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A THEOREM OF LITTLEWOOD AND LACUNARY SERIES FOR COMPACT GROUPS

ALESSANDRO FIGÀ-TALAMANCA AND DANIEL RIDER

Let G be a compact group and $f \in L^2(G)$. We prove that given $p < \infty$ there exists a unitary transformation U of $L^2(G)$ into $L^2(G)$, which commutes with left translations and such that $Uf \in L^p$. The proof is based on techniques developed by S. Helgason for a similar question. The result stated above, which is an extension of a theorem of Littlewood for the unit circle is then applied to the study of lacunary Fourier series.

The following two results concerning Fourier series of functions defined on the unit circle were proved by Littlewood [5]:

I. *Suppose that for any choice of complex numbers* α_n, *with* $|\alpha_n| = 1$, $\sum \alpha_n a_n e^{inx}$ *is the Fourier series of an integrable function (or a Fourier-Stieltjes series) then* $\sum |a_n|^2 < \infty$.

II. *Let* $\sum |a_n|^2 < \infty$. *Then given* $p < \infty$ *there exist complex numbers* α_n, *with* $|\alpha_n| = 1$, *such that* $\sum \alpha_n a_n e^{inx}$ *is the Fourier series of a function in* L^p.

Helgason [3] has generalized I to Fourier series on compact groups. Let G be a compact group with normalized Haar measure dx. If $f \in L^1(G)$ then f is uniquely represented by a Fourier series

$$f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma D_\gamma(x))$$

where Tr denotes the usual trace, Γ is the set of equivalence classes of irreducible unitary representations of G, D_γ is a representative of the class γ, d_γ is the degree of γ, and A_γ is the linear transformation given by

$$A_\gamma = \int_G f(x) D_\gamma(x^{-1}) dx \ .$$

Helgason has proved

I′. *Suppose that, for any choice of unitary transformations* U_γ *on the Hilbert space of dimension* d_γ, $\sum_{\gamma \in \Gamma} d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x))$ *is the Fourier series of an integrable function (or a Fourier-Stieltjes series) then*

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$$\sum_{\gamma\in\Gamma} d_\gamma Tr(A_\gamma A_\gamma^*) < \infty .$$

In view of the Schur-Peter-Weyl formula

$$\int_G |f(x)|^2 dx = \sum_{\gamma\in\Gamma} d_\gamma Tr(A_\gamma A_\gamma^*) ,$$

Helgason's result is an extension of I.

In this paper, using Helgason's techniques, we propose to extend II to compact groups in the same sense. That is we prove

II′. *Let* $\sum d_\gamma Tr(A_\gamma A_\gamma^*) < \infty$. *Given* $p < \infty$ *there exist unitary transformations* U_γ *such that* $\sum d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x))$ *is the Fourier series of a function in* L^p.

This is accomplished as in [3] by proving and exploiting the "lacunarity" of a certain subset of the space of irreducible unitary representations of the product group $\prod_{i\in S} U(d_i)$ where $U(d_i)$ is the group of unitary transformations of the Hilbert space of dimension d_i and S is an arbitrary index set. In the last section we discuss in general lacunary properties of subsets of the space of irreducible representations of a compact group.

2. **The main result.** For a positive integer n let $U(n)$ be the group of unitary transformations of the Hilbert space of dimension n. The normalized Haar measure on $U(n)$ will be denote by dV.

LEMMA 1. *Let A be an $n \times n$ matrix. Then for $s = 1, 2, 3, \cdots$*

$$\int_{U(n)} | Tr(AV) |^{2s} dV \leqq \frac{B(s)}{n^s} [Tr(AA^*)]^s \tag{1}$$

where $B(s)$ is a constant depending only on s.

