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ALGEBRAS AND FIBER BUNDLES

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Let A be an associative algebra and $\hat{A}_n$ the family of all equivalence classes of irreducible representations of A of dimension exactly n. Topologizing $\hat{A}_n$ as in a paper about to appear in the Transactions of the American Mathematical Society, we show that for each n, A gives rise to a fiber bundle having $\hat{A}_n$ as its base space and the $n \times n$ total matrix algebra as its fiber.

Throughout this note A will be an arbitrary fixed associative algebra over the complex field C. By a *representation* of A we understand a homomorphism T of A into the algebra of all linear endomorphisms of some complex linear space $H(T)$, the *space of* T. We write $\dim(T)$ for the dimension of $H(T)$. Irreducibility and equivalence of representations are understood in the purely algebraic sense. If T is a representation, $r \cdot T$ will be the direct sum of r copies of T. Let $\hat{A}^{(f)}$ the family of all equivalence classes of finite-dimensional irreducible representations of A; and put

$$\hat{A}^{(n)} = \{T \in \hat{A}^{(f)} \mid \dim(T) \leqq n\},\ \hat{A}_n = \{T \in \hat{A}^{(f)} \mid \dim(T) = n\} .$$

We shall usually not distinguish between representations and the equivalence classes to which they belong.

Let T be a finite-dimensional representation of A. If for each a in A $\tau(a)$ is the matrix of T_a with respect to some fixed ordered basis of $H(T)$, then $\tau: a \rightarrow \tau(a)$ is a *matrix representation of* A *equivalent to* T.

By $A^{\#}$ we mean the space of all complex linear functionals on A, and by $\mathrm{Ker}(\varphi)$ the kernel of φ. If $T \in \hat{A}^{(f)}$, we put

$$\Phi(T) = \{\varphi \in A^{\#} \mid \mathrm{Ker}(T) \subset \mathrm{Ker}(\varphi)\} .$$

An element φ of $A^{\#}$ is *associated* with T if $\varphi \in \Phi(T)$. One element of $\Phi(T)$ is of course the character χ^T of $T(\chi^T(a) = \mathrm{Trace}(T_a)$ for a in A). An element T of $\hat{A}^{(f)}$ is uniquely determined by the knowledge of one nonzero functional in $\Phi(T)$ ([2], Proposition 2).

As in [2] we equip $\hat{A}^{(f)}$ with the *functional topology* as follows: If $T \in \hat{A}^{(f)}$ and $\mathscr{S} \subset \hat{A}^{(f)}$, T belongs to the functional closure of $\mathscr{S}$ if $\Phi(T) \subset (\bigcup_{S \in \mathscr{S}} \Phi(S))^-$ where $^-$ denotes closure in the topology of pointwise convergence on A.

Our main object in this note is to prove the following fact about

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the functional topology relativized to $\hat{A}_n$:

THEOREM 1. *Fix a positive integer n; and let T be any element of $\hat{A}_n$. Then there exists a neighborhood U of T in $\hat{A}_n$, and a function τ assigning to each S in U a matrix representation τ_S of A equivalent to S, such that for each a in A the matrix-valued function*

$$S \longrightarrow \tau_S(a) \quad (S \in U)$$

is continuous on U.

This asserts (see §4) that, for each n, A gives rise to a fiber bundle with base space $\hat{A}_n$ whose fiber is the $n \times n$ total matrix algebra.

2. **Preliminary results.** The following Proposition 1 coincides with Proposition 7 of [2] (which was stated in [2] without proof). Proposition 1 is not required for what follows it; but its proof is related to later proofs.

