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VISUALIZING THE WORD PROBLEM, WITH AN APPLICATION TO SIXTH GROUPS

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The word problem in certain groups is studied in algebraic terms with a geometric background. A relator is made to correspond to a plane complex so that generators are associated with 1-cells and defining relators are associated with 2-cells of the complex. In the case of less-than-one-sixth groups, the results obtained are essentially those found by Greendlinger.

Let $\mathscr{G} = \mathscr{F}/\mathscr{N}$ where $\mathscr{N}$ is a normal subgroup of a free group $\mathscr{F}$ with fixed free generators (understood to include inverses). Let $\mathscr{N}$ be the smallest normal subgroup containing a set $\mathscr{R}$ of cyclically reduced words (defining relators for $\mathscr{G}$). Nonempty words in $\mathscr{N}$ are relators for $\mathscr{G}$. Let $\mathscr{R}$ be closed under inverses and cyclic permutations. Assume each free generator appears in at least one defining relator.

In this paper we use complexes to study how relators depend upon defining relators. A complex is determined by a finite set E of elements (called edges), a partition of E into subsets (called boundaries), a partition of E into pairs of edges, and a cyclic order for the edges in each boundary; vertices and the property of connectedness can then be defined. After a free generator is assigned to each edge (with inverse free generators assigned to paired edges), the above-mentioned cyclic orders determine words (called values) for each boundary. More precisely, some word and all its cyclic permutations are the values of a boundary.

It is shown that each relator is a value of one of the boundaries of some spherical complex (a connected complex with Euler characteristic 2) whose other boundaries have defining relators for their values. The converse is also proved: if defining relators are the values of all but one of the boundaries of a spherical complex, then a value of the remaining boundary is a relator. Thus the question of recognizing the relators in $\mathscr{G}$—the word problem in $\mathscr{G}$—can be viewed as the question of determining the words which can correspond to one boundary of a spherical complex whose other boundaries correspond to defining relators.

These results are essentially a reformulation of the first two lemmas in a paper by Van Kampen who approached the problem geometrically. The proofs given here are combinatorial in nature.

In passing from a relator to a complex, we use a system (called a

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structure) which characterizes one construction of the relator from a collection of defining relators. Structures help to define certain basic relators.

The problem of recognizing relators is reduced to finding basic relators by showing that each freely reduced relator contains a subword which is a basic relator. When W is a cyclically reduced basic relator, some subword of W is a subword of a defining relator. The number of such subwords, contained disjointly in the cyclic word W, is estimated via simple calculations using a spherical complex associated with W. The calculations are given in § 8; they were suggested by the proof of the Five Color Theorem in [1] Courant and Robbins.

This estimate is applied when W is in a group $\mathscr{G}$ which is a less-than-one-sixth group or, briefly, a sixth group. A group $\mathscr{G}$ is called a sixth group if any subword common to 2 distinct defining relators has a length which is less than one sixth of the length of both of the defining relators. As a result, W is seen to contain a subword which is more than one half of a defining relator.

Thus a nonempty cyclically reduced word is a relator in a sixth group only if the word can be shortened by replacing one of its subwords X by a shorter word Y^{-1} where XY is a defining relator. This solves the word problem for sixth groups. Other proofs have been given by Tartakovskii and Greendlinger.

Our results are contained in the following

MAIN THEOREM. *In a presented group, each freely reduced relator contains a subword which is a certain kind of relator called a basic relator.*

If a cyclically reduced word W is a basic relator for a sixth group, then either W is a defining relator or the cyclic word W contains disjointly P_k subwords which are greater than 7-k/6 of a defining relator ($k=2, 3, 4$) and the integers P_k satisfy $3P_2+2P_3+P_4\geqq 6$. Thus W contains a subword which is more than 1/2 of a defining relator.

2. **Constructing relators.** Let $W\cong V_1XV_2$ and $V\cong V_1V_2$ be words in $\mathscr{F}$. Here "$\cong$" stands for "identically equal to". We write $W\to V$ (delete X) and $V\to W$ (insert X). If also $V\to U$ (delete Y), then $W\to U$ (delete X, Y). This leads to a definition of $W\to W'$ (delete $X_1, \cdots, X_n$) and $W'\to W$ (insert $X_n, \cdots, X_1$) for $n\geqq 1$.

A word W *splits* into one or more words $W_1, \cdots, W_n$ if the W_i can be put in a sequence $W_1', \cdots, W_n'$ so that $1\to W$ (insert $W_1', \cdots, W_n'$) where 1 denotes the empty word. An *$\mathscr{R}$-word of type t* is any word which splits into t defining relators.

A product of a free generator and its inverse is a *null word*. If

W, W' are words such that either $W \cong W'$ or $W \to W'$ (delete $N_1, \cdots, N_k$) where the N_i are null words, then W *partially reduces* to W' and W' is a *partially reduced form* of W. If, in addition, no subword of W' is a null word, then W' is the freely reduced form of W. A *relator of type* t is a partially reduced $\mathscr{R}$-word of type t (i.e. a partially reduced form of an $\mathscr{R}$-word of type t).

The first lemma shows that each relator can be constructed from the empty word by insertions of defining relators, possibly followed by deletions of null words.

LEMMA 2.1. *Each relator is a partially reduced $\mathscr{R}$-word. In other words, each relator has at least one type.*

Proof. The collection of $\mathscr{R}$-words is closed under inverses and products. If W is an $\mathscr{R}$-word and x is a free generator, then it must be shown that xWx^{-1} is a partially reduced form of some $\mathscr{R}$-word W'. Suppose $1 \to W$ (insert $R_1, \cdots, R_n$) where the R_i are defining relators. Let x be the first letter in a defining relator $R \cong xY$. Put $W' \cong xWYY^{-1}x^{-1}$ so that W' partially reduces to xWx^{-1} and $1 \to W'$ (insert $R, R_1, \cdots, R_n, R^{-1}$). This completes the proof.

It can be shown if W'' is a cyclic permutation of a word W which splits into $W_1, \cdots, W_n$, then W'' splits into some cyclic permutations $W''_1, \cdots, W''_n$ of $W_1, \cdots, W_n$, respectively. Hence,

REMARK 2.1. The set of $\mathscr{R}$-words of type t is closed under cyclic permutations. The set of relators of type t is closed under cyclic permutations.

3. Structures for relators.

We need terminology for permutations of a finite set in order to define a structure. In this section, all sets are finite; 0 denotes the empty set.

Let θ be a cyclic permutation, acting on a set E. If $E \neq 0$, suppose $E = \{a_1, \cdots, a_m\}$ and either $m = 1$ with $a_1\theta = a_1$ or $m \geqq 2$ with $a_i\theta = a_{i+1} (1 \leqq i \leqq m-1)$ and $a_m\theta = a_1$. Then θ is represented by an *array* $H \cong a_1 \cdots a_m$ and by the m cyclic permutations of H. Any subword of H is said to partially represent θ. If $E = 0$, then θ is the empty permutation, represented by the empty array 1.

A set of words in $\mathscr{F}$ is associated with θ by assigning a free generator to each element in E. If x_i is assigned to a_i, then $V \cong x_1 \cdots x_m$ (a word in $\mathscr{F}$) is called the *value* of H or a value of θ. The values of θ are the cyclic permutations of V. If $E = 0$, the empty word is the only value of θ.

A cyclic permutation θ_B corresponds to each subset B of E. If

$B \neq 0$ and the elements of B form a subsequence $b_1, \cdots, b_k$ of $a_1, \cdots, a_m$, then θ_B is represented by the array $b_1 \cdots b_k$. If $B = 0$, then θ_B is the empty permutation.

