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A BRANCHING LAW FOR THE SYMPLECTIC GROUPS

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A "branching law" is derived for the irreducible tensor representations of the symplectic groups, and a relation is given between this law and the representation theory of the general linear groups.

Branching laws for the irreducible tensor representations of the general linear and orthogonal groups are well-known. Furthermore, these laws have a simple form [1]. In the case of the symplectic groups, however, the branching law becomes more complicated and is expressed in terms of a determinant. We derive this result here by brute force applied to the Weyl character formulas, though it could also have been obtained from a more sophisticated treatment of representation theory contained in some unpublished work of Kostant.

The Branching law. Let V^n be an n-dimensional vector space over the complex field. The symplectic group in n dimensions, $S_p(n/2)$, is the set of all linear transformations $a \in \mathscr{E}(V^n)$, under which a non-degenerate skew-symmetric bilinear form on $V^n \times V^n$ is invariant, [3]. If $\langle \cdot, \cdot \rangle$ is the bilinear form on $V^n \times V^n$ and $a \in \mathscr{E}(V^n)$, then

$$(1) \qquad a \in S_p(n/2) \text{ if and only if } \langle ax, ay \rangle = \langle x, y \rangle \text{ for all } x, y \in V^n .$$

It is well-known that $S_p(n/2)$ can be defined only for even dimensional spaces, ($n = 2\mu$, μ an integer). It is always possible to choose a basis e_i, e'_i, $i = 1, \cdots, \mu$ in V^n such that

$$(2) \qquad \begin{aligned} &\langle e_i, e_j \rangle = \langle e'_i, e'_j \rangle = 0 \quad 1 \leqq i, j \leqq \mu \\ &\langle e_i, e'_j \rangle = \delta_{ij} . \end{aligned}$$

We assume that the matrix realization of $S_p(\mu)$ is given with respect to such a basis [3]. The *unitary symplectic group*, $US_p(\mu)$, is defined by

$$(3) \qquad US_p(\mu) = S_p(\mu) \cap U(2\mu)$$

where $U(2\mu)$ is the group of unitary matrices in 2μ dimensions. The irreducible continuous representations of $US_p(\mu)$ can be denoted by ${}^{\mu}\omega_{f_1,\cdots,f_\mu}$, where $f_1, f_2, \cdots, f_\mu$ are integers such that $f_1 \geqq f_2 \geqq \cdots \geqq f_{\mu-1} \geqq f_\mu \geqq 0$.

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Let $a \in US_p(\mu)$. The eigenvalues of a are

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_\mu, \varepsilon_1^{-1}, \varepsilon_2^{-1}, \cdots, \varepsilon_\mu^{-1}$$

where $|\varepsilon_1| = 1$, $1 \leqq i \leqq \mu$, (see [3]). The character of the representation ${}^{\mu}\omega_{f_1,\cdots,f_\mu}$ is given by

$$(4) \qquad {}^{\mu}\chi_{f_1,\cdots,f_\mu}(a) = \frac{|\varepsilon^{l_1} - \varepsilon^{-l_1}, \varepsilon^{l_2} - \varepsilon^{-l_2}, \cdots, \varepsilon^{l_\mu} - \varepsilon^{-l_\mu}|}{|\varepsilon^{\mu} - \varepsilon^{-\mu}, \varepsilon^{\mu-1} - \varepsilon^{-\mu+1}, \cdots, \varepsilon - \varepsilon^{-1}|}$$

$l_i = f_i + \mu - i + 1$, $1 \leqq i \leqq \mu$.

The vertical lines in the numerator and denominator of (4) denote two determinants whose jth rows are obtained by replacing the symbol ε by ε_j.

Let ${}^{\mu}\omega_{f_1,\cdots,f_\mu}$ and ${}^{\mu-1}\omega_{g_1,\cdots,g_{\mu-1}}$ be irreducible representations of the groups $US_p(\mu)$, $US_p(\mu - 1)$, respectively. Denote by $R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}}$ the the multiplicity of ${}^{\mu-1}\omega_{g_1,\cdots,g_{\mu-1}}$ in the restricted representation ${}^{\mu}\omega_{f_1,\cdots,f_\mu}/US_p(\mu-1)$ which is obtained by restricting ${}^{\mu}\omega$ to the subgroup of $US_p(\mu)$ consisting of all matrices that leave the basis vectors e_μ, e'_μ fixed.