PROPOSITION 1. *Let n be a positive integer; and suppose that $\{T^{(i)}\}$ is a net of elements of $\hat{A}^{(n)}$ converging to each of the p inequivalent elements $V^1, \cdots, V^p$ of $\hat{A}^{(n)}$. Then*

$$\sum_{s=1}^{p} (\dim (V^s))^2 \leqq n^2 . \tag{1}$$

Proof. Let $m_s = \dim (V^s)$, $q = \sum_{s=1}^{p} m_s^2$. Each $\Phi(V^s)$ has dimension m_s^2, and by the Extended Burnside Theorem ([1], Theorem 27.8) the $\Phi(V^s)$ $(s = 1, \cdots, p)$ are linearly independent subspaces of A^*. Thus there are q linearly independent functionals $\varphi_1, \cdots, \varphi_q$ each of which is associated with some V^s. By the definition of the functional topology we can replace $\{T^{(i)}\}$ by a subnet, and choose for each $r = 1, \cdots, q$ and each i a functional φ_r^i in $\Phi(T^{(i)})$, such that

$$\varphi_r^i \xrightarrow[i]{} \varphi_r \, (r = 1, \cdots, q) . \tag{2}$$

Since the $\varphi_1, \cdots, \varphi_q$ are independent, (2) implies that for some i the $\varphi_1^i, \cdots, \varphi_q^i$ are independent. Since $\dim (\Phi(T^{(i)})) \leqq n^2$, it follows that $q \leqq n^2$. This proves (1).

REMARK. If A is a Banach algebra we have shown elsewhere ([2], Proposition 13) that a stronger inequality than (1) holds, namely

$$\sum_{s=1}^{p} \dim (V^s) \leqq n . \tag{3}$$

Probably (3) holds for arbitrary A, but we have not been able to prove it.

COROLLARY 1. *$\hat{A}_n$ is Hausdorff for each n.*

For each φ in $A^\ddagger$ let us define S^φ to be the natural representation of A acting in A/J, where J is the left ideal of A consisting of those a such that $\varphi(ba) = 0$ for all b in A.

LEMMA 1. *Let $\{\varphi_i\}$ be a net of elements of $A^\ddagger$, converging pointwise to an element φ of $A^\ddagger$; and suppose the S^φ, S^{φ_i} are all finite-dimensional. Then*

$$\dim(S^\varphi) \leqq \liminf_i \dim(S^{\varphi_i}) . \tag{4}$$

Further, if σ is a matrix representation of A equivalent to S^φ, there exists for each i a matrix representation σ^i of A equivalent to S^{φ_i} such that

$$\lim_i (\sigma^i(a))_{jk} = (\sigma(a))_{jk} \tag{5}$$

for all a in A and all $j, k = 1, \cdots, \dim(S^\varphi)$.

Proof. Let π be the natural map of A onto A/J, where $J = \{a \in A \mid \varphi(ba) = 0$ for all b in $A\}$; and put $m = \dim(S^\varphi)$. Every element of $(A/J)^\ddagger$ is of the form

$$\pi(a) \longrightarrow \varphi(ba) \qquad (a \in A)$$

for some b in A. Hence there are elements $a_1, \cdots, a_m\, b_1, \cdots, b_m$ of A satisfying

$$\varphi(b_j a_k) = \delta_{jk}(j, k = 1, \cdots, m) . \tag{6}$$

Since $\varphi_i \rightarrow \varphi$, (6) implies that

$$\det\{(\varphi_i(b_j a_k))_{j,k=1,\dots,m}\} \neq 0 , \tag{7}$$

and hence $\dim(S^{\varphi_i}) \geqq m$, for all large i. This proves (4).

Now the a_k, b_j could have been chosen to satisfy not only (6) but also

$$(\sigma(x))_{jk} = \varphi(b_j x a_k) \tag{8}$$

($x \in A; j, k = 1, \cdots, m$); assume this done. By (7), for each large i there are unique complex numbers $c^i_{jk}(j, k = 1, \cdots, m)$ such that the elements $b^i_j = \sum_{k=1}^m c^i_{jk} b_k$ satisfy

(9) $$\varphi_i(b_j^i a_k) = \delta_{jk} \qquad (j, k = 1, \cdots, m) .$$

By (6) and (9)