A permutation β, acting on a nonempty set E, determines a partition of E into nonempty subsets $E_1, \cdots, E_n$, called *β-orbits*: two elements a, b are in the same β-orbit if $a\beta^i = b$ for some integer i. The *β-cycles* are the restrictions of β to the sets $E_1, \cdots, E_n$. The *length* of a β-cycle is the number of elements in the corresponding β-orbit. β is a *reflection* (*pure reflection*) if the length of each β-cycle is at most 2 (exactly 2).

A *structure* $S = (E, \beta, \rho, \theta)$ consists of a nonempty set E which is acted on by a permutation β, a reflection ρ, and a cyclic permutation θ. S has *carrier* E, *reduced carrier* $F = \{a : a \in E, a\rho = a\}$, *map* θ, and *reduced map* θ_F. It is required that there exist arrays H, H_ρ, representing θ, θ_F, respectively, such that

(I) There exist arrays $H_1, \cdots, H_n$, $n \geqq 1$, representing the β-cycles, such that $1 \rightarrow H$ (insert $H_1, \cdots, H_n$).

(II) Either ρ is the identity and $H_\rho \cong H$ or there exist arrays $I_1, \cdots, I_k$, $k \geqq 1$, representing the ρ-cycles of length 2, such that $H \rightarrow H_\rho$ (delete $I_1, \cdots, I_k$).

S is said to be of type n. The members of F are *fixed* elements; the members of $E - F$ are *cancelled* elements.

If H_ρ contains a subword I, of length 2, whose elements are a, b, then $S' = (E, \beta, \sigma, \theta)$ is also a structure where $a\sigma = b$, $b\sigma = a$ and $\sigma = \rho$ except on the set $\{a, b\}$. Indeed, if H_σ is defined by $H_\rho \rightarrow H_\sigma$ (delete I), then H_σ represents the reduced map of S'. We say that S *contracts* to S' in one step.

S is an *$\mathscr{R}$-structure* (*$\mathscr{N}$-structure*) if a free generator is assigned to each element in E in such a way that the values of the ρ-cycles of length 2 are null words and the values of the β-cycles are words in $\mathscr{R}$ (in $\mathscr{N}$). When S is an $\mathscr{R}$-structure, of type n, with map θ and reduced map θ_F, then the values of θ are $\mathscr{R}$-words of type n and the values of θ_F are relators of type n.

THEOREM 3.1. *Each relator is a value of the reduced map of some $\mathscr{R}$-structure.*

Proof. Use the definition of $\mathscr{R}$-structure and Lemma 2.1.

We now turn to some more definitions concerning a structure $S = (E, \beta, \rho, \theta)$. S is called *noncancelled* if there exist fixed elements in E. S is *cancelled* if E contains only cancelled elements. In the latter case, ρ is a pure reflection.

If A is a nonempty subset of E, then A is the carrier of a substructure T whenever A is closed under β and ρ. In this case, $T =$

$(A, \gamma, \sigma, \theta_A)$ where γ, σ are the restrictions of β, ρ, respectively, to the set A. T is a proper substructure if $A \neq E$. S is *minimal* if it has no proper substructures; S is *simple* if it has no proper cancelled substructure.

THEOREM 3.2. *Each relator is a value of the reduced map of some simple* $\mathscr{R}$*-structure.*

Proof. Use previous theorem and next lemma.

LEMMA 3.1. *Each structure has the same reduced map as some simple structure.*

Proof. Consider a nonsimple structure $S = (E, \beta, \rho, \theta)$ determined by the expressions $1 \to H$ (insert $H_1, \cdots, H_n$) and $H \to H_\rho$ (delete $I_1, \cdots, I_k$) as in the definition of a structure. Suppose $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ is the maximum cancelled proper substructure of S. Let H' denote the array that results from deleting all the elements in E_1 from H.

A sequence $H'_1, \cdots, H'_m$ remains after deleting from $H_1, \cdots, H_n$ the terms which represent the β_1-cycles.

A sequence $I'_1, \cdots, I'_l$ remains after deleting from $I_1, \cdots, I_k$ the terms which represent the ρ_1-cycles. Then the expressions $1 \to H'$ (insert $H'_1, \cdots, H'_m$) and $H' \to H_\rho$ (delete $I'_1, \cdots, I'_l$) determine a simple structure having the same reduced map as S.

4. Complexes. A *complex* $C = (E, \beta, \rho)$ consists of a finite, nonempty set E which is acted on by a permutation β and a pure reflection ρ. If α is the map β, followed by ρ (i.e. $\alpha = \beta\rho$), then the α-orbits, the elements in E, and the β-orbits are the *vertices*, *edges*, and *boundaries*, respectively, of C. Whenever a free generator is assigned to each edge, the values of the β-cycles are called the values of the boundaries of C.

C is a disjoint union of 2 complexes (E_i, β_i, ρ_i) for $i = 1, 2$ if E is a disjoint union of E_1, E_2 and β_i, ρ_i are the restrictions of β, ρ, respectively, to the set E_i $(i = 1, 2)$. If this is never the case, C is said to be *connected*.

Since E is a disjoint union of the ρ-orbits and each ρ-orbit contains exactly 2 edges, the number of edges is always even. Whenever a is an edge, $a\rho$ is called the *inverse* of a. If v, $2e$, n denote the numbers of vertices, edges and boundaries of C, then $v - e + n$ is the Euler characteristic. A *spherical* complex is a connected complex with Euler characteristic 2.

Note that when $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ is a cancelled structure, then $C_1 = (E_1, \beta_1, \rho_1)$ is a complex. Furthermore, S_1 is minimal if and only if C_1 is connected.

5. From structures to complexes. We now describe a transition from a noncancelled structure S to a cancelled structure S_1; with S_1 there is associated a complex C_1.

Suppose $S = (E, \beta, \rho, \theta)$ is a noncancelled $\mathscr{R}$-structure, of type $n \geqq 1$, with $H, H_\rho, H_1, \cdots, H_n$ as in § 3. A cancelled $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $n+1$, is defined as follows.

Let $H_\rho \cong a_1 \cdots a_m$. Since S is noncancelled, H_ρ is nonempty and $m \geqq 1$. Choose m new elements $b_1, \cdots, b_m$; put $E_1 = E \cup \{b_1, \cdots, b_m\}$. θ_1 is represented by HH_{n+1} where $H_{n+1} \cong b_m \cdots b_1$. Then $H_\rho H_{n+1} \to 1$ (delete $J_1, \cdots, J_m$) where $J_i \cong a_i b_i$ ($1 \leqq i \leqq m$). The β_1-cycles are represented by $H_1, \cdots, H_n, H_{n+1}$. The β_1-cycle represented by H_{n+1} is called the *distinguished* β_1-cycle of S_1.

If ρ is the identity, then the ρ_1-cycles are represented by $J_1, \cdots, J_m$. If ρ is not the identity, then we have $H \to H_\rho$ (delete $I_1, \cdots, I_k$) where the I_i represent the ρ-cycles of length 2. In this case, $HH_{n+1} \to 1$ (delete $I_1, \cdots, I_k, J_1, \cdots, J_m$) and the I_i, J_i represent the ρ_1-cycles.

A free generator is assigned to each b_i so that the values of H_ρ and H_{n+1} are inverse words. This insures that the values of the J_i are null words and the value of H_{n+1} is a relator. The $\mathscr{N}$-structure S_1 is now complete and $C_1 = (E_1, \beta_1, \rho_1)$.

With reference to the construction of S_1, we have:

REMARK 5.1. If ab is a subword, of length 2, of some cyclic permutation of H_ρ and if $a\rho_1 = c$, $b\rho_1 = d$, then dc is a subword of some cyclic permutation of H_{n+1}. In other words, if a, b are distinct fixed elements of S and $a\theta_F = b$ where θ_F is the reduced map of S, then $b\rho_1\beta_1 = a\rho_1$.