Proof. Since dV is left and right invariant it is sufficient to prove the lemma when A is diagonal. Letting $e_1, e_2, \cdots, e_n$ be a basis for the Hilbert space on which A and V act and $a_i = \langle Ae_i, e_i\rangle$, $v_i = \langle Ve_i, e_i\rangle$ we have

$$\int_{U(n)} | Tr(AV) |^{2s} dV = \sum a_{i_1}\bar{a}_{i_2}a_{i_3}\bar{a}_{i_4} \cdots a_{i_{2s-1}}\bar{a}_{i_{2s}} \cdot \int_{U(n)} v_{i_1}\bar{v}_{i_2} \cdots v_{i_{2s-1}}\bar{v}_{i_{2s}} dV , \tag{2}$$

where the sum extends over all $i_1, i_2, \cdots, i_{2s}$ such that $1 \leqq i_j \leqq n$. Each integral in the sum is of the form

$$(3) \qquad \int_{U(n)} v_1^{j_1}\bar{v}_1^{k_1} \cdots v_n^{j_n}\bar{v}_n^{k_n} dV .$$

Now each such integral is zero unless $j_1 = k_1, \cdots, j_n = k_n$. For let W be a diagonal unitary matrix with elements α_i of modulus one on the main diagonal. Then by the invariance of dV, (3) becomes

$$\begin{aligned} &\int_{U(n)} \prod_{i=1}^n \langle Ve_i, e_i\rangle^{j_i} \langle e_i, Ve_i\rangle^{k_i} dV \\ &= \int_{U(n)} \prod_{i=1}^n \langle WVe_i, e_i\rangle^{j_i} \langle e_i, WVe_i\rangle^{k_i} dV \\ &= \prod_{i=1}^n \alpha_i^{j_i - k_i} \int_{U(n)} \prod_{i=1}^n \langle Ve_i, e_i\rangle^{j_i} \langle e_i, Ve_i\rangle^{k_i} dV . \end{aligned}$$

Thus if the integral is not zero, $\prod_{i=1}^n \alpha_i^{j_i-k_i} = 1$, for all choices of the α_i. Clearly this is possible only if $j_1 = k_1, \cdots, j_n = k_n$. Thus the sum (2) is equal to

$$(4) \qquad \sum |a_{i_1}|^2 a_{i_2}|^2 \cdots |a_{i_s}|^2 \int_{U(n)} |v_{i_1}|^2 \cdots |v_{i_s}|^2 dV .$$

We shall see that for each integer s

$$(5) \qquad \int_{U(n)} |v_i|^{2s} dV \leqq \frac{B(s)}{n^s} \qquad (s = 1, 2, \cdots)$$

where $B(s)$ depends only on s. It then follows from Hölder's inequality that the integrals in (4) are bounded by $B(s)/n^s$ so that (4) is majorized by

$$\frac{B(s)}{n^s} \sum |a_{i_1}|^2 \cdots |a_{i_s}|^2 = \frac{B(s)}{n^s} [Tr(AA^*)]^s ,$$

and the lemma will be proved.

It is sufficient to calculate (5) for $i = 1$. Let $U_1(n-1)$ be the subgroup $\{T \in U(n) : Te_1 = e_1\}$. The space $U(n)/U_1(n-1)$ of left cosets $\{\tilde{V} = VU_1(n-1) : V \in U(n)\}$ can be identified with the unit sphere Σ_n in a complex n-dimensional Hilbert space. Since v_1 is constant on these cosets

$$\int_{U(n)} |v_1|^{2s} dV = \int_{\Sigma_n} |\langle Ve_1, e_1\rangle|^{2s} d\tilde{V} = \int_{|w_1|^2 + \cdots + |w_n|^2 = 1} |w_1|^{2s} d\tilde{V}$$

where $d\tilde{V}$ is the unique normalized measure on Σ_n invariant with respect to $U(n)$ and

$$Ve_1 = w_1 e_1 + \cdots + w_n e_n .$$

If we identify Σ_n with the real $(2n-1)$ dimensional sphere S^{2n-1} in real $2n$-dimensional space and $d\tilde{V}$ with dw, the normalized invariant measure on S^{2n-1}, then

$$(6) \qquad \int_{U(n)} |v_1|^{2s} dV = \int_{x_1^2+\cdots+x_{2n}^2=1} (x_1^2 + x_2^2)^s dw \, .$$

