

UNIQUE FACTORIZATION IN POWER SERIES RINGS AND SEMIGROUPS

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In this note a short proof is given for a theorem due originally to Deckard and to Cashwell and Everett. The theorem states that every ring of power series over an integral domain R is a unique factorization domain if and only if every ring of power series over R in a finite set of indeterminates is a unique factorization domain. The proof is based on a study of the structure of the multiplicative semigroups of such rings. Much of the novelty and most of the brevity of this argument may be accounted for by the fact that Dilworth's theorem on the decomposition of partially ordered sets is invoked at a crucial point in the proof.

Suppose R is a ring and that R^+ is the additive group of R . Suppose I is a nonvoid set well ordered by $<$. If N is the additive semigroup of the nonnegative integers and M is the weak product (Chevalley [3]) $\prod_{i \in I} N$, let $R_I = \prod_{m \in M} R^+$. For f, g in R_I define

$$(fg)(m) = \sum_{k+l=m} f(k)g(l);$$

the resulting ring is called a *ring of power series* over R , and is an integral domain if R is. An f in R_I is a unit if and only if $f(0)$ is a unit of R .

Deckard [5] and Cashwell and Everett [2] proved that R_I is a unique factorization domain for every I if R_J is a unique factorization domain for every finite set J . (Thus it follows, from results of Samuel [8] and Buchsbaum [1], that R_I is a unique factorization domain if R is a regular unique factorization domain.)

It suffices to consider either the multiplicative semigroup S_I of the nonzero members of R_I or the quotient of S_I obtained by identifying associates; factorization in a cancellative semigroup being unique if and only if (Clifford [4]) both

- (1) The divisor chain condition holds.
- (2) All irreducibles are primes.

This note contains a study of factorization in a class of semigroups which, on specialization, yields a direct and concise proof of the Deckard-Cashwell and Everett theorem.

Extensions of semigroups. With I as above, suppose $\{S_i: i \in I\}$

is a family of commutative cancellative semigroups indexed by I , and that

(a) For each pair i, j of members of I , if $i < j$, S_j contains a nonvoid prime coideal (MacKenzie [7]) C_i^j and $C_i^k \subseteq C_j^k$ if $i < j < k$.

(b) If i, j are in I and $i < j$, there is an epimorphism $h_i^j: C_i^j \rightarrow S_j$ with $h_i^k = h_i^j \circ h_j^k$ if $i < j < k$.

(c) $(h_i^j)^{-1}U_i = U_j$ for $i < j$, where U_i is the group of units of S_i . Let S be the set of all f in $\bigcup_{i \in I} \prod_{j \in I, j \leq i} S_j$ for which

(d) $f(j) \in C_i^j$ for $i < j$ if and only if $I = \text{Dom } f$ or $\text{Inf Dom } f < i \in \text{Dom } f$.

(e) $f(i) = h_i^j f(j)$ if $i \in \text{Dom } f$ and $i < j$.

With the definition $(fg)(i) = f(i)g(i)$ for $i \in (\text{Dom } f) \cap (\text{Dom } g)$, S is a semigroup. Suppose, finally, that

(f) For each k in I , if there exist u_i in U_i for $i < k$ satisfying $u_i = h_i^j u_j$ whenever $i < j < k$, then there is a u in S such that $u(i) = u_i$ for $i < k$.

In view of condition (c) the u in condition (f) clearly belongs to U , the group of units of S . In the power series case a much stronger version of condition (f) holds. However, for present purposes, it is sufficient to assume (f) only for units.

It will be convenient to refer to S as the *extension* of the family $\{S_i: i \in I\}$.

Divisor chain condition. If $J \subseteq I$ and if J has no upper bound in I , J will be said to contain "arbitrarily large" members of I .

LEMMA 1. *S satisfies the divisor chain condition if, for arbitrarily large i , S_i satisfies the divisor chain condition.*

Assume, indeed, that $f^{(k)} \in S$ and $f^{(k+1)} \mid f^{(k)}$ for $k = 0, 1, 2, \dots$. If $f^{(l)} g^{(l,n)} = f^{(n)}$ for $l > n$, then, for i sufficiently large, $f^{(l)}(i) g^{(l,n)}(i) = f^{(n)}(i)$ if $l > n$. Thus for n sufficiently large, $g^{(l,n)}(i) \in U_i$ for some i if $l > n$. Hence, by (c), $g^{(l,n)} \in U$ if $l > n$ and n is sufficiently large.

Interpolation and primes. Suppose J is a subset of I containing arbitrarily large members of I and that for each i in J there is an f_i such that $f_i \in C_i^j$ and $f_i = h_i^j f_j$ if i, j in J and $i < j$. Using $h_i^k = h_i^j \circ h_j^k$ if $i < j < k$, f_i can be defined on an upper segment of I containing J , determining a unique f of S such that $f(i) = f_i$ for $i \in J$.