LEMMA 5.1. *If S is simple or minimal, then S_1 is minimal.*

Proof. Since minimal implies simple for structures, we assume S is simple. Suppose a nonempty proper subset A_1 (of E_1) is closed under β_1 and ρ_1. Then $A_2 = E_1 - A_1$ also has this property; A_1, A_2 are carriers of substructures of S_1. Thus all the elements b_i are in the same A_j, say in A_2. Therefore all the elements in A_1 are cancelled elements in S. But then A_1 is the carrier of a proper cancelled substructure of S, contary to the assumption that S is simple.

THEOREM 5.1. *For each relator W there is a cancelled, minimal $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $t \geqq 2$ and a connected complex $C_1 = (E_1, \beta_1, \rho_1)$ such that the β_1-cycles can be represented by t arrays whose values are W^{-1} and $t-1$ defining relators.*

Proof. By Theorem 3.2 there is a simple $\mathscr{R}$-structure S, of type $n \geqq 1$, where W is one of the values of the reduced map of S. Earlier we constructed a cancelled $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $n + 1$, whose β_1-cycles satisfy the desired condition. By Lemma 5.1 S_1 is minimal; hence, $C_1 = (E_1, \beta_1, \rho_1)$ is connected.

6. Spherical complexes. The relationship between relators and spherical complexes is given in Theorem 6.2 and in Theorem 6.4. Their proofs depend on Theorem 6.1 and Theorem 6.3, which are converses. Three preliminary lemmas are needed.

LEMMA 6.1. *Let $H_1, \cdots, H_n$ be arrays with disjoint sets of elements satisfying $1 \to H$ (insert $H_1, \cdots, H_n$) where H is an array and $n \geqq 2$. Suppose H has a subword I, of length 2, whose letters a, b are in H_i, H_j, respectively, for $i < j$. Then $1 \to H$ (insert $H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n$) for some array K, having subword I, such that $1 \to K$ (insert H_i, H_j).*

Proof. Let H' be the array such that $1 \to H'$ (insert $H_1, \cdots, H_i, \cdots, H_j$) and $H' \to H$ (insert $H_{j+1}, \cdots, H_n$). Then I is a subword of H'. We also have $1 \to H'$ (insert $H_1, \cdots, H_{i-1}, H_i, H_j, H_{i+1}, \cdots, H_{j-1}$). Let $H_i \cong A_1aA_2$ and $H_j = B_2bB_1$.

If $I \cong ab$, then B_2 is the empty array and we put $K \cong A_1abB_1A_2$. If $I \cong ba$, then B_1 is the empty array and we put $K \cong A_1B_2baA_2$.

LEMMA 6.2. *Let the array $abc_1 \cdots c_r$ $(r \geqq 1)$ represent a β-cycle μ corresponding to a β-orbit B of a connected complex $C = (E, \beta, \rho)$. Assume $a\rho = b$. Then C has the same Euler characteristic as some connected complex $C' = (E', \beta', \rho')$ having 2 fewer edges than C.*

Proof. Put $B' = \{c_1, \cdots, c_r\}$ and $E' = E - \{a, b\}$. Let μ' be the cyclic permutation represented by the array $c_1 \cdots c_r$. Define ρ' to be the restriction of ρ to the set E'. Define β' by putting $\beta' = \mu'$ on B' and $\beta' = \beta$ on $E' - B'$. The connectedness of C' follows from the connectedness of C. Thus, it suffices to show that C has one more vertex than C'.

Since $a\beta\rho = b\rho = a$, $\{a\}$ is a vertex of C. c_r is the only edge in E' having different images under $\beta\rho$ and $\beta'\rho'$. In fact, $c_r\beta\rho = a\rho = b$ and $c_r\beta'\rho' = c_1\rho' = c_1\rho$. Furthermore $b \neq c_1\rho$ since $a \neq c_1$ and $a\rho = b$.

Let $d = c_1\rho$, $\alpha = \beta\rho$, $\alpha' = \beta'\rho'$. There is an α-orbit V whose α-cycle is represented by an array of the form c_rbdD and V is a disjoint union of $\{b\}$ and an α'-orbit V' whose α'-cycle is represented by c_rdD. Thus C, C' have the same vertices, except that $\{a\}$ and V in C are replaced by V' in C'. This completes the proof.

LEMMA 6.3. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 2$ boundaries. Let $A = \{a_1, \cdots, a_r\}$, $B = \{b_1, \cdots, b_s\}$ be β-orbits whose β-cycles μ, ν are represented by arrays $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively. Assume $b_1\rho = a_r$. Then C has the same Euler characteristic as some connected complex $C' = (E, \beta', \rho)$ having $n - 1$ boundaries.*

Proof. Let μ' be the cyclic permutation represented by the array $a_1 \cdots a_r b_1 \cdots b_s$. Define β' by putting $\beta' = \mu'$ on the set $A \cup B$ and $\beta' = \beta$ otherwise. Then C has one more boundary than C' since 2 β-orbits A, B are replaced by one β'-orbit $A \cup B$. We must show that C' has one more vertex than C (i.e. that $\beta'\rho$ has one more orbit than $\beta\rho$).

Only $b_s = b_1\beta^{-1}$ and a_r have different images under $\beta\rho$ than under $\beta'\rho$. In fact $b_s\beta\rho = b_1\rho = a_r$ and $b_s\beta'\rho = a_1\rho$; $a_r\beta\rho = a_1\rho$ and $a_r\beta'\rho = b_1\rho = a_r$. Furthermore $a_1\rho \neq a_r$ since $a_1 \neq b_1$ and $b_1\rho = a_r$.

Let $c = a_1\rho$, $\alpha = \beta\rho$, and $\alpha' = \beta'\rho$. There is an α-orbit V whose β-cycle is represented by an array of the form $b_s a_r c D$ and V is a disjoint union of 2 α'-orbits V', V'' whose α'-cycles are represented by the arrays a_r and $b_s c D$. Thus C, C' have the same vertices, except that V is replaced by V' and V''. Therefore C' has one more vertex than C.

The connectedness of C' follows from the connectedness of C.

THEOREM 6.1. *Let $S = (E, \beta, \rho, \theta)$ be a minimal, cancelled structure of type $n \geqq 1$. Then $C = (E, \beta, \rho)$ is a spherical complex.*

Proof. Use induction on the number $2e$ of edges of C. Suppose $2e = 2$. Then $E = \{a, b\}$ and $a\rho = b$, $b\rho = a$; hence C is connected. If the β-orbits are $\{a\}$ and $\{b\}$ so that $n = 2$, then $a\beta\rho = a\rho = b$, $b\beta\rho = b\rho = a$ and $\{a, b\}$ is the only vertex. Thus $v - e + n = 1 - 1 + 2 = 2$. If $\{a, b\}$ is the only β-orbit so that $n = 1$, then $a\beta\rho = b\rho = a$, $b\beta\rho = a\rho = b$ and $\{a\}$, $\{b\}$ are the only vertices. Thus, $v - e + n = 2 - 1 + 1 = 2$.

Now assume that $2e \geqq 4$ and that the theorem holds for complexes with fewer than $2e$ edges. Let $H, H_1, \cdots, H_n$ represent θ and the β-cycles and let $I_1 \cong ab, I_2, \cdots, I_e$ represent the ρ-cycles. Assume that

$$1 \rightarrow H \text{ (insert } H_1, \cdots, H_n)$$
$$H \rightarrow 1 \text{ (delete } I_1, \cdots, I_e) .$$

Suppose a is in H_i, b is in H_j.