By Minkowski's inequality and the invariance of dw (6) is bounded by

$$2^s \int_{S^{2n-1}} x_1^{2s} dw = 2^s \frac{\Omega(S^{2n-2})}{\Omega(S^{2n-1})} \int_{-1}^{1} x_1^{2s}(1 - x_1^2)^{n-1}(1 - x_1^2)^{-1/2} dx_1$$

where $\Omega(S^m) = \dfrac{2\pi^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}\right)}$ is the Euclidean surface area of the real m-dimensional unit sphere. Thus the integral in (6) is bounded by

$$2^s \frac{2\pi^{(2n-1)/2}}{\Gamma\left(n - \frac{1}{2}\right)} \cdot \frac{\Gamma(n)}{2\pi^{(2n)/2}} \cdot \frac{\Gamma\left(s + \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)}{\Gamma(n + s)} \leqq \frac{B(s)}{n^s}$$

which proves (5).

COROLLARY 2. *Let J be the canonical representation $U \to U$ of $U(n)$ and $J_{s,t}$ be the tensor product of J, s times and $\tilde{J}$, the conjugate representation, t times. $J_{s,t}$ decomposes into at most $B(s + t)$ irréducible components. If $s \neq t$ then none of the components is the identity representation.*

Proof. If χ_T is the character of the representation T, then $\chi_{J_{s,t}}(V) = (\chi_J(V))^s\overline{(\chi_J(V))}^t = (Tr(V))^s\overline{(Tr(V))}^t$. Thus by the lemma

$$\int_{U(n)} |\chi_{J_{s,t}}(V)|^2 dV \leqq B(s + t) \, ,$$

which proves the first statement.

The number of times the identity representation occurs in $J_{s,t}$ is

$$\int_{U(n)} \chi_{J_{s,t}}(V) dV = \int_{U(n)} (Tr(V))^s\overline{(Tr(V))}^t dV = 0$$

if $s \neq t$ by the statement following (3).

LEMMA 3. *Let $G = \prod_{i \in S} U(d_i)$ be a product of unitary groups $U(d_i)$. Let $F(V)$ be a function on G of the form*

$$F(V) = \sum_{i \in S} d_i Tr(A_i V_i)$$

where A_i is a $d_i \times d_i$ matrix and V_i is the projection of V on $U(d_i)$. Then

$$\int_G |F(V)|^{2s} dV \leqq B(s)\left(\int |F(V)|^2 dV\right)^s$$

where dV is the normalized Haar measure on G.

Proof. It suffices to prove the lemma when

$$F(V) = \sum_{i=1}^{N} d_i Tr(A_i V_i) .$$

Then

$$(7) \quad \int_G |F(V)|^{2s} dV$$
$$= \sum \int_{U(d_i)\times\cdots\times U(d_n)} d_{i_1} Tr(A_{i_1} V_{i_1}) d_{i_2} \overline{Tr(A_{i_2} V_{i_2})} \cdots d_{i_{2s}} \overline{(A_{i_{2s}} V_{i_{2s}})} dV$$

where the sum extends over all $i_1, \cdots, i_{2s}$ such that $1 \leqq i_j \leqq N$. By the corollary the only terms in the sum which do not vanish are those of the form

$$(8) \qquad \int_{U(d_i)\times\cdots\times U(d_n)} d_{i_1}^2 |Tr(A_{i_1} V_{i_1})|^2 \cdots d_{i_s}^2 |Tr(A_{i_s} V_{i_s})|^2 dV .$$