(10) $$\lim_i c_{jk}^i = \delta_{jk} .$$

In view of (4) and (9), there are elements $a_{m+1}^i, \cdots, a_{p_i}^i, b_{m+1}^i, \cdots, b_{p_i}^i$ of A (where $p_i = \dim(S^{\varphi_i})$), such that

(11) $$\varphi_i(b_j^i a_k^i) = \delta_{jk}$$

for all large i and all $j, k = 1, \cdots, p_i$; (here we agree that $a_j^i = a_j$ for $j = 1, \cdots, m$). Now, if $j, k = 1, \cdots, p_i$ and $x \in A$, define

$$(\sigma^i(x))_{jk} = \varphi_i(b_j^i x a_k^i) .$$

From (8), (10), and (11), we verify that σ^i is a matrix representation equivalent to S^{φ_i} and that (5) holds. This completes the proof.

The following corollary was stated without proof as Proposition 8 of [2].

COROLLARY 2. *For each positive integer n, the map $T \to \chi^T (T \in \hat{A}_n)$ is a homeomorphism of $\hat{A}_n$ into $A^\sharp$ (the latter having the topology of pointwise convergence on A).*

Proof. Obviously $\chi^T \to T$ is continuous. To prove that $T \to \chi^T$ is continuous, we shall suppose that $T, \{T^i\}$ are elements of $\hat{A}_n$ and that $\varphi_i \xrightarrow[i]{} \chi^T$ pointwise on A, where for each i φ_i is associated with T^i; and we shall prove that $\chi^{T^i} \xrightarrow[i]{} \chi^T$ pointwise on A. Clearly this is sufficient.

By [2], Proposition 1, $S^{\chi^T} \cong n \cdot T$ and $S^{\varphi_i} \cong r_i \cdot T^i$, where $r_i \leqq n$. By (4) $r_i = n$ for all large i. Hence by (5) $\chi^T(a) = 1/n$ Trace $(S_a^\varphi) = \lim_i 1/n$ Trace $(S_a^{\varphi_i}) = \lim_i \chi^{T^i}(a)$ for all a in A. So $\chi^{T^i} \to \chi^T$, and the corollary is proved.

If M is any finite-dimensional complex linear space, the family $\mathscr{F}$ of all linear subspaces of M of fixed dimension r ($r \leqq \dim(M)$) has a natural compact topology. Indeed, if G is the unitary group on M (with respect to some fixed inner product), and G_0 is the subgroup of G which leaves stable some fixed L in $\mathscr{F}$, then $\mathscr{F}$ is in one-to-one correspondence with G/G_0, and the (compact) topology of $\mathscr{F}$ which makes this correspondence a homeomorphism is independent of the inner product and of L.

If p is any positive integer, M_p will be the $p \times p$ total matrix algebra over the complexes. Fix a positive integer n; and let $\mathscr{L}$ be the family of all those subalgebras A of M_{n^2} which contain 1 and are isomorphic with M_n. For each A in $\mathscr{L}$ let A' be the commuting algebra of A in M_{n^2}:

$$A' = \{a \in M_{n^2} \mid ab = ba \text{ for all } b \text{ in } A\} .$$

It is well known that $A' \in \mathscr{L}$ and that $A'' = A$ whenever $A \in \mathscr{L}$.

LEMMA 2. *The map $A \to A'$ is continuous on $\mathscr{L}$ to $\mathscr{L}$ (with the topology discussed above).*

Proof. If not, then, by the compactness of the space $\mathscr{M}$ of all n^2-dimensional subspaces of M_{n^2}, one can find a net $\{A_i\}$ of elements of $\mathscr{L}$ such that $A_i \to A$, $A'_i \to B$, where $A \in \mathscr{L}$, $B \in \mathscr{M}$, $A' \neq B$. Choose an element b of B which is not in A', and let a be any element of A. Then for each i we can choose an a_i in A_i and b_i in A'_i so that $a_i \to a$, $b_i \to b$. Since $a_i b_i = b_i a_i$, passing to the limit we obtain $ab = ba$, whence $b \in A'$, a contradiction.