Case 1. $(i = j)$ Then I_1 is a subword of H_i. Let β_1 be the β-cycle represented by H_i. It cannot happen that $H_i \cong I_1$ since then $C_1 = (E_1, \beta_1, \rho_1)$ is a subcomplex of C where $E_1 = \{a, b\}$ and I_1 represents the only ρ_1-cycle. Also S is minimal so C is connected; hence $C = C_1$. This is contrary to $2e \geqq 4$. Therefore, some cyclic permutation of H_i is of the form $abc_1 \cdots c_r$ $(r \geqq 1)$.

A minimal, cancelled structure $S' = (E', \beta', \rho', \theta')$ is determined as follows. Let $H', H_1, \cdots, H_{i-1}, H'_i, H_{i+1}, \cdots, H_n$ represent θ' and the β'-cycles, and $I_2, \cdots, I_e$ represent the ρ'-cycles, where

$$\begin{aligned} H &\rightarrow H' \text{ (delete } I_1) \\ H_i &\rightarrow H'_i \text{ (delete } I_1) \\ 1 &\rightarrow H' \text{ (insert } H_1, \cdots, H_{i-1}, H'_i, H_{i+1}, \cdots, H_n) \\ H' &\rightarrow 1 \quad \text{(delete } I_2, \cdots, I_e) . \end{aligned}$$

The complexes $C' = (E', \beta', \rho')$ and C have the same Euler characteristic and C' is connected by Lemma 6.2.

Case 2. $(i \neq j)$ Suppose $i < j$ (Treatment of $j < i$ is similar.) A minimal, cancelled structure $S' = (E, \beta', \rho, \theta)$ is determined as follows. Let $H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n)$ represent the β'-cycles where K has the subword I_1 and

$$\begin{aligned} 1 &\rightarrow H \text{ (insert } H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n) \\ 1 &\rightarrow K \text{ (insert } H_i, H_j) . \end{aligned}$$

This is possible by Lemma 6.3.

The complexes $C' = (E, \beta', \rho)$ and C have the same Euler characteristic and C' is connected (by Lemma 6.1). In fact, some cyclic permutation of K, H_i, and H_j are of the forms $a_1 \cdots a_r b_1 \cdots b_s$, $a_1 \cdots a_r$, and $b_1 \cdots b_s$, respectively, where $I_1 \cong a_r b_1$. Now S' and C' can be treated as in Case 1, since I_1 is a subword of K.

Thus, in either one or two steps, we can always find a new minimal, cancelled structure whose associated connected complex has $2(e-1)$ edges such that the original and new complexes have the same Euler characteristic. By the induction assumption, the new complex has Euler characteristic 2; hence, so does the original complex. This completes the proof.

THEOREM 6.2. *For each relator W there is some spherical complex C with $n \geqq 2$ boundaries such that a free generator is assigned to each edge (with inverse free generators assigned to inverse edges), W^{-1} is a value of one of the boundaries, and defining relators are the values of the remaining boundaries.*

Proof. Use Theorem 5.1 and Theorem 6.1.

THEOREM 6.3. *Let $C = (E, \beta, \rho)$ be a spherical complex with $n \geqq 1$ boundaries. Then there exists some minimal, cancelled structure $S = (E, \beta, \rho, \theta)$.*

Proof. Use induction on n. We first prove the case $n = 1$. Here β itself is the only β-cycle. This case will be proved by induction on the number $2e$ of edges. When $2e = 2$, we have $\beta = \rho$ and we take $\theta = \beta$.

Now assume $n = 1$, $2e \geqq 4$ and the theorem holds for complexes having one boundary and fewer than $2e$ edges. There must be a vertex containing just one edge since, if not, we have $2e \geqq 2v$ and $v - e + 1 = 2$ (where v is the number of vertices). But this implies $e \geqq v$ and $v = 1 + e$ which is impossible. If $\{a\}$ is a vertex, let $b = a\rho$. Then $a\beta = b$ since $a\beta\rho = a$. Thus β is represented by some array $H \cong I_1 H'$ where $I_1 \cong ab$.

A connected complex $C' = (E', \beta', \rho')$ with $2e - 2$ edges and 1 boundary is defined by $E' = E - \{ab\}$ if we take ρ' to be the restriction of ρ to E' and put $\beta' = \beta_A$ with $A = E'$. Now apply the induction assumption to C'. There exists a minimal, cancelled structure $S' = (E', \beta', \rho', \theta')$. There exist an array X representing $\theta' = \beta'$ and arrays $I'_2, \cdots, I'_e$ representing the ρ'-cycles such that $X \rightarrow 1$ (delete $I'_2, \cdots, I'_e$). But since H' is a cyclic permutation of X, there exist arrays $I_2, \cdots, I_e$ representing the ρ'-cycles such that $H' \rightarrow 1$ (delete $I_2, \cdots, I_e$).

Since $H \rightarrow H'$ (delete I_1), we have that $\theta = \beta$ is represented by an array H satisfying $H \rightarrow 1$ (delete $I_1, I_2, \cdots, I_e$). Thus $S = (E, \beta, \rho, \theta)$ is a cancelled structure which is minimal since C is connected.

Now suppose $n \geqq 2$. Assume that the theorem holds for complexes having fewer than n boundaries. We need only consider the case that there exist two edges a, b, in different boundaries, such that $a\rho = b$. For if an edge and its image under ρ are always in the same boundary, then one boundary E_1 consists of the edges in some subcomplex which must be the whole complex C, by the connectedness of C. But then $n = 1$.

Thus, we can choose two β-cycles μ, ν represented by arrays $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively, such that $a_r\rho = b_1$. Form a connected complex $C' = (E, \beta', \rho)$, having $n - 1$ boundaries, as in Lemma 6.3. The induction assumption implies that there is a minimal, cancelled structure $S' = (E, \beta', \rho, \theta)$. Here one of the β'-cycles μ' is represented by the array $a_1 \cdots a_r b_1 \cdots b_s$. There exist arrays $H, H_1, \cdots, H_{n-1}$ representing θ and the $n - 1$ β'-cycles such that $1 \rightarrow H$ (insert $H_1, \cdots, H_{n-1}$). Then $a_1 \cdots a_r b_1 \cdots b_s$ is a cyclic permutation of H_i, for some i. Thus $1 \rightarrow H_i$ (insert A, B) or $1 \rightarrow H_i$ (insert B, A) for some arrays A, B which are cyclic permutations of $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively. In either case, H splits into $H_1, \cdots, H_{i-1}, A, B, H_{i+1}, \cdots, H_{n-1}$ which represent the β-cycles. Thus $S = (E, \beta, \rho, \theta)$ is a cancelled structure which is minimal since C is connected.

THEOREM 6.4. *Let $C = (E, \beta, \rho)$ be a spherical complex with $n \geqq 2$*

boundaries such that a free generator is assigned to each edge (with inverse free generators assigned to inverse edges). If all but one of the boundaries have values which are defining relators, then each value of the remaining boundary is a relator.

Proof. A minimal, cancelled structure $S = (E, \beta, \rho, \theta)$ exists by Theorem 6.3. Suppose an array H represents θ. Since H splits into arrays representing the ρ-cycles, W splits into null words so that W is a relator. Since H splits into arrays representing the β-cycles, W splits into $n-1$ defining relators and a word K (a value of the "remaining" β-cycle). Since W is a relator, K must be a relator.

7. Sides of nontrivial complexes. In this section each complex $C = (E, \beta, \rho)$ is nontrivial (i.e. has $n \geqq 3$ boundaries). When C is also spherical, we show that each β-cycle can be represented by an array which is broken up into a product $X_1 \cdots X_t$ $(t \geqq 1)$ where each X_i has certain properties. The X_i will be called sides. In order to define sides, we classify the edges of C. Let a be an edge.

If either $a\rho\beta = a$ or $a\rho\beta\rho\beta \neq a$, then a is *initial*. If either $a\beta\rho = a$ or $a\beta\rho\beta\rho \neq a$, then a is *final*. Thus, if a is initial, final, or neither, then $a\rho$ is final, initial, or neither, respectively. Also, if a is initial, then $a\beta^{-1}$ is final; if a is final, then $a\beta$ is initial.

