aligned} \tag{7}$$

on E, F and G.

Suppose now that a C^2 quadratic form

$$\Omega = Edx^2 + 2Fdxdy + Gdy^2$$

is given on a Riemann surface R. Suppose also that $K(E, F, G) > 0$. Then the previous discussion establishes

$$\mu(E, F, G)(dx^2 + dy^2)$$

as the only possible II for a C^3 R_2 conformal imbedding of R in E^3 with $\Omega = I$. Thus, if such an imbedding exists, $\mu(E, F, G)$ must be positive and C^1, while (7) must be valid. On the other hand, if $\mu(E, F, G)$ is a positive C^1 function, and if (7) does hold, then both (5) and $K = K(E, F, G)$ are valid, with $\mu = \mu(E, F, G)$. Thus the fundamental theorem of surface theory immediately implies the following result.

THEOREM 2. *If $\Omega = Edx^2 + 2Fdxdy + Gdy^2$ is a C^2 quadratic form on R, then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions) of a C^3 R_2 conformal imbedding of R in E^3 with $\Omega = I$ are that $K(E, F, G)$ be positive, that $\mu(E, F, G)$ be positive and C^1, and that* (7) *be valid.*

4. Finally, consider an oriented surface S which is C^4 imbedded in E^3 so that $K < 0$. We may introduce C^3 disothermal coordinates x, y locally on S, so that

$$II' = \mu'(dx^2 + dy^2)$$

with $x + iy$ a conformal parameter on R_2', and $\mu' > 0$ a C^1 function of x and y. Since $H'II' = KI - HII$,

$$
\begin{aligned}
HL + H'\mu' &= KE \,, \\
HN + H'\mu' &= KG \,, \\
HM &= KF \,.
\end{aligned} \tag{8}
$$

But we show in [2] that on R_2',

$$L = -N \,. \tag{9}$$

Thus

$$K = \frac{-(L^2 + M^2)}{EG - F^2} < 0$$

must be given by the theorem egregium expression $K(E, F, G)$. Using (8), we obtain

$$
\begin{aligned}
HL &= K(E - G) \,, \\
HM &= KF \,,
\end{aligned} \tag{10}
$$

so that

$$H^2(L^2 + M^2) = K^2\langle (E - G)^2 + F^2 \rangle \,.$$

Division of this last equation by $(L^2 + M^2) = - K(E, F, G)(EG - F^2) \neq 0$ yields

$$H = \pm \sqrt{\frac{-K(E, F, G)}{EG - F^2}\{E - G)^2 + F^2\}} \,. \tag{11}$$

Thus H vanishes if and only if $(E - G) + iF = 0$. Of course, where $H \neq 0$, the orientation of S determines the sign of H. On the other hand, where $H \neq 0$, we may set

$$
\begin{aligned}
L(E, F, G) &= (G - E)\sqrt{\frac{-K(E, F, G)(EG - F^2)}{\{(E - G)^2 + F^2\}}} \,, \\
M(E, F, G) &= -F\sqrt{\frac{-K(E, F, G)(EG - F^2)}{\{(E - G)^2 + F^2\}}} \,,
\end{aligned}
$$

and

$$N(E, F, G) = -L(E, F, G) \,.$$

Using (10), we conclude that so long as $H \neq 0$,

$$
\begin{aligned}
L &= \pm L(E, F, G) \,, \\
M &= \pm M(E, F, G) \,, \\
N &= \pm N(E, F, G) \,,
\end{aligned} \tag{12}
$$

with plus or minus signs consistently chosen in accordance with the sign of H. Thus we have established our original claim that I on R_2' determines II (if $H \neq 0$, and sign H is known).

Note, however, that the Codazzi-Mainardi equations on R_2' which read

$$\begin{aligned} L_y - M_x &= L(\Gamma^1_{12} + \Gamma^2_{11}) + M(\Gamma^2_{12} - \Gamma^1_{11}) \\ L_x + M_y &= L(\Gamma^1_{22} + \Gamma^2_{12}) + M(\Gamma^2_{22} - \Gamma^1_{12}) \end{aligned} \tag{13}$$

are not affected by the sign of H. (In particular, if L, M and N solve (13), so will $-L$, $-M$ and $-N$.) Thus, whichever the choice of signs in (12), the Codazzi-Mainardi equations yield the following conditions

$$\begin{aligned} &\{L(E, F, G)\}_y - \{M(E, F, G)\}_x \\ &\qquad = L(E, F, G)(\Gamma^1_{12} + \Gamma^2_{11}) + M(E, F, G)(\Gamma^2_{12} - \Gamma^1_{11}) , \\ &\{L(E, F, G)\}_x - \{M(E, F, G)\}_y \\ &\qquad = L(E, F, G)(\Gamma^1_{22} + \Gamma^2_{12}) + M(E, F, G)(\Gamma^2_{22} - \Gamma^1_{12}) , \end{aligned} \tag{14}$$

on E, F and G, wherever $(E - G) + iF \neq 0$.

Suppose now that a C^2 quadratic form

$$\Omega = Edx^2 + 2Fdxdy + Gdy^2$$

is given on a Riemann surface R. Suppose also that $K(E, F, G) < 0$ while $(E - G) + iF \neq 0$. Then the previous discussion establishes

$$L(E, F, G)(dx^2 - dy^2) + 2M(E, F, G)dxdy \tag{15}$$

and

$$-L(E, F, G)(dx^2 - dy^2) - 2M(E, F, G)dxdy \tag{16}$$

as the only possible forms which could serve as II for a C^3 R_2' conformal imbedding of R in E^3 with $\Omega = I$. Thus, if such an imbedding exists, $L(E, F, G)$ and $M(E, F, G)$ must be C^1 functions, while (14) must be valid. Finally, should such an imbedding exist with one choice (15) or (16) for II, composition with a reflection of S in a plane will leave I invariant while yielding the remaining choice for II. On the other hand, if $L(E, F, G)$ and $M(E, F, G)$ are C^1 functions, and if (14) does hold, then, given either choice (15) or (16) for II, both (13) and $K = K(E, F, G)$ are valid. Thus the fundamental theorem of surface theory immediately implies the following result.

Theorem 3. *If $\Omega = Edx^2 + 2Fdxdy + Gdy^2$ is a C^2 quadratic form on R with $(E - G) + iF \neq 0$, then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions and reflections in planes) of a C^3 R_2' conformal imbedding of R in*

E^3 with $\Omega = I$ are that $K(E, F, G)$ be negative, that $L(E, F, G)$ and $M(E,F,G)$ be C^1 functions, and that (14) *be valid.*

5. We close by noting a pair of statements of the type one gets by slight rewording of the results described above. Isometric oriented surfaces imbedded C^4 in E^3 so that $H > 0$ and $K > 0$ are congruent if and only if the isometry between them is conformal between their R_2 structures. Similarly, such surfaces on which $H \neq 0$ and $K < 0$ are congruent if and only if the isometry between them is conformal between their R_2' structures and preserves the sign of H.

The weakness of these results amply illustrates the sense in which II, while inessential on R_2 or R_2', is of fundamental importance in determining the imbedding of a surface, as distinct from the R_2 or R_2' conformal imbedding of a Riemann surface. None-the-less, more significant applications of Theorems 1, 2 and 3 should be possible.

References

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