THEOREM 1. *$R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}} = 0$ unless $f_i \geqq g_i \geqq f_{i+2}$, $1 \leqq i \leqq \mu - 2$. If these conditions are fulfilled, then*

$$(5) \quad R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}} = \left|\begin{array}{llllll}
f_1-g_1+1, & f_1-g_2+2, & f_1-g_3+3, & \cdots f_1-g_{\mu-2}+\mu-2, & f_1-g_{\mu-1}+\mu-1, & f_1+\mu \\
D_{21} & f_2-g_2+1, & f_2-g_3+2, & \cdots f_2-g_{\mu-2}+\mu-3, & f_2-g_{\mu-1}+\mu-2, & f_2+\mu-1 \\
0 & D_{32} & f_3-g_3+1, & \cdots f_3-g_{\mu-2}+\mu-4, & f_3-g_{\mu-1}+\mu-3, & f_3+\mu-2 \\
0 & 0 & D_{43} & \cdot & \cdot & \cdot \\
0 & 0 & 0 \quad \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & f_{\mu-2}-g_{\mu-2}+1, & f_{\mu-2}-g_{\mu-1}+2, & f_{\mu-2}+3 \\
 & & & D_{\mu-1,\mu-2} & f_{\mu-1}-g_{\mu-1}+1, & f_{\mu-1}+2 \\
0 & 0 & 0 \quad \cdots & 0 & D_{\mu,\mu-1} & f_\mu+1
\end{array}\right|$$

where $D_{i+1,i} = \max[f_{i+1} - g_i, 0]$, $1 \leqq i \leqq \mu - 1$.

Proof. The dependence of the character of the representation ${}^{\mu}\omega_{f_1,\cdots,f_\mu}$ on the eigenvalues $\varepsilon_1, \cdots, \varepsilon_\mu$ of $a \in US_p(\mu)$ can be explicitly exhibited by setting

$${}^{\mu}\chi_{f_1,\cdots,f_\mu}(a) = {}^{\mu}\chi_{f_1,\cdots,f_\mu}(\varepsilon_1, \cdots, \varepsilon_\mu) .$$

The restriction of ${}^{\mu}\omega_{f_1,\dots,f_\mu}$ to $US_p(\mu-1)$ can be accomplished by requiring that $\varepsilon_\mu = 1$. It follows that

$$(6)\qquad {}^{\mu}\chi_{f_1,\dots,f_\mu}(\varepsilon_1,\cdots,\varepsilon_{\mu-1},1) = \sum_{g_1,\cdots,g_{\mu-1}} R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}}\chi_{g_1,\dots,g_{\mu-1}}(\varepsilon_1,\cdots,\varepsilon_{\mu-1}) .$$

We will calculate the constants $R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}}$ by carrying out the decomposition in equation (6). If we take the limit as $\varepsilon_\mu \to 1$ in the character formula (4), we get

$$(7)\qquad {}^{\mu}\chi_{f_1,\dots,f_\mu}(\varepsilon_1,\cdots,\varepsilon_{\mu-1},1) = \frac{\begin{vmatrix} \varepsilon_1^{l_1}-\varepsilon_1^{-l_1} & \varepsilon_1^{l_2}-\varepsilon_1^{-l_2} & \cdots & \varepsilon_1^{l_\mu}-\varepsilon_1^{-l_\mu} \\ \varepsilon_2^{l_1}-\varepsilon_2^{-l_1} & \varepsilon_2^{l_2}-\varepsilon_2^{-l_2} & \cdots & \varepsilon_2^{l_\mu}-\varepsilon_2^{-l_\mu} \\ \vdots & \vdots & & \vdots \\ \varepsilon_{\mu-1}^{l_1}-\varepsilon_{\mu-1}^{-l_1} & \varepsilon_{\mu-1}^{l_2}-\varepsilon_{\mu-1}^{-l_2} & \cdots & \varepsilon_{\mu-1}^{l_\mu}-\varepsilon_{\mu-1}^{-l_\mu} \\ l_1 & l_2 & \cdots & l_\mu \end{vmatrix}}{\begin{vmatrix} \varepsilon_1^{\mu}-\varepsilon_1^{-\mu} & \varepsilon_1^{\mu-1}-\varepsilon_1^{-\mu+1} & \cdots & \varepsilon_1-\varepsilon_1^{-1} \\ \varepsilon_2^{\mu}-\varepsilon_2^{-\mu} & \varepsilon_2^{\mu-1}-\varepsilon_2^{-\mu+1} & & \varepsilon_2-\varepsilon_2^{-1} \\ \vdots & \vdots & & \vdots \\ \varepsilon_{\mu-1}^{\mu}-\varepsilon_{\mu-1}^{-\mu} & \varepsilon_{\mu-1}^{\mu-1}-\varepsilon_{\mu-1}^{-\mu+1} & \cdots & \varepsilon_{\mu-1}-\varepsilon_{\mu-1}^{-1} \\ \mu & \mu-1 & \cdots & 1 \end{vmatrix}}$$