An array $X \cong a_1 \cdots a_r$ $(r \geqq 1)$, which partially represents a β-cycle, is a *side* if a_1 is the only initial edge in X and a_r is the only final edge in X. If $X \cong a_1 \cdots a_r$ is a side, then the array $Y \cong b_r \cdots b_1$, where $a_i\rho = b_i$ $(1 \leqq i \leqq r)$, is called the *inverse* of X.

LEMMA 7.1. *If $X \cong a_1 \cdots a_r$ is a side, so is its inverse $Y \cong b_r \cdots b_1$.*

Proof. It suffices to check that Y partially represents a β-cycle when $r \geqq 2$. i.e. $b_{i+1}\beta = b_i$ for $1 \leqq i \leqq r-1$. Indeed, $b_{i+1}\beta = a_{i+1}\rho\beta = a_i\beta\rho\beta = b_i\rho\beta\rho\beta = b_i$. The last equality holds since b_i is not initial for $1 \leqq i \leqq r-1$.

LEMMA 7.2. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 3$ boundaries. Then each boundary contains at least one initial edge and at least one final edge (possibly the same edge).*

Proof. Suppose the array $A \cong a_1 \cdots a_r$, $r \geqq 1$, represents a β-cycle so that $\{a_1, \cdots, a_r\}$ is a boundary. Let $B \cong b_r \cdots b_1$ be the inverse of A. Suppose all the a_i are not final. Then all the b_i are not initial.

When $r \geqq 2$, $b_{i+1}\beta = b_i$ for $1 \leqq i \leqq r-1$ as in the proof of the previous lemma. $b_1\beta = a_1\rho\beta = a_r\beta\rho\beta = b_r\rho\beta\rho\beta = b_r$. When $r = 1$,

$a_1\beta = a_1$ and $b_1\beta = b_1\rho\beta\rho\beta = b_1$. In either case, $E_1 = \{a_1, \cdots, a_r, b_1, \cdots, b_r\}$ is closed under β and ρ. Hence $C_1 = (E_1, \beta_1, \rho_1)$ is a subcomplex where β_1, ρ_1 are the restrictions of β, ρ to E_1. C_1 must be the whole complex by the connectedness of C. But C_1 has just two boundaries: $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots, b_r\}$. This contradicts $n \geqq 3$. Thus some a_i is final and then $a_i\beta$ is initial.

LEMMA 7.3. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 3$ boundaries. Then each β-cycle can be represented by a product $X_1 \cdots X_t$ $(t \geqq 1)$ where each X_i is a side. This representation is unique to within a cyclic permutation of these sides.*

Proof. Let μ be a β-cycle. Choose an array M, representing μ, so that the first letter of M is an initial edge. (Then the last letter of M is a final edge.) Therefore $M \cong X_1 \cdots X_t$, $t \geqq 1$, where an edge in M is initial (final) if and only if it is the first (last) letter in some X_i. The essential uniqueness of this representation follows from the fact that each edge can be placed uniquely in one of four classes: initial but not final, neither initial nor final, final but not initial and both initial and final. This completes the proof.

Vertices containing exactly 2 edges are called *nonessential*; all other vertices are *essential*. If the inverse arrays $X \cong a_1 \cdots a_r$ and $Y \cong b_r \cdots b_1$ $(r \geqq 2)$ are sides, then $\{a_i, b_{i+1}\}$ are nonessential vertices for $1 \leqq i \leqq r - 1$ since $a_i\beta\rho = a_{i+1}\rho = b_{i+1}$ and $b_{i+1}\beta\rho = b_i\rho = a_i$. The next lemma shows that all nonessential vertices arise in this way.

LEMMA 7.4. *If $\{a_1, b_2\}$ is a nonessential vertex of a complex $C = (E, \beta, \rho)$ and if $a_2 = a_1\beta$, $b_1 = b_2\beta$, then a_1a_2 and b_2b_1 are subwords of sides.*

Proof. $a_2\rho = a_1\beta\rho = b_2$; $b_1\rho = b_2\beta\rho = a_1$. We must show that a_1 is not final and a_2 is not initial. Indeed, $a_1\beta\rho \neq a_1$ since $b_2 \neq a_1$; $a_1\beta\rho\beta\rho = a_2\rho\beta\rho = b_2\beta\rho = a_1$. Also, $a_2\rho\beta \neq a_2$ since $a_2\rho\beta = b_2\beta = b_1$ and $b_1\rho = a_1 \neq b_2 = a_2\rho$. Similarly, b_2 is not final and b_1 is not initial. This completes the proof.

The relationships between essential vertices, final edges, and sides can now be given.

LEMMA 7.5. *Let $C = (E, \beta, \rho)$ be a nontrival complex. An edge is in an essential vertex if and only if the edge is final. An edge is final if and only if it is the last letter in some side.*

Proof. Let a be an edge. Suppose a is in an essential vertex V. If $V = \{a\}$, then $a\beta\rho = a$ and a is final. If V contains at least 3 edges, then a, $b = a\beta\rho$ and $c = b\beta\rho$ are distinct edges. Thus, $a\beta\rho\beta\rho = c \neq a$; hence, a is final.

Now suppose a is final. If $a\beta\rho = a$, then $\{a\}$ is a vertex. If $a\beta\rho\beta\rho \neq a$, then $b = a\beta\rho \neq a$ and $c = b\beta\rho \neq a$. Also $a \neq b$ and the fact that $\beta\rho$ is a one-to-one map imply that $b = a\beta\rho \neq b\beta\rho = c$. Therefore there is an essential vertex containing a, b, c among its edges. The second statement of Lemma 7.5 follows from the proof of Lemma 7.3.

THEOREM 7.1. *Let $C = (E, \beta, \rho)$ be the connected complex associated with a cancelled, minimal $\mathscr{N}$-structure $S = (E, \beta, \rho, \theta)$, of type $n \geqq 3$. Assume that the values of the β-cycles are cyclically reduced words. Let $2s$, w denote the number of sides and the number of essential vertices of C. Then there is no vertex containing just one edge, $2s \geqq 3w$, and $w - s + n = 2$.*

Proof. If $\{a\}$ were a vertex, then $a\beta\rho = a$; hence $a\beta = a\rho$. Let $b = a\beta$. Then ab partially represents some β-cycle μ. Since $a\rho = b$, the value of ab is a null word which is a subword of a value of μ. This contradicts the assumption that the values of the β-cycles are cyclically reduced words. Hence, there is no vertex $\{a\}$.

Therefore each essential vertex contains at least 3 edges. Using Lemma 7.5 and the resulting fact that there is a one-to-one correspondence between final edges and sides, we get $2s \geqq 3w$.

We know that $v - e + n = 2$ where v, $2e$ are the numbers of vertices and edges of C. We show that $v - e = w - s$ by letting each pair of inverse sides (of length $m \geqq 2$) replace $2m$ edges and $m - 1$ nonessential vertices. In fact, if $X \cong a_1 \cdots a_m$, $Y \cong b_m \cdots b_1$ are inverse sides ($m \geqq 2$), then the letters in X, Y are the discarded edges and $\{a_i, b_{i+1}\}$ for $1 \leqq i \leqq m - 1$ are the discarded vertices. Thus each step reduces both v and e by $m - 1$. Lemma 7.4 assures us that each nonessential vertex (if any) will be discarded in this process. After a finite number of steps, we have discarded all edges which are not sides and all nonessential vertices. Thus $v - e = w - s$ and $w - s + n = 2$.