By Hölder's inequality (8) is majorized by

$$d_{i_1}^2 \cdots d_{i_s}^2 \left[\int_{U(d_{i_1})} |Tr(A_{i_1} V_{i_1})|^{2s} dV_{i_1}\right]^{1/s} \cdots \left[\int_{U(d_{i_s})} |T_r(A_{i_s} V_{i_s})|^{2s} dV_{i_s}\right]^{1/s} .$$

which by Lemma 1 is majorized by

$$d_{i_1}^2 \cdots d_{i_s}^2 B(s) \frac{Tr(A_{i_1} A_{i_1}^*)}{d_{i_1}} \cdots \frac{Tr(A_{i_s} A_{i_s}^*)}{d_{i_s}} .$$

Hence the left side of (7) is bounded by

$$B(s)\left[\sum_1^N d_i Tr(A_i A_i^*)\right]^s = B(s)\left[\int_G |f(V)|^2 dV\right]^s ,$$

where the equality follows from the Peter-Weyl formula.

Now let G be an arbitrary compact group and Γ be the set of equivalence classes of irreducible representations of G. Let d_γ be the degree of the class γ. Then $\boldsymbol{G} = \prod_{\gamma\in\Gamma} U(d_\gamma)$ is a compact group which can be thought of as the group of unitary transformations of $L^2(G)$ into $L^2(G)$ which commute with left translations. That is, if V is such a transformation then V corresponds to the element $\{V_\gamma\} \in \boldsymbol{G}$ such that

$$Vf(x) \sim \sum_{\gamma\in\Gamma} d_\gamma Tr(V_\gamma A_\gamma D_\gamma(x))$$

whenever $f(x) \sim \sum_{\gamma\in\Gamma} Tr(A_\gamma D_\gamma(x)) \in L^2(G)$.

THEOREM 4. *Let* $f \in L^2(G)$ *and* $p < \infty$, *then for almost every* $V \in \boldsymbol{G}$, $Vf \in L^p(G)$.

Proof. Let $Vf(x) = f(V, x) = \sum d_\gamma Tr(V_\gamma A_\gamma D_\gamma(x))$. Then $f(V, x)$ can be considered as a function on $\boldsymbol{G} \times G$. For fixed $x \in G$ we have by Lemma 2 that

$$\int_{\boldsymbol{G}} |f(V, x)|^{2s} dV \leqq B(s)\left[\int_{\boldsymbol{G}} |f(V, x)|^2 dV\right]^s = B(s)\left[\sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma A_\gamma^*)\right]^s$$
$$= B(s)\left[\int_G |f(x)|^2 dx\right]^s,$$

so that

$$\int_{\boldsymbol{G}}\int_G |f(V, x)|^{2s} dx dV = \int_G \int_{\boldsymbol{G}} |f(V, x)|^{2s} dV dx$$
$$\leqq B(s)\left[\int_G |f(x)|^2 dx\right]^s.$$

Therefore if $f \in L^2(G)$, then for almost every $V \in \boldsymbol{G}$, $\int_G |Vf(x)|^{2s} dx < \infty$. Letting $s > p/2$ we obtain the theorem.

We remark for later use that for some V

$$\| Vf \|_p \leqq \| Vf \|_{2s} \leqq 2B(s) \| f \|_2 .$$

Indeed the set of V for which

$$\int_G |Vf(x)|^{2s} dx > 2B(s)\left[\int_G |f(x)|^2 dx\right]^s$$

cannot be of measure one.