Set $s_i(j) = (\varepsilon_i)^j - (\varepsilon_i)^{-j}$, $1 \leqq i \leqq \mu-1$

$$d_i = \varepsilon_i + \varepsilon_i^{-1} - 2,\ 1 \leqq i \leqq \mu - 1 .$$

It is easy to verify the formula

$$(8)\qquad s_i(n-1)d_i = s_i(n) - 2s_i(n-1) + s_i(n-2) .$$

Also, the relation

$$(9)\qquad s_i(n) = d_i[s_i(n-1) + 2s_i(n-2) + \cdots + ks_i(n-k) + \cdots + (n-1)s_i(1)] + ns_i(1)$$

can be established by induction on (8).

Consider the determinant in the denominator of equation (7). Using obvious abbreviations, we have

$$(10)\qquad \begin{vmatrix} s(\mu), s(\mu-1), \cdots s(2), s(1) \\ \mu\ ,\ \mu-1\ , \cdots\ 2\ ,\ 1 \end{vmatrix} =$$

$$= \begin{vmatrix} s(\mu)-s(\mu-1), s(\mu-1)-s(\mu-2), \cdots, s(2)-s(1), s(1) \\ 1 \quad , \quad 1 \quad ,\cdots, \quad 1 \quad , 1 \end{vmatrix}$$

$$= \begin{vmatrix} s(\mu)-2s(\mu-1)+s(\mu-2),\ s(\mu-1)-2s(\mu-2)+s(\mu-3), \\ \cdots, s(2)-2s(1),\ s(1) \\ 0 \quad , \quad 0 \quad \cdots, \quad 0 \quad , \ 1 \end{vmatrix}$$

$$= | s(\mu)-2s(\mu-1)+s(\mu-2),\ s(\mu-1)-2s(\mu-2)+s(\mu-3),\ \cdots, s(2)-2s(1) |'$$

$$= \prod_{i=1}^{\mu-1} d_i \,|\, s(\mu-1),\ s(\mu-2),\ \cdots, s(2),\ s(1) \,|' .$$

Equation (8) was used in the last step of (10). The quantity $|\cdot|'$ stands for a determinant of order $\mu - 1$.

Now, consider the numerator of (7).

We have

(11)

$$\begin{vmatrix} s(l_1),\ (s(l_2),\ \cdots, s(l_\mu) \\ l_1 \ , \ l_2 \ , \cdots, \ l_\mu \end{vmatrix} = \text{(using (9))}$$

$$= \begin{vmatrix} d[s(l_1-1)+\cdots+(l_1-1)s(1)]+l_1 s(1),\ \cdots, d[s(l_\mu-1)+\cdots+(l_\mu-1)s(1)] \\ +l_\mu s(1) \\ l_1 \quad ,\cdots, \quad l_\mu \end{vmatrix}$$

$$= l_\mu \Big| d\Big\{[s(l_1-1)+\cdots+(l_1-1)s(1)] - \frac{l_1}{l_\mu}[s(l_\mu-1)+\cdots+(l_\mu-1)s(1)]\Big\}, \cdots$$

$$\cdots, d\Big\{[s(l_{\mu-1}-1)+\cdots+(l_{\mu-1}-1)s(1)]$$

$$- \frac{l_{\mu-1}}{l_\mu}[s(l_\mu-1)+\cdots+(l_\mu-1)s(1)]\Big\}\Big|' .$$

Set $q_j(i) = s_j(l_i - 1) + 2s_j(l_i - 2) + \cdots + (l_i - 1)s_j(1)$, $1 \leqq i \leqq \mu$, $1 \leqq j \leqq \mu - 1$. Then, we find that the numerator of (7) is equal to

$$(12)\quad l_\mu \prod_{i=1}^{\mu-1} d_i \Big| q(1) - \frac{l_1}{l_\mu} q(\mu),\ q(2) - \frac{l_2}{l_\mu} q(\mu),\ \cdots,\ q(\mu - 1) - \frac{l_{\mu-1}}{l_\mu} q(\mu) \Big|'$$

$$= \prod_{i=1}^{\mu-1} d_i \{ l_\mu \,|\, q(1), q(2), \cdots, q(\mu - 1) \,|'$$

$$- l_1 \,|\, q(\mu), q(2), q(3), \cdots, q(\mu - 1) \,|'$$

$$- l_2 \,|\, q(1), q(\mu), q(3), \cdots, q(\mu - 1) \,|'$$

$$- \cdots - l_{\mu-1} \,|\, q(1), q(2), \cdots, q(\mu - 2), q(\mu) \,|' \} .$$

Dividing the last expression in (12) by the last expression in (10), we cancel the factor $\prod_{i=1}^{\mu-1} d_i$. Thus, to calculate $R^{f_1, \cdots, f_\mu}_{g_1, \cdots, g_{\mu-1}}$ it only remains to expand the determinants in (12) as linear combinations of determinants of the form

$$(13)\qquad | s(h_1), s(h_2), \cdots, s(h_{\mu-1}) |',\ h_1 > \cdots > h_{\mu-1} > 0 .$$