8. Calculations. Let $S = (E, \beta, \rho, \theta)$ be a noncancelled, minimal $\mathscr{R}$-structure, of type $n \geqq 2$, with reduced map θ_F. Assume that the values of θ_F are cyclically reduced words. Let $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ be a cancelled, minimal $\mathscr{N}$-structure, of type $n + 1$, associated with S. (Thus the values of the β_1-cycles are cyclically reduced words.) Suppose that the distinguished β_1-cycle has m sides in the complex $C_1 = (E_1, \beta_1, \rho_1)$.

Consider a side X of a nondistinguished β_1-cycle of C_1. X will be called a *fixed side* whenever the inverse of X is a side of the distinguished β_1-cycle. In such a case, the letters in X are all fixed elements in E.

Let B_k^i denote the number of nondistinguished boundaries having k sides, i sides of which are fixed; put $B_k = \sum_i B_k^i$. Then we have

$$(1)\qquad n = \sum_{k=1}^{\infty} B_k ; \qquad 2s = m + \sum_{k=1}^{\infty} kB_k$$

$$m = \sum_{k=1}^{\infty} \sum_{1 \leqq i \leqq k} iB_k^i .$$

From Theorem 7.1 applied to C_1 we get $6w - 6s + 6(n + 1) = 12$ and $4s \geqq 6w$. Therefore

$$(2)\qquad 6n - 2s \geqq 6 .$$

From (1) and (2) we get

$$\sum_{k=1}^{5} (6 - k)B_k \geqq m + 6 + \sum_{k=7}^{\infty} (k - 6)B_k$$

and

$$(3)\qquad \sum_{k=1}^{5} (6 - k)B_k \geqq m + 6 .$$

Now expand the left hand side of (3):

$$(4)\qquad \sum_{k=1}^{5} (6 - k)B_k = \sum_{k=1}^{4} (5 - k)B_k^1 + \sum_{k=1}^{5} B_k^1 + \sum_{k=1}^{5} (6 - k)B_k^0 + \sum_{k=2}^{5} \sum_{i=2}^{k} (6 - k)B_k^i .$$

Further,

$$(5)\qquad \sum_{k=2}^{5} \sum_{i=2}^{k} (6 - k)B_k^i \leqq 2B_2^2 + B_3^2 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i .$$

This can be seen as follows:
When (i, k) is neither (2, 2) nor (2, 3), we have $(6 - k) \leqq i$.
When $i = k = 2$, $(6 - k)B_k^i = 2B_2^2 + iB_k^i$.
When $i = 2$, $k = 3$, $(6 - k)B_k^i = B_3^2 + iB_k^i$.
Now use (3), (4), and (5) to get:

$$\sum_{k=1}^{4} (5 - k)B_k^1 + \sum_{k=1}^{5} (6 - k)B_k^0 + \sum_{k=1}^{5} B_k^1 + 2B_2^2 + B_3^2 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i \geqq m + 6 .$$

But

$$m \geqq \sum_{k=1}^{5} B_k^1 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i .$$

Therefore,

$$(6) \qquad \sum_{k=1}^{4}(5-k)B_k^1 + \sum_{k=1}^{5}(6-k)B_k^0 + 2B_2^2 + B_3^2 \geqq 6 \ .$$

9. Minimal relators. A *minimal* relator of type n is a value of the reduced map of a minimal $\mathscr{R}$-structure of type n (i.e. a minimal structure, of type n, which is also an $\mathscr{R}$-structure). Similarly a nonminimal relator corresponds to a nonminimal $\mathscr{R}$-structure.

We aim to show that each relator splits into minimal relators. We prove this by showing that an analogous situation holds for the reduced map of a structure S and the reduced maps $\theta_1, \cdots, \theta_r$ of the minimal substructures of S. This requires the following.

DEFINITION. Let $\theta, \theta_1, \cdots, \theta_r$ be cyclic permutations acting on sets $E, E_1, \cdots, E_r$, respectively, such that $E = E_1 \cup \cdots \cup E_r$ is a disjoint union ($r \geqq 1$). θ *splits* into $\theta_1, \cdots, \theta_r$ if the θ_i can be put in a sequence $\theta'_1, \cdots, \theta'_r$ and if arrays $H, H_1, \cdots, H_r$, representing $\theta, \theta'_1, \cdots, \theta'_r$, respectively, can be chosen so that $1 \rightarrow H$ (insert $H_1, \cdots, H_r$).

THEOREM 9.1. *The reduced map of any structure S splits into the reduced maps of the minimal substructures of S.*

The proof of Theorem 9.1 requires a lemma.

LEMMA 9.1. *Suppose the structure $S = (E, \beta, \rho, \theta)$ contracts to the structure $S' = (E, \beta, \sigma, \theta)$ in one step. If S satisfies Theorem 9.1, so does S'.*

Proof. By assumption there exist arrays H_ρ, H_σ representing the reduced maps of S, S', respectively, such that $H_\rho \rightarrow H_\sigma$ (delete I) for some array I, of length 2, whose elements are a, b. $\sigma = \rho$ except on the set $\{a, b\}$; $a\sigma = b$, $b\sigma = a$. Let $H_\rho \cong XIY$ and $H_\sigma \cong XY$.

If $S_i = (E_i, \beta_i, \rho_i, \theta_i)$ are the minimal substructures of S ($1 \leqq i \leqq r$), then there exist arrays $M_1, \cdots, M_r$ representing the reduced maps of $S_1, \cdots, S_r$, respectively, such that $1 \rightarrow H_\rho$ (insert $M_1, \cdots, M_r$). Suppose $a \in E_i$, $b \in E_j$.

Case 1. ($i = j$) Since E_i is closed under β and σ, E_i is the carrier of a substructure S'_i of S'. The fact that S_i is minimal implies that each nonempty proper subset A (of E_i) is not closed under both β and ρ; hence A is not closed under both β and σ. Thus S'_i is a minimal substructure of S'.

Let $M_i \cong PIQ$. Then the possibly empty array $M'_i \cong PQ$ represents the reduced map of S'_i. Finally, $1 \rightarrow H_\sigma$ (insert $M_1, \cdots, M_{i-1}, M'_i, M_{i+1}, \cdots, M_r$).

Case 2. $(i < j)$ By Lemma 6.1, there is an array K such that $1 \to H_\rho$ (insert $M_1, \cdots, M_{i-1}, K, M_{i+1}, \cdots, M_{j-1}, M_{j+1}, \cdots, M_r$), $1 \to K$ (insert M_i, M_j), and K is of the form $K \cong PIQ$.

E_i, E_j, and hence $E_i \cup E_j$ are closed under β and ρ. Then $E_i \cup E_j$ is closed under σ and is the carrier of a substructure S_i' of S'. Since E_i, E_j, and each nonempty proper subset of either E_i or E_j are not closed under both β and σ, we have that S_i' is a minimal substructure of S'.

The possibly empty array $K' \cong PQ$ represents the reduced map of S_i'; $1 \to H_\sigma$ (insert $M_1, \cdots, M_{i-1}, K', M_{i+1}, \cdots, M_{j-1}, M_{j+1}, \cdots, M_r$). This completes the proof of Lemma 9.1.

Now Theorem 9.1 can be proved. Let $S = (E, \beta, \rho, \theta)$ be a structure with k ρ-cycles of length 2. If $k = 0$, then ρ is the identity, the β-orbits are the carriers of the minimal substructures of S, and θ is the reduced map of S. Theorem 9.1 holds in this case since θ splits into the β-cycles (by the definition of a structure).

If $k \geqq 1$, then there exist structures $T_0 = (E, \beta, \rho_0, \theta), \cdots, T_k = (E, \beta, \rho_k, \theta)$ where ρ_0 is the identity and $\rho_k = \rho$, $T_k = S$ such that T_i contracts to T_{i+1} in one step $(0 \leqq i \leqq k-1)$. Use Lemma 9.1 and the fact that T_0 satisfies Theorem 9.1 to get that $T_k = S$ satisfies Theorem 9.1. This completes the proof.