We will also use the following

REMARK 5. Let $f \in C(G)$ be a continuous function such that for all self adjoint $V \in \boldsymbol{G}$, $Vf \in C(G)$, then $f(x) \sim \sum d_\gamma Tr(A_\gamma D_\gamma(x))$ with $\sum d_\gamma Tr(|A_\gamma|) < \infty$ ($|A_\gamma|$ is the absolute value of the matrix A_γ). Indeed letting $\tilde{f}(x) = \overline{f(x^{-1})}$ we can write $f = (f + \tilde{f})/2 + i(f - \tilde{f})/2i = f_1 + if_2$. If $f_i(x) \sim \sum d_\gamma Tr(A_{\gamma, i} D_\gamma(x))$ $(i = 1, 2)$ then $A_{\gamma, i}^* = A_{\gamma, i}$. Therefore there exists a self adjoint $V = \{V_\gamma\} \in \boldsymbol{G}$ such that $A_{\gamma, i} V_\gamma = |A_{\gamma, i}|$. Thus $\sum d_\gamma Tr(|A_{\gamma, i}| D_\gamma(x))$ is continuous so that applying a method of summation as in [4, 8.3] we obtain that the partial sums of $\sum d_\gamma Tr(|A_{\gamma, i}|) = \sum d_\gamma Tr(|A_{\gamma, i}| D_\gamma(e))$ are bounded. Thus $\sum d_\gamma Tr(|A_\gamma|) \leqq \sum d_\gamma Tr(|A_{\gamma, 1}|) + \sum d_\gamma Tr(|A_{\gamma, 2}|) < \infty$.

We shall call a series $\sum d_\gamma Tr(A_\gamma D_\gamma(x))$ satisfying $\sum d_\gamma Tr(|A_\gamma|) < \infty$ an *absolutely convergent* series. The space of such functions will be denoted by $A(G)$. It is easy to see that $A(G)$ consists of functions of the type $f * g$ with $f, g \in L^2(G)$. The space $A(G) = L^2(G) * L^2(G)$ has been studied in [1].

3. **Lacunary Fourier series.** Given a compact group G we shall say that a subset $E \subseteqq \Gamma$ of the set of irreducible unitary representation of G is a *Sidon set* if it satisfies the following property:

A. $\sum d_\gamma Tr(|A_\gamma|) < \infty$ *whenever* $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$ *is the Fourier series of a continuous function* (*cf.* [6, 5.7]).

A set $E \subset \Gamma$ will be called a set of type $\Lambda(p)$ (or $E \in \Lambda(p)$) for $p > 1$ if it satisfies

B. *If* $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$ *is the Fourier series of an integrable function then it is the Fourier series of a function in* L^p (*cf.* [8]).

If B is a space of functions on G and $E \subseteqq \Gamma$ we will denote by B_E those functions in B with a series of the form $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$. It is seen as in [8, 1.4] that $E \in \Lambda(p)$ if, for some $r < p$, $L^r_E = L^p_E$. Clearly $\Lambda(p_1) \subseteqq \Lambda(p_2)$ if $p_1 \geqq p_2$.

If $\boldsymbol{G} = \prod_{i \in S} \boldsymbol{U}(d_i)$ then S can be thought of as the set of irreducible representations of $\boldsymbol{G}$ consisting of the projections of $\boldsymbol{G}$ onto the $\boldsymbol{U}(d_i)$. Lemma 2 shows that $S \in \Lambda(p)$ for every $p < \infty$. It is a simple matter to prove that S is also a Sidon set. Indeed, if $f(V) = \sum_{i \in S} d_i Tr(A_i V_i)$ is a continuous function belonging to $C_S(\boldsymbol{G})$ and if $U = \{U_i\} \in \boldsymbol{G}$ then

$$Uf(V) = \sum_{i \in S} d_i Tr(A_i U_i V_i) = \text{left translation of } f \text{ by } U,$$

is also continuous. It suffices to pick the U_i so that $A_i U_i = |A_i|$ to obtain that $\sum_{i \in S} d_i Tr(|A_i|) < \infty$.

We shall now establish a characterization of sets of type $\Lambda(p)$ which will imply that every Sidon set is a $\Lambda(p)$ set for every $p < \infty$. For a group G denote by $\mathscr{R}_p = \mathscr{R}_p(G)$ the algebra of operators on $L^p(G)$ generated in the weak operator topology by the operators $\{R_y : y \in G\}$ where $R_y f(x) = f(xy)$. We shall use the fact [2, Th. 6] that $\mathscr{R}_p$ is (isometric and isomorphic to) the dual space of a Banach space A^p of continuous functions on G. $A^2 = A(G)$ the space of functions with absolutely convergent Fourier series [1].