LEMMA 3. *Let A be in $\mathscr{L}$, and let e be a minimal nonzero idempotent in A. Then there is a neighborhood U of A in $\mathscr{L}$, and a continuous function w on U to M_{n^2} such that*

(i) $w(A) = e$, *and*

(ii) *for each B in U $w(B)$ is a minimal nonzero idempotent in B.*

Proof. Choose an element a of A whose spectrum in A is $\{1, 2, \cdots, n\}$, and such that the spectral idempotent (in A) corresponding to the eigenvalue 1 of a is precisely e; that is,

$$e = ((n-1)!)^{-1}(2-a)(3-a)\cdots(n-a) . \tag{12}$$

Introducing a Hilbert space inner product into M_{n^2} in an arbitrary manner and projecting, we can construct a continuous function α on $\mathscr{L}$ to M_{n^2} such that $\alpha(A) = a$ and $\alpha(B) \in B$ for each B in $\mathscr{L}$. Let $\sigma(B)$ be the spectrum of $\alpha(B)$ (considered as an element either of B or of M_{n^2}). Since α is continuous, $\sigma(B)$ is continuous as a function of B. Thus there is a neighborhood U of A in $\mathscr{L}$, and n continuous complex functions $\lambda_1, \cdots, \lambda_n$ on U such that

(i) $\lambda_r(A) = r$ $(r = 1, \cdots, n)$,

(ii) for each B in U the $\lambda_1(B), \cdots, \lambda_n(B)$ are all distinct, and

(iii) $\sigma(B) = \{\lambda_1(B), \cdots, \lambda_n(B)\}$ for each B in U. Now, for B in U, put

$$w(B) = \prod_{j=2}^{n} (\lambda_j(B) - \lambda_1(B))^{-1}(\lambda_j(B)\cdot 1 - \alpha(B)) .$$

Clearly w is continuous on U, $w(B) \in B$ for each B in U, and $w(A) = e$.

Since $w(B)$ is the spectral idempotent corresponding to the eigenvalue $\lambda_1(B)$ of $\alpha(B)$ (which has multiplicity 1), $w(B)$ is a minimal idempotent of B for each B in U.

LEMMA 4. *If $A \in \mathscr{L}$, there is a neighborhood U of A in $\mathscr{L}$, and a continuous function w on U to M_{n^2}, such that, for each B in U, $w(B)$ is a minimal idempotent of the commuting algebra of B.*

Proof. This follows immediately from Lemmas 2 and 3.

3. Proof of Theorem 1. We have seen ([2], Proposition 1) that $S^{\chi^T} \cong n \cdot T$. Thus, putting $m = n^2$, we may choose elements $a_1, \cdots, a_m$, $b_1, \cdots, b_m$ of A as in the proof of Lemma 1 so that

$$\chi^T(b_j a_k) = \delta_{jk} (j, k = 1, \cdots, m) .$$

Since $S \rightarrow \chi^S$ is continuous on $\hat{A}_n$ (Corollary 2), there is a neighborhood U' of T in $\hat{A}_n$ such that $\det (\chi^S(b_j a_k))_{j,k} \neq 0$ for S in U'. Thus, as in the proof of Lemma 1, for each S in U' we find unique complex numbers $c_{jk}(S)$ such that the elements $b_j(S) = \sum_{k=1}^m c_{jk}(S) b_k$ satisfy

$$\chi^S(b_j(S) a_k) = \delta_{jk} \tag{13}$$

$(j, k = 1, \cdots, m; S \in U')$. We now set

$$(\sigma_S(x))_{jk} = \chi^S(b_j(S) x a_k)$$

$(j, k = 1, \cdots, m; S \in U'; x \in A)$, and verify as in the proof of Lemma 1 that, for S in U', σ_S is a matrix representation of A equivalent to $n \cdot S$. Since $S \rightarrow \chi^S$ is continuous (Corollary 2), the $c_{jk}(S)$ are continuous in S on U', and so

$$S \longrightarrow \sigma_S(x) \quad \text{is continuous on } U' \tag{14}$$

for each x in A.