Since each relator is a value of the reduced map of some $\mathscr{R}$-structure, we have

COROLLARY 9.1. *Each relator splits into minimal relators.*

The next 3 lemmas will be useful later.

LEMMA 9.2. *A nonminimal relator, of type $n \geqq 2$, splits into relators having types smaller than n.*

Proof. Observe that a relator of type 1 is necessarily minimal. Use Theorem 9.1 and the fact that a nonminimal structure, of type $n \geqq 2$, has minimal substructures whose types have sum n.

LEMMA 9.3. *Let $S = (E, \beta, \rho, \theta)$ be a structure. If the array $H \cong ac_1 \cdots c_r bD$, $r \geqq 1$, represents θ and if the fixed elements a, b satisfy $a\beta = b$, then $\{c_1, \cdots, c_r\}$ is closed under β and ρ.*

Proof. There exist arrays $H_1, \cdots, H_n$ representing the β-cycles $\mu_1, \cdots, \mu_n$, respectively, such that $1 \to H$ (insert $H_1, \cdots, H_n$). Since $a\beta = b$, ab is a subword of H_i for some i, $1 \leqq i \leqq n$. Since ab is not a subword of H, we have $i < n$. The set $\{c_1, \cdots, c_r\}$ must be the union of the β-orbits corresponding to some subsequence of $\mu_{i+1}, \cdots,$

μ_n. Hence, $\{c_1, \cdots, c_r\}$ is closed under β.

Since a, b are fixed elements, we have that $\{c_1, \cdots, c_r\}$ is closed under ρ.

LEMMA 9.4. *Let a, b be fixed elements of a minimal structure $S = (E, \beta, \rho, \theta)$ with reduced map θ_F. If $a\beta = b$, then $a\theta = b$ and $a\theta_F = b$.*

Proof. If $a\theta \neq b$, then there is an array $ac_1 \cdots c_r b$, $r \geqq 1$, which partially represents θ. Lemma 9.3 implies that $\{c_1, \cdots, c_r\}$ is the carrier of a proper substructure of S. This is impossible since S is minimal. Thus, $a\theta = b$. But then $a\theta_F = b$ since a, b are fixed elements.

10. **Asymmetric relators.** Let W be an $\mathscr{R}$-word with $1 \rightarrow W$ (insert $R_1, \cdots, R_n$) where the R_i are defining relators. We always consider just one mode of performing the insertions (if there is more than one). Since each letter of W originates from a letter of one of the R_i, there is a one-to-one correspondence between the letters in W and the letters in $R_1, \cdots, R_n$.

Let $X \cong X_1xX_2$ and $Y \cong Y_2yY_1$ be any two of the R_i. Suppose that x, y correspond to the letters u, v in W; that u, v can cancel with each other during free reduction of W; and that the words X_2X_1x and yY_1Y_2 are inverses. Then we say that u, v can cancel *symmetrically* or that W is a *symmetric* $\mathscr{R}$-word.

In this situation, either u, v are adjacent in W or u, v are separated by a nonempty subword (of W) which freely reduces to 1. We indicate this by saying that u, v can cancel either *immediately* or *eventually*; W is either immediately or eventually symmetric. If no two letters of W can cancel symmetrically during free reduction of W, then W is an *asymmetric* $\mathscr{R}$-word. Finally, an asymmetric (symmetric) relator of type t is a partially reduced asymmetric (symmetric) $\mathscr{R}$-word of type t.

LEMMA 10.1. *If a word W splits into $t \geqq 2$ defining relators, two of which are X, Y, then W splits into two words U, V such that*
U splits into $p \geqq 1$ defining relators, one of which is X,
V splits into $q \geqq 1$ defining relators, one of which is Y,
and $p + q = t$.

Proof. Use induction on t. The lemma holds for $t = 2$ with $U \cong X$, $V \cong Y$. Let $t \geqq 3$ and assume the lemma is true for smaller t. Suppose $1 \rightarrow W$ (insert $R_1, \cdots, R_t$) and $X \cong R_i$, $Y \cong R_j$ for $i < j$. Let W' be the word such that $1 \rightarrow W'$ (insert $R_1, \cdots, R_{t-1}$) and $W' \rightarrow W$ (insert R_t).

If $j = t$, choose $U \cong W'$, $V \cong R_t$. If $j < t$, then by the induction assumption W' splits into two words U', V' which split into p' defining relators and q' defining relators among which are X and Y, respectively, where $p' + q' = t - 1$. We can choose U, V so that either $U \cong U'$ and $V' \to V$ (insert R_t) or $V \cong V'$ and $U' \to U$ (insert R_t).

LEMMA 10.2. *An eventually symmetric $\mathscr{R}$-word W, of type $t \geqq 2$, is freely equal to some immediately symmetric $\mathscr{R}$-word W', of type t.*

Proof. Suppose $1 \to W$ (insert $R_1, \cdots, R_t$) where the R_k are defining relators. Let W contain the letters u, v which can eventually cancel symmetrically during free reduction of W. Suppose that u, v correspond to the letters x, y in $R_i \cong X_1xX_2$, $R_j \cong Y_2yY_1$. Apply the previous lemma with $X \cong R_i$, $Y \cong R_j$ to find the words U, V. Then U, V have cyclic permutations U', V', respectively, such that the product U', V' is a cyclic permutation of W.

Let $U' \cong M_1mM_2$ and $V' \cong N_2 \cong N_2nN_1$ where m, n correspond to x, y, respectively. Since u, v can cancel in W, either N_1M_1 or M_2N_2 freely reduces to 1. Thus W has a cyclic permutation $mM_2N_2nN_1M_1$ which partially reduces to either M_2N_2 or N_1M_1.

Put $W'' \cong M_2M_1mnN_1N_2$ which is an $\mathscr{R}$-word of type t. In fact, M_2M_1m is a cyclic permutation of U and is an $\mathscr{R}$-word of the same type as U by Remark 2.1. Similarly, nN_1N_2 and V are $\mathscr{R}$-words of the same type. Thus W'' is a product of $\mathscr{R}$-words whose types have sum t.

Either W'' partially reduces to M_2M_1 or W'' has a cyclic permutation which partially reduces to N_1M_1. Thus W'' has a cyclic permutation W' which is freely equal to W.

LEMMA 10.3. *Let W be a word which splits into $t \geqq 2$ defining relators $R_1, \cdots, R_t$. If two letters u, v in W can immediately cancel symmetrically, then W also splits into $t - 2$ defining relators and one or more null words.*

Proof. Let $R_i \cong X_1xX_2$ and $R_j \cong Y_2yY_1$ where x, y correspond to u, v, respectively. By assumption, X_2X_1x and yY_1Y_2 are inverses so that $X_1xyY_1Y_2X_2$ and $X_1Y_1Y_2yxX_2$ freely reduce to 1.

The proof of Lemma 6.1 shows that W splits into $t - 2$ defining relators and a word U. Either $U \cong X_1xyY_1X_2$ (with $Y_2 \cong 1$) or $U \cong X_1Y_2yxX_2$ (with $Y_1 \cong 1$). In either case, U freely reduces to 1 so that U splits into one or more null words. Thus, W splits into $t - 2$ defining relators and one or more null words.

LEMMA 10.4. *Suppose* $1 \to U$ (insert X, Y) *where* X, Y *are relators of types* $p, q \geqq 0$ *with the understanding that a relator of type* 0 *is a null word. Let* U *have a subword* N *which is a null word whose letters* u, v *correspond to a letter in* X *and a letter in* Y, *respectively. Let* V *be defined by* $U \to V$ (delete N). *Then* V *is a relator of type* $p + q$.