The isomorphism between $\mathscr{R}_p$ and the dual space of A^p is given by $T \to \varphi_T$ where $\varphi_T(f) = Tf(e)$. This correspondence is well defined because every $T \in \mathscr{R}_p$ maps each element of A^p into a continuous function, indeed an element of A^p. We also have that $\mathscr{R}_p$ consists exactly of those bounded operators on L^p which commute with left translations.

Now if $T \in \mathscr{R}_p$, $p > 2$, then $T \in \mathscr{R}_2$ and $\| T \|_{\mathscr{R}_2} \leqq M \| T \|_{\mathscr{R}_p}$ where M is a constant depending only on p. For if $f \in L^2$ then by Theorem

3 there exists a unitary transformation U commuting with right translations (and therefore with elements of $\mathscr{R}_p$) such that $Uf \in L^p$. We can also choose U such that $\| Uf \|_p \leqq 2B(s) \| f \|_2$ where $B(s)$ is the constant appearing in Lemma 1 and $s > p/2$ (cf. the remarks following the proof of Theorem 4).

We then have that $TUf \in L^p$ and $U^*TUf = TU^*Uf = Tf \in L^2$. Also $\| Tf \|_2 = \| U^*TUf \|_2 \leqq \| TUf \|_2 \leqq \| TUf \|_p \leqq \| T \|_{\mathscr{R}_p} \| Uf \|_p \leqq \| T \|_{\mathscr{R}_p} M \| f \|_2$ where $M = 2B(s)$. Therefore $\| T \|_{\mathscr{R}_2} \leqq M \| T \|_{\mathscr{R}_p}$. This implies that $A^2 \subseteqq A^p$ and $\| \; \|_{A^p} \leqq M \| \; \|_{A^2}$. It is now a simple matter to prove:

THEOREM 6. *Let $E \subseteqq \Gamma$ be a set of irreducible unitary representations of G and $p > 2$. The following are equivalent:*
(a) *E is a set of type $\Lambda(p)$.*
(b) *If $T \in \mathscr{R}_2$ there exists $S \in \mathscr{R}_p$ such that $Tf = Sf$ for all $f \in L^p_E$.*
(c) *If $f \in A^p_E$ then $f \in A^2 = A(G)$.*
(d) *Every closed subspace of L^p_E which is invariant under left translations is the range of a projection P belonging to $\mathscr{R}_p$ which is self-adjoint in the sense that $P_\gamma = P^*_\gamma$ for each $\gamma \in \Gamma$.*

Proof. Let $E \in \Lambda(p)$. Then $L^p_E = L^2_E$ so that by the open mapping theorem there exists B such that $\| f \|_p \leqq B \| f \|_2$ for $f \in L^2_E$. As L^2_E is invariant under right and left translations there exists a projection P_E of L^2 onto L^2_E which commutes with right and left translations. If $T \in \mathscr{R}_2$ let $S = TP_E$, then $\| Sf \|_p \leqq \| T \| \| P_E f \|_p \leqq \| T \| B \| P_E f \|_2 \leqq B \| T \| \| f \|_p$. Thus $S \in \mathscr{R}_p$ and (a) implies (b).

Now assume (b) holds. If $f \in L^2_E$ then by Theorem 3 there exists $U \in \mathscr{R}_2$ such that $Uf \in L^p$; clearly $Uf \in L^p_E$. Let $S \in \mathscr{R}_p$ be such that $Sg = U^*g$ for all $g \in L^p_E$; then $SUf = U^*Uf = f \in L^p$. Hence $L^p_E = L^2_E$ so that (b) implies (a).

We now show that (a) and (b) imply (d). Indeed if (a) holds the projection P_E of L^2 onto L^2_E is bounded in L^p. Suppose the $Y \subseteqq L^p_E$ is invariant under left translations, let P_Y be the projection (belonging to $\mathscr{R}_2$) of L^2 onto the left invariant subspace of L^2 generated by Y. By (b) there exists $S \in \mathscr{R}_p$ with $S = P_Y$ on L^p_E. Then $P_E S = P_Y$ so that $P_Y \in \mathscr{R}_p$.