LEMMA 2. *If $p \in S$ and if, for arbitrarily large i , $p(i)$ is a prime in S_i , then p is a prime in S .*

If f, g are in S and $p \mid fg$, then for sufficiently large i , $p(i) \mid f(i)g(i)$; and for arbitrarily large i , $p(i) \mid f(i)$ or $p(i) \mid g(i)$. If $p(i)q_i = f(i)$ for arbitrarily large i , then $h_i^j q_j = q_i$ if $i < j$ for i, j in an unbounded subset of I , so there is a q in S for which $pq = f$.

Identification of associates. If \bar{S}_i and \bar{C}_i^j are, respectively, the quotients of S_i and C_i^j obtained by identifying associates and if $\bar{h}_i^j: \bar{C}_i^j \rightarrow \bar{S}_i$ are the corresponding homomorphisms generated by the h_i^j , it follows that conditions (a)-(f) remain valid for these "reduced" semi-groups. Let \bar{S} be the quotient of S and let S' be the extension of the \bar{S}_i .

LEMMA 3. *\bar{S} and S' are isomorphic.*

For f in S , write \bar{f} for the member of \bar{S} containing f and $\bar{f}(i)$ for the member of \bar{S}_i containing $f(i)$, and define the map $\varphi: \bar{S} \rightarrow S'$ by $(\varphi\bar{f})(i) = \bar{f}(i)$, $i \in \text{Dom } f$. It is easy to see that φ is a monomorphism. An induction argument shows that φ is surjective. Indeed, suppose $g \in S'$: to find f in S satisfying $\bar{f}(i) = g(i)$ for $i \in \text{Dom } g$. Let $i_g = \text{Inf Dom } g$, pick f_{i_g} in $g(i_g)$ and suppose that for $i_g \leq i < k$ there are f_i in $g(i)$ satisfying $h_i^j f_j = f_i$ whenever $i_g \leq i < j < k$. If $s \in g(k)$, then, for $i_g \leq i < k$, $h_i^k s = u_i f_i$ where $u_i \in U_i$. Here $u_i = h_i^j u_j$ for $i < j < k$, and by condition (f) there is a u in U satisfying $h_i^k(u(k)) = u_i$ for $i < k$. If $f_k = [u(k)]^{-1}s$, then $h_i^k(f_k) = u_i^{-1}h_i^k(s) = f_i$. By induction, there is an f in S satisfying $f(i) = f_i$ and $\varphi\bar{f} = g$.

Reducibility. If $f \in S$ say that f is *normal* if, for arbitrarily large i , $f(i)$ has a unique representation in S_i as a product of irreducibles.

LEMMA 4. *If f is normal then f is reducible in S if, for arbitrarily large i , $f(i)$ is reducible in S_i .*

In view of Lemma 3 associates may be identified. Suppose, then, that J is an unbounded subset of I and that for each j in J , $f(j)$ has a unique representation

$$f(j) = a_j^{(1)} \cdots a_j^{(n_j)}$$

where each $a_j^{(k)}$ is an irreducible in S_j . If $i \in J$ and $i < j$

$$f(i) = \prod_1^{n_i} a_i^{(l)} = \prod_1^{n_j} h_i^j a_j^{(l)};$$

so $n_i \geq n_j$ and for sufficiently large j in J , n_j is a constant n depend-

ing only on f . By hypothesis, $n \geq 2$. Hence, for each i in some unbounded subset K of I , $f(i)$ is a product of exactly n irreducibles. In the set P of all irreducible factors of $f(i)$ with $i \in K$ define

$$a < b \text{ if } a = h_i b \text{ for some } i, j \text{ in } K, i < j.$$

The relation $<$ is transitive by condition (b) and thus defines a partial order in P . However each subset of P with more than n members contains a comparable pair and thus, by a theorem of Dilworth [6], P is the union of n or fewer chains. At least one such chain is unbounded and determines in S a proper factor of f .

THEOREM. *In S factorization into irreducibles is unique if, for arbitrarily large i , factorization into irreducibles in S_i is unique.*

Proof. By Lemma 1, S satisfies the divisor chain condition. It remains to show that in S all irreducibles are primes. Suppose $p \in S$ and p is irreducible. Then, by Lemma 4, $p(i)$ is irreducible in S_i for some i , since every member of S is normal. In view of condition (c), $p(i)$ is irreducible in S_i for all sufficiently large i . But then, $p(i)$ is prime in S_i for i arbitrarily large and, by Lemma 2, p is prime in S .

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