Set $p_i = g_i + \mu - i$, $1 \leqq i \leqq \mu - 1$. Then $R^{f_1, \cdots, f_\mu}_{g_1, \cdots, g_{\mu-1}}$ will be the coefficient of the determinant

$$| s(p_1), s(p_2), \cdots, s(p_{\mu-1}) |'$$

in the expansion of (12). It is straightforward to show that

$$R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}} = \sum_\sigma sgn\sigma \langle l_{\sigma(1)} - p_1\rangle \langle l_{\sigma(2)} - p_2\rangle \cdots \langle l_{\sigma(\mu-1)} - p_{\mu-1}\rangle\, l_{\sigma(\mu)} \tag{14}$$

where the sum is taken over all permutations σ of the integers $1, 2, \cdots, \mu$. The quantity

$$\langle l_i - p_j\rangle = \begin{cases} l_i - p_j & \text{if } l_i - p_j \geqq 0 \\ 0 & \text{if } l_i - p_j < 0 \,. \end{cases} \tag{15}$$

Thus,

$$R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}} = \begin{vmatrix} \langle l_1 - p_1\rangle, & \langle l_1 - p_2\rangle \cdots \langle l_1 - p_{\mu-1}\rangle, & l_1 \\ \langle l_2 - p_1\rangle, & \langle l_2 - p_2\rangle \cdots \langle l_2 - p_{\mu-1}\rangle, & l_2 \\ \vdots & \vdots \qquad\quad \vdots & \vdots \\ \langle l_\mu - p_1\rangle, & \langle l_\mu - p_2\rangle \cdots \langle l_\mu - p_{\mu-1}\rangle, & l_\mu \end{vmatrix} \tag{16}$$

An analysis of expression (16) yields the theorem.

COROLLARY. $R^{f_1,\cdots,f_{\mu-1},0}_{g_1,\cdots,g_{\mu-2},0} = R^{f_1,\cdots,f_{\mu-1}}_{g_1,\cdots,g_{\mu-2}}$

Proof. Direct verification from expression (5).

It is well-known that the continuous irreducible representations of the $n \times n$ unitary group $U(n)$ can be denoted by ${}^n\nu_{f_1,\cdots,f_n}$ where the integers $f_1, f_2, \cdots, f_n$ can take on all values consistent with $f_1 \geqq f_2 \geqq \cdots f_n$, [3]. We make the assumption that $f_n \geqq 0$.

$U(n)$ contains a subgroup $G(n-2) = U(n-2) \dotplus E_2$ where E_2 is the 2×2 unit matrix, which is obviously isomorphic to $U(n-2)$. (see [1], page 16 for the notation). We identify $G(n-2)$ and $U(n-2)$ by this isomorphism. Thus the irreducible continuous representations of $G(n-2)$ will be denoted by ${}^{n-2}\nu_{g_1,\cdots,g_{n-2}}$.

Denote by $M^{f_1,\cdots,f_n}_{g_1,\cdots,g_{n-2}}$ the multiplicity of ${}^{n-2}\nu_{g_1,\cdots,g_{n-2}}$ in the restricted representation ${}^n\nu_{f_1,\cdots,f_n}/G(n-2)$. The quantity $M^{f_1,\cdots,f_n}_{g_1,\cdots,g_{n-2}}$ can be computed from the Weyl character formula for the irreducible representations of $U(n)$ in the same way as we have done for the irreducible representations of $US_p(\mu)$. We give only the results of this computation.

THEOREM 2. *Let* $M^{f_1,\cdots,f_{\mu+1}}_{g_1,\cdots,g_{\mu-1}}$ *be the multiplicity of* ${}^{\mu-1}\nu_{g_1,\cdots,g_{\mu-1}}$ *in* ${}^{\mu+1}\nu_{f_1,\cdots,f_{\mu+1}}/U(\mu-1)$ *as defined above.*
Then

$$M^{f_1,\cdots,f_\mu,0}_{g_1,\cdots,g_{\mu-1}} = R^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-1}} .$$

COROLLARY. $M^{f_1,\cdots,f_\mu}_{g_1,\cdots,g_{\mu-2}} = R^{f_1,\cdots,f_\mu}_{g_1,\cdots,\mu-2,0}$.

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