Suppose (d) holds and let U be a unitary self adjoint element of $\mathscr{R}_2$. Then $U^2 = I$ so that $P = (U + I)/2$ is a projection which commutes with left translations. Let Y be the subspace of L^p_E generated by $PL^2_E \cap L^p_E$. Then Y is invariant under left translations so that by (d) there is a self-adjoint projection of L^p onto Y commuting with left translations. Clearly this projection is PP_E so that $PP_E \in \mathscr{R}_p$. Hence $UP_E = (2P - I)P_E \in \mathscr{R}_p$. Therefore $UP_E f$ is continuous for every $f \in A^p$. In particular if $f(x) \sim \sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x)) \in A^p_E$ then $UP_E f = Uf$ is continuous. Therefore by Remark 4, $A_E^p \subseteqq A(G)$.

Finally since $A \subseteqq A^p$ with $\|f\|_{A^p} \leqq M\|f\|_A$, (c) implies that $A_E = A_E^p$, so that, by the closed graph theorem, $\|f\|_A \leqq B\|f\|_{A^p}$ for each $f \in A_E^p$. Each $T \in \mathscr{R}_2$ defines therefore a continuous linear functional on A_E^p by $Tf(e)$. The Hahn-Banach extension of this functional determines, in view of the duality between A^p and $\mathscr{R}_p$, an element $S \in \mathscr{R}_p$ such that $Sf(x) = (SL_xf)(e) = (TL_xf)(e) = Tf(x)$ for $f \in A_E^p$; therefore $S = T$ on L_E^p. Thus (c) implies (b) and the theorem is proved.

REMARK 7. It suffices for condition (d) to be true that every closed left invariant subspace of L_E^p is the range of a projection. Indeed the argument used in [7, Th. 1] will show that such a projection can be chosen to be left invariant (and therefore belonging to $\mathscr{R}_p$).

THEOREM 8. *$E \subseteqq \Gamma$ is a Sidon set if and only if for each $T \in \mathscr{R}_2$ there exists a finite measure μ on G such that $Tf = f * \mu$ for each $f \in L_E^2$.*

Proof. One applies the same duality argument used in the proof of Theorem 6 (cf. also [6, 5.7.3]. Assume first that E is a Sidon set. Then given $T \in \mathscr{R}_2$, define a linear functional F on C_E by $F(f) = Tf(e)$. Then F is well defined, since $f \in C_E \Rightarrow f \in A$; by the closed graph theorem F is continuous and has a Hahn-Banach extension to all of $C(G)$. That is, by the Riesz representation theorem there exists a bounded measure μ satisfying

$$F(f) = \int_G f(x^{-1})d\mu(x) \quad \text{for all} \quad f \in C_E \ .$$

Since T commutes with left translations $Tf = f * \mu$ for all $f \in C_E$. Conversely let E satisfy the hypothesis of the theorem, to prove that E is a Sidon set, let $f \in C_E$ and let T be an unitary element of $\mathscr{R}_2$. By hypothesis there exists a measure μ such that $Tf = f * \mu$. Hence $Tf \in C(G)$ and by Remark 5, $f \in A$.

COROLLARY 9. *Every Sidon set is a $\Lambda(p)$ set for every p.*

Proof. If μ is a bounded measure and $R_\mu f = f * \mu$, then $R_\mu \in \mathscr{R}_p$ for every p. Therefore, by Theorem 8 if E is a Sidon set condition (b) of Theorem 6 holds.

REMARK 10. In [4, 9.2] a sufficient condition for a set $E \subseteqq \Gamma$ to be a Sidon set is given. This condition includes the requirements that the degrees of the representations of E be bounded. The fact that for $\prod_{i \in S} U(d_i)$, S is a Sidon set shows that this requirement is not necessary.

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