Since $\sigma_S \cong n \cdot S$, Burnside's Theorem asserts that the range $\sigma_S(A)$ of σ_S belongs to $\mathscr{L}$. Further, it follows from (14) that $S \rightarrow \sigma_S(A)$ is continuous on U' (in the topology of n^2-dimensional subspaces discussed in § 2). Thus, by Lemma 4, there is a neighborhood U'' of T contained in U', and a function w on U'' to M_m such that, for each S in U'', $w(S)$ is a minimal idempotent of the commuting algebra of $\sigma_S(A)$.

We now consider M_m is acting on C^m (the space of complex m-tuples). Let $v_1, \cdots, v_m$ be a basis of C^m such that $v_1, \cdots, v_n$ is a basis of range $(w(T))$. By the continuity of w there will be a neighborhood U of T contained in U'' such that

$$w(S)v_1, \cdots, w(S)v_n, v_{n+1}, \cdots, v_m \tag{15}$$

is a basis of C^m for each S in U (the first n vectors of (15) being, of course, a basis of range $(w(S))$). Now for each S in U and x in A let $\rho_S(x)$ be the matrix of $\sigma_S(x)$ with respect to the ordered basis (15), and let $\tau_S(x)$ be the $n \times n$ matrix consisting of the first n rows and columns of $\rho_S(x)$. Since $w(S)$ is a minimal idempotent of the commuting algebra of $\sigma_S(A)$, σ_S restricted to range $(w(S))$ is an irreducible subrepresentation of σ_S and so is equivalent to S. Thus, for each S in U, τ_S is a matrix representation of A equivalent to S. Further, since $S \to w(S)$ is continuous on U, the basis (15) varies continuously with S on U; and therefore by (14) we conclude that $S \to \tau_S(x)$ is continuous on U for each x in A. This completes the proof of Theorem 1.

4. Fiber bundles associated with A. Fix a positive integer n, and let G_n be the group of all algebraic automorphisms of the total matrix algebra M_n. We are going to describe to within equivalence a fiber bundle B_n with base space $\hat{A}_n$, fiber M_n, and group G_n. To do so, it is sufficient to specify an open covering of $\hat{A}_n$, and to define on the overlap of any two sets in the covering the G_n-valued "coordinate transformation functions" ([3], §§ 2, 3). As our open covering we take the set of all the $U = U_T$ ($T \in \hat{A}_n$) of Theorem 1. If $T, T' \in \hat{A}_n$, the coordinate transformation function $\Gamma_{T,T'}$ on $U_T \cap U_{T'}$ will assign to each S in $U_T \cap U_{T'}$ the following automorphism of M_n:

$$\Gamma_{T,T'}(S) : \tau_S^{(T)}(a) \longrightarrow \tau_S^{(T')}(a) \qquad (a \in A) .$$

(Here $\tau^{(T)}$ is the τ of Theorem 1). The property $\Gamma_{T,T''} = \Gamma_{T,T''} \circ \Gamma_{T,T'}$ (on $U_T \cap U_{T'} \cap U_{T''}$) obviously holds; and the continuity of the maps $S \to \tau_S^{(T)}(a)$ and $S \to \tau_S^{(T')}(a)$ assures us that $\Gamma_{T,T'}$ is continuous. Thus we have defined a fiber bundle of the required kind; its equivalence class clearly depends only on A.

Thus, if the algebra A has a large supply of finite-dimensional irreducible representations, the structure of the fiber bundles $B_n (n = 1, 2, \cdots)$ constitutes a significant feature of the structure of A. We hope in a later paper to discuss the structure of these bundles for certain special kinds of algebras associated with locally compact groups having "large" compact subgroups.

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