Proof. If $p = q = 0$, then X, Y and hence V are null words. If $p > 0$, $q = 0$, then $V \cong X$. If $p = 0$, $q > 0$, then either $V \cong Y$ or V is a cyclic permutation of Y.

Finally, if $p > 0$, $q > 0$, then X, Y are partially reduced forms of $\mathscr{R}$-words P, Q of types p, q, respectively. Then U is a partially reduced form of an $\mathscr{R}$-word M, of type $p + q$, such that $1 \to M$ (insert P, Q). Thus U is a relator of type $p + q$; hence so is V.

LEMMA 10.5. *If a word* W *splits into null words and/or relators having types whose sum is* $t \geqq 1$, *then this is also true for each word* W' *which is freely equal to* W.

Proof. It suffices to check the cases when W' is obtained from W by a single insertion or deletion of a null word N. If $W \to W'$ (insert N), then W' satisfies the lemma.

Now suppose $W \to W'$ (delete N). By assumption $1 \to W$ (insert $W_1, \cdots, W_r$) where $W_1, \cdots, W_r$ are null words and/or relators having types whose sum is t. Let W_k have type t_k with $t_k = 0$ if W_k is a null word. The lemma holds when each W_k is a null word since then W' also splits into null words. Therefore, assume some W_k is not a null word so that $t_1 + \cdots + t_r = t$.

One possibility is that the letters in N correspond to letters in the same W_i so that $W_i \to W'_i$ (delete N) for some word W'_i. If $t_i = 0$, W'_i is the empty word. If $t_i \geqq 1$, W'_i is either empty or a relator of type t_i. In any case, $1 \to W'$ (insert $W_1, \cdots, W_{i-1}, W'_i, W_{i+1}, \cdots, W_r$).

The other possibility is that the letters in N correspond to letters in two words W_i, W_j so that $r \geqq 2$. Lemma 6.1 implies that W splits into $r - 2$ W_k's, having types whose sum is $t - t_i - t_j$, and a word U which splits into W_i, W_j. Then W' splits into the same $r - 2$ W_k's and a word V such that $U \to V$ (delete N). By the previous lemma, V is a relator of type $t_i + t_j$. This completes the proof.

LEMMA 10.6. *A symmetric relator* W, *of type* $t \geqq 2$, *splits into null words and/or relators having types smaller than* t.

Proof. Let W be a partially reduced form of a symmetric $\mathscr{R}$-word V of type t. By Lemma 10.2 V is freely equal to an immediately symmetric $\mathscr{R}$-word V' of type t. By Lemma 10.3 either $t = 2$ and V' splits into null words or $t \geqq 3$ and V' splits into null words and relators having types whose sum is $t - 2$, (since a defining relator is a relator of type 1). By Lemma 10.5, W splits into null words and/or relators having types whose sum is $t - 2$. This implies Lemma 10.6.

THEOREM 10.1. *Each relator splits into null words and/or asymmetric relators.*

Proof. Let W be a relator of type $t \geqq 1$. When $t = 1$, W is a defining relator which is an asymmetric relator. Use induction on t. Let $t \geqq 2$ and assume the theorem for relators of type smaller than t. Theorem 10.1 then follows from Lemma 10.6.

11. Proof of Main Theorem. In order to solve the word problem in the presented group $\mathscr{G}$, it suffices to be able to recognize the asymmetric, minimal relators which we call *basic* relators.

THEOREM 11.1. *Each relator splits into null words and/or basic relators.*

Proof. Use Lemma 9.2, Lemma 10.6 and the fact that a relator of type 1 (a defining relator) is a basic relator. This completes proof.

We now consider a basic relator in a sixth group. More specifically, consider a cyclically reduced relator W which is a value of the reduced map of a minimal, noncancelled $\mathscr{R}$-structure $S = (E, \beta, \rho, \theta)$, of type $n \geqq 2$. Then some cyclic permutation of W is the freely reduced form of an $\mathscr{R}$-word V of type n, where V is a value of θ. We assume that V is an asymmetric $\mathscr{R}$-ward so that W is an asymmetric relator. The structure S characterizes one method of freely reducing V to a word which is a cyclic permutation of W. As usual, let $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ be the cancelled $\mathscr{N}$-structure associated with S; $C_1 = (E_1, \beta_1, \rho_1)$. Note that C_1 has no vertex containing just one edge (by Theorem 7.1).

In this situation, consider the B_k^i of § 8. The following lemma implies that $B_1^1 = B_2^2 = B_3^2 = 0$.

LEMMA 11.1. *Let $S = (E, \beta, \rho, \theta)$ be a noncancelled, minimal structure with associated cancelled structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$. Let $C_1 = (E_1, \beta_1, \rho_1)$ and assume that C_1 has no vertex containing just one edge. Suppose the product XY of nonempty arrays partially represents a nondistinguished β_1-cycle and X, Y are both sides in C_1.*

Then X, Y are not both fixed sides. Also, there is no nondistinguished β_1-cycle which is represented by one fixed side.

Proof. Suppose X, Y are fixed sides. This assumption together with the fact that XY partially represents a nondistinguished β_1-cycle imply that XY partially represents θ_F, the reduced map of S. Let a be the last letter in X; let b be the first letter in Y. Since Y is a side of C_1, b is an initial edge. Also, since $\{b\}$ cannot be a vertex of C_1, we have $b\rho_1\beta_1 \neq b$.

Since a, b are fixed elements of S and $a\beta = a\beta_1 = b$, we have $a\theta_F = b$ by Lemma 9.4. By Remark 5.1 $b\rho_1\beta_1 = a\rho_1$. Hence $b\rho_1\beta_1\rho_1\beta_1 = a\rho_1\rho_1\beta_1 = a\beta_1 = b$. This contradicts the fact that b is an initial edge of C_1. Thus, both X and Y cannot be fixed sides.

Now let Z be a fixed side, representing a nondistinguished β_1-cycle. If Z is of length $\geqq 2$, let a, b be the last and first letters of Z, respectively, so that $a \neq b$. We get a contradiction as before.

If Z is of length 1 and $Z \cong a$, then $a\beta = a\beta_1 = a$ and $a\rho = a$. Hence, $\{a\}$ is the carrier of a proper substructure of S, which is again a contradiction. This completes the proof.

Let the arrays MX and YN represent nondistinguished β_1-cycles μ, ν. respectively. Assume that the values of MX, YN are the defining relators R_1, R_2, respectively, and that X, Y are inverse sides.

If $\mu \neq \nu$, then R_1, R_2 are not inverses since V is asymmetric. Hence, R_1 and R_2^{-1} are distinct defining relators with a common subword (the value of X). The less-than-one-sixth property implies that

$$(*) \qquad l(X) < \frac{1}{6}\, l(MX) \quad \text{and} \quad l(Y) < \frac{1}{6}\, l(YN)\,.$$

It is also possible that $\mu = \nu$. In this case R_1, R_2 are cyclic permutations of one another. Once again (*) will hold provided that R_1, R_2 are not inverses. But this proviso holds.

LEMMA 11.2. *If T is a nonempty cyclically reduced word, then no cyclic permutation of T is the word T^{-1}.*

Proof. Let $U \cong T_2T_1$ be a cyclic permutation of $T \cong T_1T_2$. If $U \cong T^{-1}$, then $T_1 \cong T_1^{-1}$, $T_2 \cong T_2^{-1}$; hence T_1, T_2 are empty words, contradiction.

Thus, for C_1, we also have $B_k^0 = 0$ for $1 \leqq k \leqq 6$. From (6) in § 8, we get $3B_2^1 + 2B_3^1 + B_4^1 \geqq 6$. This implies the Main Theorem with $P_k = B_k^1$, $k = 2, 3, 4$.

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